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Interest rate modeling with fractional Lévy process

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Abstract We propose a new interest rate model driven by fractional Lévy process. We derive the joint characteristic function for spot rate and its integral, which enables us to obtain the analytical formula for the prices of bonds and interest rate derivatives. We numerically study a particular type of long memory interest rate model, namely fractional normal inverse Gaussian (NIG) model. We show that the higher fractional integration parameter leads to the slower decay of term structure of volatilities. We also find the long memory parameter has significant effects on the prices of bonds and interest rate derivatives.

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1 Introduction

The term structure of interest rates is, perhaps, the most important entity in finance as it describes the relationship between the yields on a default free discount bond and its maturity. It is a key concept in economic and financial

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theory, and in the risk-neutral valuation and hedging of interest rate contingent claims. Many models of the term structure are based on the assumption that all information about the economy is contained in a finite-dimensional vector of state variables whose dynamics are governed by stochastic processes. In the pioneering work by Vasicek [1] a univariate diffusion process is proposed for modeling the unobservable instantaneous interest rate (spot rate). Cox, Ingersoll and Ross (CIR) [2] propose the square-root process for the spot rate in a general equilibrium framework in order to introduce heteroscedasticity in the spot rate dynamics. Duffie and Kan [3] present the most general affine class model, which nests the Vasicek and CIR models. A common feature of these models is to describe the randomness in the interest rate process by Brownian motion.

In recent years there is a growing body of literature that explicitly incorporates jumps in modeling the term structure of interest rates. For example, Das [4] extends the Vasicek model to a jump diffusion model and finds strong evidence of jumps in the daily Federal Funds rate. Jump diffusion process is a special case of Lévy process, while the general Lévy setting permits more flexible jump structures. Eberlein and Raible [5] develop a Lévy forward model and Eberlein et al. [6] extend it to time-inhomogeneous Lévy process. Hainaut and MacGilchrist [7] propose an interest rate model driven by a particular Lévy process, the normal inverse Gaussian (NIG) process. They show that an NIG process provides a better fit of bond returns than those driven by a Brownian motion.

One important implication of the existing popular interest rate models is that the spot rate process has the property that the correlation between spot rates n periods apart goes to zero exponentially. In this case, Backus and Zin [8] shows that the yield on an n-period bond converges to a constant: the variance of the yield on a long bond goes to zero exponentially, as well. However, the implication that long term rates are constant seems to be at odds with the data. Although the yield curve generally flattens out as the maturity increases, the long term rates are only marginally less volatile than short term rates but certainly their volatility does not appear to vanish. Backus and Zin [8] then propose a fractional ARMA model for spot rate as a resolution of the discrepancy between theory and evidence on long term rates. Golinski and Zaffaroni [9] present a two-factor fractional ARIMA term structure model which allows for long memory. They find that extension of the model from short memory to long memory factors gives a substantial improvement in terms of fit of the model and forecast errors.

The idea of modeling financial processes as long memory stochastic processes is not new in the literature. For example, it has been empirically observed that the autocorrelation function of the squared returns on stock price is usually characterized by its slow decay towards zero. This decay is neither exponential, as in short memory processes, nor implies a unit root, as in integrated processes. Consequently, it has been suggested that the squared returns may be modeled as a long memory process, whose autocorrelations decay at a hyperbolic rate. In this direction, Comte and Renault [10] propose a continuous time fractional stochastic volatility model. They assume that the stochastic volatility is driven by a fractional Ornstein-Uhlenbeck process; that is the standard Ornstein-Uhlenbeck process where a Brownian motion is replaced by a fractional Brownian Motion. Comte, Coutin and Renault [11] consider a fractional affine stochastic volatility model, where the volatility process is driven by a fractional square root process.

In this paper, we extend the works of Backus and Zin [8], Comte and Renault [10] and Comte, Coutin and Renault [11]. We consider an interest rate model driven by fractional Lévy process. We derive the joint characteristic function of spot rate and its integral. Then we evaluate a closed-form solution for bond price and interest rate derivatives such as bond options, caps and Asian options on short rate using Fourier inversion techniques. To the best of our knowledge, this is the first paper that provides the formula for interest rate derivatives based on a fractional Lévy process.

The structure of the paper is as follows. We start with a brief overview of Lévy process and introduce fractionally integrated process driven by the Lévy process in section 2. With these preliminaries and tools, we define a fractional Lévy interest rate model in section 3. The main result, the joint characteristic function of spot rate and its integral, is also included in this section. We provide the closed form solutions to the prices of bonds and interest rate derivatives such as bond options, caps and Asian options in section 4. Numerical results based on a particular Lévy process, normal inverse Gaussian (NIG) process, are given in section 5.

2 Preliminaries and tools

2.1 Lévy process: definition and properties

In this section Lévy processes are introduced together with several important definitions and properties. See Sato [12] for a more exhaustive treatment on Lévy processes.

Let $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ be a filtered probability space which satisfies the usual conditions.

Definition 2.1 A càdlàg (sample paths that are almost surely continuous from the right and have limits from the left), adapted, real valued stochastic process $L = \{L(t)\}_{t\geq 0}$ with $L(0) = 0$ a.s. is called a Lévy process if the following conditions are satisfied:

- L has independent increments, i.e. $L(t)-L(s)$ is independent of $\mathcal{F}_s$ for any $0 \leq s < t$;
- L has stationary increments, i.e. for any s, $t \geq 0$ the distribution $L(t+s)-L(t)$ does not depend on t;
- L is stochastically continuous, i.e. for every $t \geq 0$ and $\epsilon > 0$:

$$\lim_{s\uparrow t} P(|L(t) - L(s)| > \epsilon) = 0.$$

A Lévy process $L(t)$ is infinitely divisible, which indicates that the characteristic function of marginal random variable $L(t)$ can be expressed as follows:

$$\Phi_{L(t)}(u) = \mathrm{E}[\exp(iuL(t))] = \exp(t\psi_{L(1)}(u)),$$

where $\psi_{L(1)}(u)$ is the characteristic exponent of the Lévy process at unit time.

The Lévy-Khintchine formula (see Schoutens [13]) gives the expression for characteristic exponent $\psi_{L(1)}(u)$ as follows:

$$\psi_{L(1)}(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty} (\exp(iux) - 1 - iux1_{|x|<1})\nu(dx),$$

where $\gamma \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$ and ν is a positive measure satisfying:

$$\int_{-\infty}^{+\infty} \min(1, |x|^2)\nu(dx) < \infty.$$

The measure ν is called the Lévy measure of the distribution L. Throughout the whole paper we will always assume that ν additionally satisfies:

$$\int_{-\infty}^{+\infty} |x|^2\nu(dx) < \infty.$$

Furthermore, we impose the restriction that $\mathrm{E}[L(1)] = 0$.

In this paper we will work with two-sided Lévy process $L = \{L(t)\}_{t\in\mathbb{R}}$ constructed by taking two independent copies $\{L_1(t)\}_{t\geq 0}$, $\{L_2(t)\}_{t\geq 0}$ of a one-sided Lévy process and setting

$$L(t) = \begin{cases} L_1(t), & \text{if } t \geq 0, \\ -L_2(-t-), & \text{if } t < 0. \end{cases}$$

We consider the stochastic integral with respect to Lévy process. The following theorem of Rajput and Rosinski [14] will be used in the paper:

Theorem 2.1 *For $f \in \mathcal{L}^2[0, T]$ the integral $\int_0^T f(t)dL(t)$ exists as an $\mathcal{L}^2(\Omega)$-limit of approximating step functions. Moreover, we have for $u \in \mathbb{R}$:*

$$\mathrm{E}\left[\exp\left(iu\int_0^T f(t)dL(t)\right)\right] = \exp\left(\int_0^T \psi_{L(1)}(uf(t))dt\right).$$

2.2 Fractionally integrated process driven by the Lévy process

In this section, we first provide definitions of fractional calculus and integration, which we will require in the following discussions.

Definition 2.2 Let $f \in \mathcal{L}^1(\mathbb{R})$ and $d > 0$, the Riemann-Liouville fractional integrals on $\mathbb{R}$ are defined as

$$(I_+^d f)(s) = \frac{1}{\Gamma(d)} \int_{-\infty}^{s} f(u)(s-u)^{d-1} du,$$

and

$$(I_-^d f)(s) = \frac{1}{\Gamma(d)} \int_{s}^{\infty} f(u)(u-s)^{d-1} du.$$

Definition 2.3 Let $0 < d < 1$, the Riemann-Liouville fractional derivatives on $\mathbb{R}$ can be defined as

$$(D_+^d f)(u) = \frac{1}{\Gamma(1-d)} \frac{d}{du} \int_{-\infty}^{u} f(s)(u-s)^{-d} ds,$$

and

$$(D_-^d f)(u) = -\frac{1}{\Gamma(1-d)} \frac{d}{du} \int_{u}^{\infty} f(s)(s-u)^{-d} ds.$$

Let $0 < d < 1$. Following Samko, Kilbas and Marichev [15], we know that the fractional derivative D_+^d and fractional integration I_+^d have the following properties:

- For any $f \in \mathbb{R}$, we have
$$D_+^d I_+^d f = f;$$
- For any h such that $h = I_{a+}^d f$, we have
$$I_+^d D_+^d h = h.$$

The operators D_-^d and I_-^d have properties corresponding to those of D_+^d and I_+^d.

We now define the fractional Lévy process as follows:

Definition 2.4 Let $L = \{L(t)\}$ be a two-sided Lévy process on $\mathbb{R}$. For fractional integration parameter $0 < d < 0.5$, a stochastic process

$$M^{(d)}(t) = \frac{1}{\Gamma(d+1)} \int_{-\infty}^{\infty} [(t-s)_+^d - (-s)_+^d] dL(s), \quad t \in \mathbb{R},$$

is called a fractional Lévy process.

In terms of the fractional operators, fractional Lévy process can be rewritten as

$$M^{(d)}(t) = \int_{-\infty}^{\infty} (I_-^d \chi_{[0,t]})(s) dL(s), \quad t \in \mathbb{R},$$

where the indicator $\chi_{[a,b]}$ is given by $(a,\, b \in \mathbb{R})$

$$\chi_{[a,b]}(t) = \begin{cases} 1, & a \le t < b, \\ -1, & b \le t < a, \\ 0, & \text{otherwise.} \end{cases}$$

Assume $g \in \mathcal{L}^1(\mathbb{R})$. We obtain for $0 < d < 0.5$, the fractionally integrated kernel

$$g^{(d)}(t) := (I_+^d g)(t) = \int_{-\infty}^{t} \frac{(t-s)^{d-1}}{\Gamma(d)} g(s)ds, \quad t \in \mathbb{R}. \tag{2.1}$$

Definition 2.5 Let $0 < d < 0.5$. Then the fractionally integrated process $Y^{(d)} = \{Y^{(d)}(t)\}$ driven by the Lévy process L is defined by

$$Y^{(d)}(t) = \int_{-\infty}^{t} g^{(d)}(t-u)dL(u), \quad t \in \mathbb{R},$$

where the fractionally integrated kernel $g^{(d)}$ is given in (2.1).

Marquardt [16] proves the following theorem:

Theorem 2.2 *Suppose $Y^{(d)} = \{Y^{(d)}(t)\}$ is the fractionally integrated process $Y^{(d)}(t) = \int_{-\infty}^{t} g^{(d)}(t-u)dL(u)$, $t \in \mathbb{R}$, with $g^{(d)} \in \mathcal{L}^2(\mathbb{R})$. Then $Y^{(d)}$ can be represented as*

$$Y^{(d)}(t) = \int_{-\infty}^{t} g(t-u)dM^{(d)}(u), \quad t \in \mathbb{R}, \tag{2.2}$$

with

$$g(x) = (D_+^d g^{(d)})(x) = \frac{1}{\Gamma(1-d)} \frac{d}{dx} \int_{-\infty}^{x} g^{(d)}(s)(x-s)^{-d} ds, \quad x \in \mathbb{R}.$$

On the other hand, if $Y^{(d)}$ is given by (2.2) with $g \in \mathcal{L}^1(\mathbb{R})$, then $Y^{(d)}$ can be rewritten as $Y^{(d)}(t) = \int_{-\infty}^{t} g^{(d)}(t-u)dL(u)$, $t \in \mathbb{R}$, where $g^{(d)}(x) = (I_+^d g)(x)$.

3 Fractional Lévy short rate model

3.1 Definition

Let $r(t)$ denote the spot rate at time t. We define the Lévy short rate model as

$$r(t) = \alpha + \beta X^{(d)}(t), \tag{3.1}$$

where $\alpha, \beta \in \mathbb{R}$ and $0 < d < \frac{1}{2}$. The process $X^{(d)}(t)$ is the Riemann-Liouville order-d fractional integration of $X(t)$:

$$X^{(d)}(t) = (I_+^d X)(t) = \int_{-\infty}^{t} \frac{(t-s)^{d-1}}{\Gamma(d)} X(s)ds, \tag{3.2}$$

where $X(t)$ is a latent factor modeled as a Lévy driven Ornstein-Uhlenbeck process:

$$dX(t) = -\kappa X(t)dt + \sigma dL(t), \tag{3.3}$$

where $\kappa > 0$ and $\sigma > 0$. $L(t)$ is a Lévy process under risk neutral probability Q.

By imposing the constraint that $L(t)$ is a standard Brownian motion, we will have a fractional Brownian motion driven interest rate model studied by Comte and Renault [10] for modeling stochastic volatility. By imposing the constraint that $d = 0$, we obtain a Lévy driven short rate model, which includes the model by Hainaut and MacGilchrist [7] as a special case. If we further constrain $L(t)$ to be a Brownian motion, we will have the classical Vasicek model.

In the following theorem, we can show that $X^{(d)}(t)$ is a fractional Lévy driven Ornstein-Uhlenbeck process.

Theorem 3.1 *Suppose $X^{(d)}(t)$ is the process defined in (3.2) and (3.3). Then we can alternatively represent $X^{(d)}(t)$ as*

$$dX^{(d)}(t) = -\kappa X^{(d)}(t)dt + \sigma dM^{(d)}(t). \tag{3.4}$$

Proof Define $g(t) = \sigma \exp(-\kappa t)1_{[0,\infty)}(t)$ and $g^{(d)}(t) = (I^d_+ g)(t)$. We have

$$\begin{aligned}
X^{(d)}(t) &= \int_{-\infty}^{t} \frac{(t-s)^{d-1}}{\Gamma(d)} X(s)ds \\
&= \int_{-\infty}^{t}\int_{-\infty}^{s} \frac{(t-s)^{d-1}}{\Gamma(d)} \exp(-\kappa(s-u))\sigma dL(u)ds \\
&= \int_{-\infty}^{t}\int_{u}^{t} \frac{(t-s)^{d-1}}{\Gamma(d)} \exp(-\kappa(s-u))\sigma ds dL(u) \\
&= \int_{-\infty}^{t}\int_{0}^{t-u} \frac{(t-u-s)^{d-1}}{\Gamma(d)} \exp(-\kappa s)\sigma ds dL(u) \\
&= \int_{-\infty}^{t} g^{(d)}(t-u)dL(u) \\
&= \int_{-\infty}^{t} g(t-u)dM^{(d)}(u) \\
&= \int_{-\infty}^{t} \exp(-\kappa(t-u))\sigma dM^{(d)}(u).
\end{aligned}$$

Hence, $X^{(d)}$ is the solution of stochastic differential equation (3.4).

□

We also can show that $r(t)$ in the fractional Lévy short rate model is a long memory process.

Theorem 3.2 *Let $r(t)$ be defined in (3.1), (3.2) and (3.3). Then*

$$\gamma_r(h) = \mathrm{Cov}(r(t+h), r(t)) \sim \frac{\beta^2\sigma^2}{\kappa^2}\frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)}\mathrm{E}[L(1)^2]|h|^{2d-1}, \quad as\ h \to \infty.$$

Proof We know

$$X^{(d)}(t) = \int_{-\infty}^{t} g(t-u) dM^{(d)}(u),$$

where $g(t) = \sigma \exp(-\kappa t) 1_{[0,\infty)}(t)$.

$$\begin{aligned}\mathrm{Cov}(r(t+h), r(t)) &= \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \beta^2 \mathrm{E}[L(1)^2] \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(t+h-u) g(t-u) |u-v|^{2d-1} du dv \\ &= \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \beta^2 \mathrm{E}[L(1)^2] \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(s) g(w) |h-s+w|^{2d-1} ds dw.\end{aligned}$$

It follows that

$$\gamma_r(h) \sim \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \beta^2 \mathrm{E}[L(1)^2] \left(\int_{-\infty}^{\infty} g(u) du \right)^2 |h|^{2d-1}, \quad as\ h \to \infty.$$

□

3.2 The joint characteristic function of spot rate and its integral

Let $\mathcal{F}_t$ be the filtration generated by $L(\cdot)$. To obtain the analytical formula for prices of bond and interest rate derivatives, we follow the approach of Chacko and Das [17] and derive the conditional joint characteristic function $\Phi(u, v; t, \tau)$ of $r(t+\tau)$ and $\int_t^{t+\tau} r(s)ds$, that is

$$\Phi(u, v; t, \tau) = \mathrm{E}^Q \left[\exp \left(iu \int_t^{t+\tau} r(s) ds + ivr(t+\tau) \right) \mid \mathcal{F}_t \right] \tag{3.5}$$

under the probability measure Q. This will be achieved by representing the fractional Lévy process $X^{(d)}(t)$ in (3.2) in terms of the Ornstein-Uhlenbeck process via fractional integration. We compute $\Phi(u, v; t, \tau)$ in the following theorem:

Theorem 3.3 *For the model given by (3.1), (3.2) and (3.3), the conditional joint characteristic function $\Phi(u, v; t, \tau)$ of $r(t+\tau)$ and $\int_t^{t+\tau} r(s)ds$ under the probability measure Q is given by*

$$\begin{aligned}&\Phi(u, v; t, \tau) \\ &= \exp \Bigg[iu\alpha\tau + iv\alpha + \int_0^{\tau} \psi_{L(1)} \left(\beta\sigma(uE_{\tau-w}(d+1, -\kappa) + vE_{\tau-w}(d, -\kappa)) \right) dw \\ &\quad + i\beta \int_{-\infty}^{t} \left(u \frac{(t+\tau-s)^d - (t-s)^d}{\Gamma(d+1)} + v \frac{(t+\tau-s)^{d-1}}{\Gamma(d)} \right) X(s) ds \\ &\quad + i\beta(uE_{\tau}(d+1, -\kappa) + vE_{\tau}(d, -\kappa)) X(t) \Bigg],\end{aligned} \tag{3.6}$$

where $\psi_{L(1)}(u)$ is the characteristic exponent of $L(1)$ and $E_t(d,a)$ is a Miller-Ross function defined by

$$E_t(d,a) = t^d \exp(at)\Gamma^*(d,at),$$

where $\Gamma^*(d,at)$ is an incomplete Gamma function defined by

$$\Gamma^*(v,t) = \frac{1}{\Gamma(v)t^v}\int_0^t \xi^{v-1}\exp(-\xi)d\xi.$$

Proof Note first that $X^{(d+1)}(t) = \int_{-\infty}^t X^{(d)}(s)ds$. From (3.2) we can decompose the integral of $X^{(d)}(\cdot)$ as

$$\begin{aligned}\int_t^{t+\tau} X^{(d)}(s)ds &= \int_{-\infty}^{t+\tau} X^{(d)}(s)ds - \int_{-\infty}^{t} X^{(d)}(s)ds \\ &= X^{(d+1)}(t+\tau) - X^{(d+1)}(t) \\ &= \int_{-\infty}^{t+\tau} \frac{(t+\tau-s)^d}{\Gamma(d+1)}X(s)ds - \int_{-\infty}^{t} \frac{(t-s)^d}{\Gamma(d+1)}X(s)ds \\ &= \frac{1}{\Gamma(d+1)}\int_{-\infty}^{t}[(t+\tau-s)^d-(t-s)^d]X(s)ds \\ &\quad + \frac{1}{\Gamma(d+1)}\int_t^{t+\tau}(t+\tau-s)^d X(s)ds.\end{aligned} \tag{3.7}$$

Using (3.1) and (3.7), we can compute $\Phi(u,v;t,\tau)$ as

$$\begin{aligned}\Phi(u,v;t,\tau) &= \mathrm{E}^Q\left[\exp\left(iu\int_t^{t+\tau} r(s)ds + ivr(t+\tau)\right) \mid \mathcal{F}_t\right] \\ &= \mathrm{E}^Q\left[\exp\left(iu\int_t^{t+\tau}(\alpha+\beta X^{(d)}(s))ds + iv(\alpha+\beta X^{(d)}(t+\tau))\right) \mid \mathcal{F}_t\right] \\ &= \exp(iu\alpha\tau + iv\alpha)\mathrm{E}^Q\Bigg[\exp\Bigg(\frac{iu\beta}{\Gamma(d+1)}\int_{-\infty}^t((t+\tau-s)^d-(t-s)^d)X(s)ds \\ &\quad + \frac{iu\beta}{\Gamma(d+1)}\int_t^{t+\tau}(t+\tau-s)^d X(s)ds + \frac{iv\beta}{\Gamma(d)}\int_{-\infty}^{t+\tau}(t+\tau-s)^{d-1}X(s)ds\Bigg) \mid \mathcal{F}_t\Bigg] \\ &= \exp\left(iu\alpha\tau + iv\alpha + i\beta\int_{-\infty}^t\left(u\frac{(t+\tau-s)^d-(t-s)^d}{\Gamma(d+1)} + v\frac{(t+\tau-s)^{d-1}}{\Gamma(d)}\right)X(s)ds\right) \\ &\quad \times \mathrm{E}^Q\left[\exp\left(i\beta\int_t^{t+\tau}\left(u\frac{(t+\tau-s)^d}{\Gamma(d+1)} + v\frac{(t+\tau-s)^{d-1}}{\Gamma(d)}\right)X(s)ds\right) \mid \mathcal{F}_t\right].\end{aligned} \tag{3.8}$$

We can compute the last line in (3.8) as

$$
\begin{aligned}
&\mathrm{E}^Q\left[\exp\left(i\beta\int_t^{t+\tau}\left(u\frac{(t+\tau-s)^d}{\Gamma(d+1)}+v\frac{(t+\tau-s)^{d-1}}{\Gamma(d)}\right)X(s)ds\right)\mid\mathcal{F}_t\right]\\
&=\mathrm{E}^Q\Bigg[\exp\Bigg(i\beta\int_t^{t+\tau}\left(u\frac{(t+\tau-s)^d}{\Gamma(d+1)}+v\frac{(t+\tau-s)^{d-1}}{\Gamma(d)}\right)(\exp(-\kappa(s-t))X(t)\\
&\quad+\int_t^s\exp(-\kappa(s-w))\sigma dL(w))ds\Bigg)\mid\mathcal{F}_t\Bigg]\\
&=\exp\left(i\beta\int_t^{t+\tau}\left(u\frac{(t+\tau-s)^d}{\Gamma(d+1)}+v\frac{(t+\tau-s)^{d-1}}{\Gamma(d)}\right)\exp(-\kappa(s-t))dsX(t)\right)\\
&\quad\times\mathrm{E}^Q\left[\exp\left(i\beta\int_t^{t+\tau}\left(u\frac{(t+\tau-s)^d}{\Gamma(d+1)}+v\frac{(t+\tau-s)^{d-1}}{\Gamma(d)}\right)\int_t^s\exp(-\kappa(s-w))\sigma dL(w)ds\right)\mid\mathcal{F}_t\right].
\end{aligned}
\tag{3.9}
$$

We can compute the last line in (3.9) as

$$
\begin{aligned}
&\mathrm{E}^Q\left[\exp\left(i\beta\int_t^{t+\tau}\left(u\frac{(t+\tau-s)^d}{\Gamma(d+1)}+v\frac{(t+\tau-s)^{d-1}}{\Gamma(d)}\right)\int_t^s\exp(-\kappa(s-w))\sigma dL(w)ds\right)\mid\mathcal{F}_t\right]\\
&=\mathrm{E}^Q\left[\exp\left(i\beta\int_t^{t+\tau}\int_w^{t+\tau}\left(u\frac{(t+\tau-s)^d}{\Gamma(d+1)}+v\frac{(t+\tau-s)^{d-1}}{\Gamma(d)}\right)\exp(-\kappa(s-w))\sigma dsdL(w)\right)\mid\mathcal{F}_t\right]\\
&=\exp\left(\int_t^{t+\tau}\psi_{L(1)}\left(\beta\int_w^{t+\tau}\left(u\frac{(t+\tau-s)^d}{\Gamma(d+1)}+v\frac{(t+\tau-s)^{d-1}}{\Gamma(d)}\right)\exp(-\kappa(s-w))\sigma ds\right)dw\right).
\end{aligned}
\tag{3.10}
$$

Finally, using (3.8), (3.9), (3.10) and the fact (see Miller and Ross [18]) that

$$
\int_0^t\frac{(t-s)^{d-1}}{\Gamma(d)}\exp(as)ds=E_t(d,a),
$$

we obtain (3.6).

□

Next, we define

$$
p(t,\tau)=A+B\int_{-\infty}^t\frac{(t+\tau-s)^d-(t-s)^d}{\Gamma(d+1)}X(s)ds+CX(t),
$$

where A, B and $C\in\mathbb{R}$. X is defined in (3.3).

We can also derive the conditional joint characteristic function $\Psi(u,v;t,\tau,\hat{\tau},A,B,C)$ of $r(t+\tau)$ and $p(t+\tau,\hat{\tau})$, that is

$$
\Psi(u,v;t,\tau,\hat{\tau},A,B,C)=\mathrm{E}^Q\left[\exp\left(iu\int_t^{t+\tau}r(s)ds+ivp(t+\tau,\hat{\tau})\right)\mid\mathcal{F}_t\right]
\tag{3.11}
$$

under the probability measure Q. Using the similar technique to the proof of theorem 3.3, we can compute $\Psi(u,v;t,\tau,\hat{\tau},A,B,C)$ in the following theorem:

Theorem 3.4 *For the model given by (3.1), (3.2) and (3.3), the conditional joint characteristic function* $\Psi(u, v; t, \tau, \hat{\tau}, A, B, C)$ *of* $r(t+\tau)$ *and* $p(t+\tau, \hat{\tau})$ *under the probability measure* Q *is given by*

$$\begin{aligned}
&\Psi(u, v; t, \tau, \hat{\tau}, A, B, C) \\
&= \exp\Bigg[iu\alpha\tau + ivA + \int_0^\tau \psi_{L(1)}((u\beta - vB)\sigma E_{\tau-w}(d+1, -\kappa) + vB\sigma(E_{\tau+\hat{\tau}-w}(d+1, -\kappa) \\
&\quad - \exp(-\kappa(\tau - w))E_{\hat{\tau}}(d+1, -\kappa)) + vC\sigma \exp(-\kappa(\tau - w)))dw \\
&\quad + i\int_{-\infty}^t \frac{u\beta((t+\tau-s)^d - (t-s)^d) + vB((t+\tau+\hat{\tau}-s)^d - (t+\tau-s)^d)}{\Gamma(d+1)} X(s)ds \\
&\quad + i((u\beta - vB)E_\tau(d+1, -\kappa) + ivB(E_{\tau+\hat{\tau}}(d+1, -\kappa) - \exp(-\kappa\tau)E_{\hat{\tau}}(d+1, -\kappa)) \\
&\quad + ivC\exp(-\kappa\tau))X(t)\Bigg].
\end{aligned} \tag{3.12}$$

4 Pricing of bond and interest rate derivatives

4.1 Bond prices

Given the conditional joint characteristic function, we can obtain the closed form solution for the bond prices.

Let $P(t, \tau)$ represent the time t price of zero coupon bond that matures after a period of time τ. Using Theorem 3.3, we have

$$\begin{aligned}
P(t, \tau) &= \mathrm{E}^Q\left[\exp(-\int_t^{t+\tau} r(s)ds) \mid \mathcal{F}_t\right] \\
&= \Phi(i, 0; t, \tau) \\
&= \exp[-A(\tau) - B(t, \tau, X) - C(\tau)X(t)],
\end{aligned} \tag{4.1}$$

where

$$A(\tau) = \alpha\tau - \int_0^\tau \psi_{L(1)}(i\beta\sigma E_{\tau-w}(d+1, -\kappa))dw, \tag{4.2}$$

$$B(t, \tau, X) = \beta\int_{-\infty}^t \frac{(t+\tau-s)^d - (t-s)^d}{\Gamma(d+1)} X(s)ds, \tag{4.3}$$

$$C(\tau) = \beta E_\tau(d+1, -\kappa). \tag{4.4}$$

The continuous compounded yields to maturity of discount bonds are given by

$$y(t, \tau) = \frac{A(\tau)}{\tau} + \frac{B(t, \tau, X)}{\tau} + \frac{C(\tau)}{\tau} X(t). \tag{4.5}$$

Note that the yields function of the fractional Lévy model is inherently different from that of the affine term structure model in that it is non-Markovian. As a result, the yields will be dependent on the whole history of the process X through the term $B(t, \tau, X)$. When $d = 0$, this term will disappear, then the yields will be a linear function of the current factor.

4.2 Interest rate derivatives prices

4.2.1 General case

Let $f(t+\tau)$ denote the interest rate derivative payoff function at time $t+\tau$ and $C(t)$ be the derivative price at time t. We have

$$C(t+\tau) = (f(t+\tau) - K)1_{f(t+\tau)\geq K},$$

where K is the strike price. We can obtain the time t price of derivative as

$$\begin{aligned} C(t) &= \mathrm{E}^Q\left[\exp\left(-\int_t^{t+\tau} r(s)ds\right) C(t+\tau) \mid \mathcal{F}_t\right] \\ &= \mathrm{E}^Q\left[\exp\left(-\int_t^{t+\tau} r(s)ds\right) (f(t+\tau) - K)1_{f(t+\tau)\geq K} \mid \mathcal{F}_t\right] \\ &= \mathrm{E}^Q\left[\exp\left(-\int_t^{t+\tau} r(s)ds\right) f(t+\tau)1_{f(t+\tau)\geq K} \mid \mathcal{F}_t\right] \\ &\quad - K\mathrm{E}^Q\left[\exp\left(-\int_t^{t+\tau} r(s)ds\right) 1_{f(t+\tau)\geq K} \mid \mathcal{F}_t\right] \\ &= \Pi_0(t)\Pi_1(t) - KP(t,\tau)\Pi_2(t), \end{aligned} \tag{4.6}$$

where

$$\Pi_0(t) = \mathrm{E}^Q\left[\exp\left(-\int_t^{t+\tau} r(s)ds\right) f(t+\tau) \mid \mathcal{F}_t\right],$$

$$\Pi_1(t) = \mathrm{E}^Q\left[\frac{\exp\left(-\int_t^{t+\tau} r(s)ds\right) f(t+\tau)1_{f(t+\tau)\geq K}}{\mathrm{E}^Q\left[\exp\left(-\int_t^{t+\tau} r(s)ds\right) f(t+\tau) \mid \mathcal{F}_t\right]} \mid \mathcal{F}_t\right],$$

$$\Pi_2(t) = \mathrm{E}^Q\left[\frac{\exp\left(-\int_t^{t+\tau} r(s)ds\right) 1_{f(t+\tau)\geq K}}{\mathrm{E}^Q\left[\exp\left(-\int_t^{t+\tau} r(s)ds\right) \mid \mathcal{F}_t\right]} \mid \mathcal{F}_t\right].$$

It is clear that $\Pi_1(t)$ and $\Pi_2(t)$ are two probabilities. We need to evaluate $\Pi_0(t)$, $\Pi_1(t)$ and $\Pi_2(t)$ to obtain the derivatives prices. In the next sections we will examine several types of derivatives.

4.2.2 Bond options

A call option on a discount bond at strike K is the right but not the obligation to purchase a discount bond with remaining maturity $\hat{\tau}$ on the expiration date of the option. The option payoff is

$$C(t+\tau) = (P(t+\tau,\hat{\tau}) - K)1_{P(t+\tau,\hat{\tau})\geq K}.$$

The time t price can be calculated from equation (4.6). We have

$$\Pi_0(t) = P(t, \tau + \hat{\tau}).$$

To calculate $\Pi_1(t)$, we first calculate

$$\begin{aligned}\widetilde{\Pi}_1(t) &= \frac{1}{\Pi_0(t)}\mathrm{E}^Q\left[\exp\left(-\int_t^{t+\tau} r(s)ds\right)P(t+\tau,\hat{\tau})\exp(iu\log(P(t+\tau,\hat{\tau}))) \mid \mathcal{F}_t\right]\\ &= \frac{1}{\Pi_0(t)}\mathrm{E}^Q\left[\exp\left(-\int_t^{t+\tau} r(s)ds\right)\exp((1+iu)(-A(\hat{\tau})-B(t+\tau,\hat{\tau},X)-C(\hat{\tau})X(t+\tau))) \mid \mathcal{F}_t\right]\\ &= \frac{1}{\Pi_0(t)}\Psi(i,-1+u;t,\tau,\hat{\tau},-A(\hat{\tau}),-\beta,-C(\hat{\tau})),\end{aligned}$$

where $A(\cdot)$, $B(\cdot,\cdot,\cdot)$ and $C(\cdot)$ are defined in (4.2), (4.3) and (4.4) respectively.

Similarly, to calculate $\Pi_2(t)$, we first calculate

$$\begin{aligned}\widetilde{\Pi}_2(t) &= \frac{1}{P(t,t+\tau)}\mathrm{E}^Q\left[\exp\left(-\int_t^{t+\tau} r(s)ds\right)\exp(iu\log(P(t+\tau,\hat{\tau}))) \mid \mathcal{F}_t\right]\\ &= \frac{1}{P(t,t+\tau)}\mathrm{E}^Q\left[\exp\left(-\int_t^{t+\tau} r(s)ds\right)\exp(iu(-A(\hat{\tau})-B(t+\tau,\hat{\tau},X)-C(\hat{\tau})X(t+\tau))) \mid \mathcal{F}_t\right]\\ &= \frac{1}{P(t,t+\tau)}\Psi(i,u;t,\tau,\hat{\tau},-A(\hat{\tau}),-\beta,-C(\hat{\tau})).\end{aligned}$$

Then $\Pi_1(t)$ and $\Pi_2(t)$ can be obtained by inverting $\widetilde{\Pi}_1(t)$ and $\widetilde{\Pi}_2(t)$ respectively:

$$\Pi_j(t) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{1}{iu}\exp(-iu\log(K))\widetilde{\Pi}_j(t)\right)du, \quad j=1,2.$$

Using the similar calculations, we also can obtain the prices for bond futures options.

4.2.3 Caps

An interest rate cap is an option that pays off when the terminal interest rate exceeds the strike price K. If the cap contract matures at time $t+\tau$, its payoff is given by

$$C(t+\tau) = (r(t+\tau)-K)1_{r(t+\tau)\geq K}.$$

The time t price can be calculated from equation (4.6). We have

$$\begin{aligned}\Pi_0(t) &= \mathrm{E}^Q\left[\exp\left(-\int_t^{t+\tau} r(s)ds\right)r(t+\tau) \mid \mathcal{F}_t\right]\\ &= \left\{\frac{\partial}{\partial u}\mathrm{E}^Q\left[\exp\left(-\int_t^{t+\tau} r(s)ds\right)\exp(ur(t+\tau)) \mid \mathcal{F}_t\right]\right\}_{u=0}\\ &= \left\{\frac{\partial}{\partial u}\Phi(i,-iu;t,\tau)\right\}_{u=0}.\end{aligned}$$

To calculate $\Pi_1(t)$, we first calculate

$$\begin{aligned}
\widetilde{\Pi}_1(t) &= \frac{1}{\Pi_0(t)} \mathrm{E}^Q \left[\exp\left(-\int_t^{t+\tau} r(s)ds \right) r(t+\tau) \exp(iur(t+\tau)) \mid \mathcal{F}_t \right] \\
&= \frac{1}{\Pi_0(t)} \frac{1}{i} \frac{\partial}{\partial u} \mathrm{E}^Q \left[\exp\left(-\int_t^{t+\tau} r(s)ds \right) \exp(iur(t+\tau)) \mid \mathcal{F}_t \right] \\
&= \frac{1}{\Pi_0(t)} \frac{1}{i} \frac{\partial}{\partial u} \Phi(i, u; t, \tau).
\end{aligned}$$

Similarly, to calculate $\Pi_2(t)$, we first calculate

$$\begin{aligned}
\widetilde{\Pi}_2(t) &= \frac{1}{P(t, t+\tau)} \mathrm{E}^Q \left[\exp\left(-\int_t^{t+\tau} r(s)ds \right) \exp(iur(t+\tau)) \mid \mathcal{F}_t \right] \\
&= \frac{1}{P(t, t+\tau)} \Phi(i, u; t, \tau).
\end{aligned}$$

Then $\Pi_1(t)$ and $\Pi_2(t)$ can be obtained by inverting $\widetilde{\Pi}_1(t)$ and $\widetilde{\Pi}_2(t)$ respectively:

$$\Pi_j(t) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re \left(\frac{1}{iu} \exp(-iuK) \widetilde{\Pi}_j(t) \right) du, \quad j = 1, 2. \tag{4.7}$$

Using the similar calculations, we also can obtain the prices for other derivatives such as yield caps, floors and yield combo options.

4.2.4 Asian options

The Asian option on the short rate has the following payoff function:

$$C(t+\tau) = \left(\frac{1}{\tau} \int_t^{t+\tau} r(s)ds - K \right) 1_{\frac{1}{\tau} \int_t^{t+\tau} r(s)ds \geq K}.$$

The time t price can be calculated from equation (4.6). We have

$$\begin{aligned}
\Pi_0(t) &= \mathrm{E}^Q \left[\exp\left(-\int_t^{t+\tau} r(s)ds \right) \frac{1}{\tau} \int_t^{t+\tau} r(s)ds \mid \mathcal{F}_t \right] \\
&= \left\{ \frac{\partial}{\partial u} \mathrm{E}^Q \left[\exp\left(-\int_t^{t+\tau} r(s)ds \right) \exp\left(\frac{u}{\tau} \int_t^{t+\tau} r(s)ds \right) \mid \mathcal{F}_t \right] \right\}_{u=0} \\
&= \left\{ \frac{\partial}{\partial u} \mathrm{E}^Q \left[\exp\left(\left(\frac{u}{\tau} - 1 \right) \int_t^{t+\tau} r(s)ds \right) \mid \mathcal{F}_t \right] \right\}_{u=0} \\
&= \left\{ \frac{\partial}{\partial u} \Phi\left(i \left(1 - \frac{u}{\tau} \right), 0; t, \tau \right) \right\}_{u=0}.
\end{aligned}$$

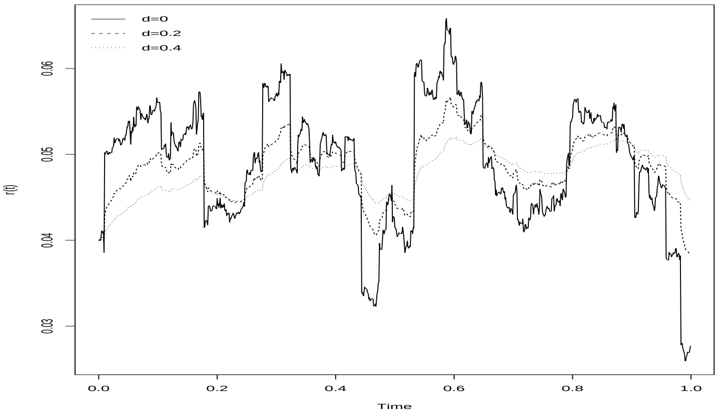

Fig. 5.1 Simulation of fractional NIG process for different integration parameters. $r(t) = \alpha + X^{(d)}(t)$, $X^{(d)}(t) = \int_{-\infty}^{t} \frac{(t-s)^{d-1}}{\Gamma(d)} X(s)ds$ and $dX(t) = -\kappa X(t)dt + \sigma dL(t)$. $\alpha = 0.04$, $\kappa = 0.5$, $\sigma = 0.04$, $a = 5$, $b = 0.2$, $c = 5$ and $x(t) = 0$ for $t \le 0$.

$$\begin{aligned}\widetilde{\Pi}_1(t) &= \frac{1}{\Pi_0(t)} \mathrm{E}^Q \left[\exp\left(-\int_t^{t+\tau} r(s)ds \right) \frac{1}{\tau} \int_t^{t+\tau} r(s)ds \exp\left(iu\frac{1}{\tau} \int_t^{t+\tau} r(s)ds \right) \mid \mathcal{F}_t \right] \\ &= \frac{1}{\Pi_0(t)} \frac{1}{i} \frac{\partial}{\partial u} \mathrm{E}^Q \left[\exp(-\int_t^{t+\tau} r(s)ds) \exp\left(iu\frac{1}{\tau} \int_t^{t+\tau} r(s)ds \right) \mid \mathcal{F}_t \right] \\ &= \frac{1}{\Pi_0(t)} \frac{1}{i} \frac{\partial}{\partial u} \Phi\left(\frac{u}{\tau} + i, 0, t, \tau \right).\end{aligned}$$

$$\begin{aligned}\widetilde{\Pi}_2(t) &= \frac{1}{P(t,t+\tau)} \mathrm{E}^Q \left[\exp\left(-\int_t^{t+\tau} r(s)ds \right) \exp\left(iu\frac{1}{\tau} \int_t^{t+\tau} r(s)ds \right) \mid \mathcal{F}_t \right] \\ &= \frac{1}{P(t,t+\tau)} \Phi\left(\frac{u}{\tau} + i, 0, t, \tau \right).\end{aligned}$$

Then $\Pi_1(t)$ and $\Pi_2(t)$ can be obtained by inverting $\widetilde{\Pi}_1(t)$ and $\widetilde{\Pi}_2(t)$ using equation (4.7).

Using the similar technique, we also can obtain the prices for Asian options on the yields.

5 Numerical example

In this section, we numerically study a term structure model driven by a specific type of Lévy process. We choose the normal inverse Gaussian (NIG) process. An NIG random variable L follows a 4-parameter (a, b, c, m) probability law, and its characteristic exponent (see Schoutens [13]) is given by

$$\psi_{L(1)}(u; a, b, c) = -c(\sqrt{a^2 - (b + iu)^2} - \sqrt{a^2 - b^2}) + ium,$$

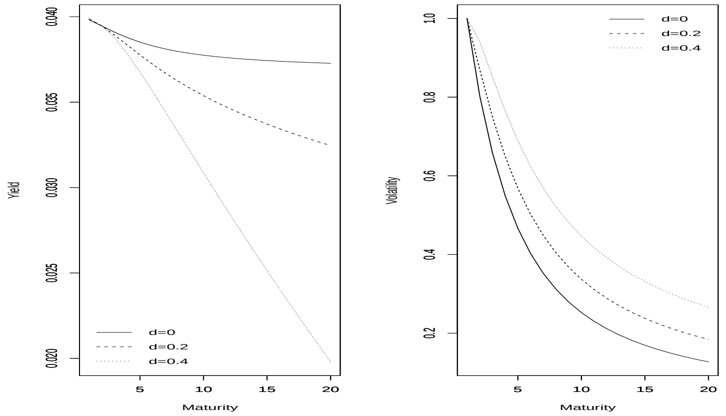

Fig. 5.2 The yield curves and the term structure of volatility of fractional NIG process for different integration parameters. The left panel is the yield curve and the right panel is the term structure of volatility. $r(t) = \alpha + X^{(d)}(t)$, $X^{(d)}(t) = \int_{-\infty}^{t} \frac{(t-s)^{d-1}}{\Gamma(d)} X(s)ds$ and $dX(t) = -\kappa X(t)dt + \sigma dL(t)$. $\alpha = 0.04$, $\kappa = 0.5$, $\sigma = 0.04$, $a = 5$, $b = 0.2$, $c = 5$ and $x(t) = 0$ for $t \leq 0$.

where $a > 0$, $|b| < a$, $c > 0$ and $m \in \mathbb{R}$.

The moments of an NIG random variable L are given by

$$Mean(L(1)) = \frac{bc}{\sqrt{a^2 - b^2}} + m,$$

$$Variance(L(1)) = \frac{a^2 c}{\sqrt{(a^2 - b^2)^3}},$$

$$Sknewness(L(1)) = \frac{3b}{a\sqrt{c}\sqrt[4]{a^2 - b^2}},$$

$$Kurtosis(L(1)) = 3\left(1 + \frac{a^2 + 4b^2}{ca^2\sqrt{a^2 - b^2}}\right).$$

In our numerical example, we set $m = -\frac{bc}{\sqrt{a^2-b^2}}$ so that $\mathrm{E}[L(1)] = 0$. Figure 5.1 shows a single simulated path of r for $d = 0$, $d = 0.2$ and $d = 0.4$ respectively. For three paths we have used the parameters similar to those from Hainaut and MacGilchrist [7] and the same sequence of random numbers. We find that the integration parameter d influences the smoothness of the interest rate process. The greater d is, the smoother the path of r is.

We calculate the compounded yields to maturity of discount bonds using formula (4.5). Figure 5.2 shows that the term structure of interest rate has an downward sloping for each integration parameter d. However, the greater d is, the steeper the slope is. We also compute the term structure of volatilities for different integration parameters. Figure 5.2 plots the term structure of

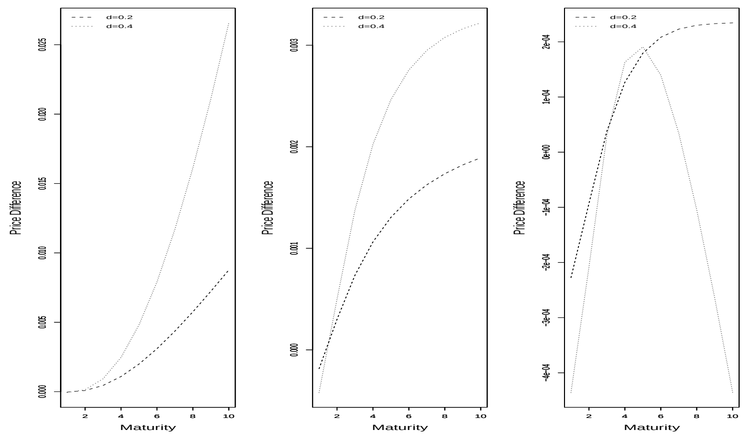

Fig. 5.3 Effects of time to maturity on the derivatives price differences between the fractional NIG model and NIG model. The left panel is for the bond option price, the middle panel is for the cap price and the right panel is for the Asian option price. $r(t) = \alpha + X^{(d)}(t)$, $X^{(d)}(t) = \int_{-\infty}^{t} \frac{(t-s)^{d-1}}{\Gamma(d)} X(s)ds$ and $dX(t) = -\kappa X(t)dt + \sigma dL(t)$. $\alpha = 0.04$, $\kappa = 0.5$, $\sigma = 0.04$, $a = 5$, $b = 0.2$, $c = 5$, $K = 0.03$ and $x(t) = 0$ for $t \leq 0$.

volatility as a function of maturity and d. It seems the volatility curve flattens out slower for the model with higher d than the model with lower d. This is consistent with the findings of Backus and Zin [8]. This permits that the volatility persists even for long yields. This also shows that the long memory model has the potential to capture the term structure of volatility better than the short memory models.

Figure 5.3 shows that the effect of integration parameter d on the bond option prices. It seems that the effect is not a linear function of maturity. Instead, the term structure of the price difference between the fractional NIG model and NIG model has a convex shape. The effect of d is more significant for longer maturities. Figure 5.3 also plots the effect of integration parameter d on the cap prices. The effect is similar to the case of bond option prices but the term structure of the price difference between the fractional NIG model and NIG model has a concave shape. Finally, We also calculate the Asian option prices for different maturities. Figure 5.3 shows the term structure of the price difference between the fractional NIG model and NIG model can have more complex shape. The effect of d is not necessarily a monotone function of maturity.

6 Conclusion

In this paper we have proposed a new term structure of interest rate model based on the fractional Lévy process. We derive the joint characteristic function of spot rate and its integral, which makes it possible for us to obtain the closed form solutions to the prices of bonds and interest rate derivatives. In our numerical study, we focus on a specific type of Lévy process, namely NIG process. Based on our fractional NIG model, we find that the higher integration parameter leads to the slower decay of term structure of volatility. It implies that the long memory interest model has the potential to capture the persistence of the interest rate volatilities better than the corresponding short memory models. We also find that the integration parameter has significant effects on bond yields and interest derivatives prices.

Our goal in this paper is to study the way to incorporate the long memory into the existing term structure of interest rate models. As a result, we only consider one-factor model for simplicity. It would be interesting to extend our model framework to multi-factor models. We will leave this development for further research.

References

[1] O. Vasicek, “An Equilibrium Characterization of the Term Structure”, Journal of Financial Economics, vol. 5, 1977, pp. 177-188.
[2] J. C. Cox, J. E. Ingersoll and S. A. Ross, “A Theory of the Term Structure of Interest Rates”, Econometrica, vol. 53, 1985, pp. 385-407.
[3] D. Duffie and R. Kan, “A Yield-Factor Model of Interest Rates”, Mathematical Finance, vol. 6, 1996, pp. 379-406.
[4] S. Das, “The Surprise Element: Jumps in Interest Rates”, Journal of Econometrics, vol. 106, 2002, pp. 27-65.
[5] E. Eberlein and S. Raible, “Term Structure Models Driven by General Lévy Processes”, Mathematical Finance, vol. 9, 1999, pp. 31-53.
[6] E. Eberlein, J. Jacod and S. Raible, “Lévy Term Structure Models: No-Arbitrage and Completeness”, Finance and Stochastics, vol. 9, 2005, pp. 67-88.
[7] D. Hainaut and R. Macgilchrist, “An Interest Rate Tree Driven by a Lévy Process”, Journal of Derivatives, vol. 18, 2010, pp. 33-45.
[8] D. K. Backus and S. E. Zin, “Long-Memory Inflation Uncertainty: Evidence from the Term Structure of Interest Rates”, Journal of Money Credit and Banking, vol. 25, 1993, pp. 681-700.
[9] A. Golinski and P. Zaffaroni, “Long Memory Affine Term Structure Models”, Working Paper, University of York, 2012.
[10] F. Comte and E. Renault, “Long Memory in Continuous-Time Stochastic Volatility Models”, Mathematical Finance, vol. 73, 1998, pp. 101-149.
[11] F. Comte, L. Coutin and E. Renault, “Affine Fractional Stochastic Volatility Models”, Annal of Finance, vol. 8, 2012, pp. 337-378.
[12] K. Sato, “Lévy Processes and Infinitely Divisible Distribution”, Cambridge Press, 1999.
[13] W. Schoutens, ”Lévy Processes in Finance”, Wiley Finance, 2003.
[14] B. S. Rajput and J. Rosinski, “Spectral Representation of Infinitely Divisible Processes”, Journal of Theoretical Probability, vol. 82, 1989, pp. 453-487.
[15] S. G. Samko, A. A. Kilbas and O. I. Marichev, “Fractional Integrals and Derivatives, Theory and Applications”, Gordon and Breach Science Publishers, 1993.
[16] T. Marquardt, “Fractional Lévy Processes with an Appication to Long Memory Moving Average Processes”, Bernoulli, vol. 12, 2006, pp. 1009-1126.
[17] S. Das and G. Chacko, “Pricing Interest Rate Derivatives: a General Approach”, Review of Financial Studies, vol. 15, 2002, pp. 195-241.
[18] K. S. Miller and B. Ross, “An Introduction to Fractional Calculus and Fractional Differential Equations”, Wiley-Interscience, 1993.