Journal's Title
Vol.x (xx) (20xx), xxx - xxx

On the space of square-integrable Hilbert C^*-module-valued maps on compact groups

M. Todjro[1], Y. Mensah and V.S.K. Assiamoua

Various properties of $L_2(G, \mathcal{M})$ the space of the square-integrable $\mathcal{M}$-valued functions on the compact group G where $\mathcal{M}$ is a Hilbert C^*-module are established. In particular a pre-Hilbert space structure is constructed on it. Similar results are obtained for its discrete analogue via the Fourier transform of vector-valued functions.

1 Introduction

The aim of this paper is to scrutinize some properties of Bochner-square-integrable Hilbert C^*-module-valued maps defined on compact groups. Introduced by Kaplansky in the first half of 1950's, the concept of Hilbert C^*-module is straightforward generalization of the notion of Hilbert space. Roughly speaking, it consists to endow a vector space with an inner product which takes values not in the field of complex numbers but in a C^*-algebra. The results were applied by Kaplansky to solve certains problems in operator theory such as the structure of derivations of AW^*-algebras [10]. At the present time, the challenge is to invesgate for Hilbert C^*-modules the analogues of various facts true for Hilbert spaces. However the methods to handle problems related to Hilbert C^*-modules may not be easy to foresee since they are not orthogonally complemented in general like Hilbert spaces.

On the other hand, square integrable functions are almost ubiquitous in mathematics and its applications. They are encountered for instance in signal processing (signals with finite energy), in stochastic calculus (random variables with finite second moment), in mathematical physiscs (concrete realisation of some Hilbert spaces)...

[1]Corresponding author email address: todjrom7@gmail.com

2010 *MSC.* Primary 43A77, 43A15; Secondary 46C50.

Key words and phrases. Hilbert C^*-module, compact group, Fourier transform, square-integrable function.

The vector valued functions are useful tools in the study of some geometrical properties of Banach spaces. Here in particular we are interested in studying some properties of the space of square integrable Hilbert C^*-module-valued functions on compact groups.

The rest of the paper is structured as follows. Section 2 contains definitions and facts about Hilbert C^*-modules. Section 3 furnishes basic notions on the representation theory of groups with emphasis on compact groups. Section 4 gives some facts about Fourier transform of vector-valued functions on compact groups. We establish our main results in section 5.

2 Hilbert C^*-modules

In this section we recall the definition of a Hilbert C^*-module and some properties related to its norm. For more details, we refer to [11].

Definition 2.1 *Let $\mathcal{A}$ be a C^*-algebra. A pre-Hilbert $\mathcal{A}$-module is a vector space $\mathcal{M}$ which is a right $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued inner product $(x,y) \mapsto \langle x,y\rangle : \mathcal{M}\times\mathcal{M} \to \mathcal{A}$ such that the following conditions are satisfied:*

1. $\forall x,y,z \in \mathcal{M}, \forall \alpha,\beta \in \mathbb{C}$, $\langle x, \alpha y + \beta z\rangle = \alpha\langle x,y\rangle + \beta\langle x,z\rangle$.
2. $\forall x,y \in \mathcal{M}$, $\forall a \in \mathcal{A}$, $\langle x, ya\rangle = \langle x,y\rangle a$.
3. $\forall x,y \in \mathcal{M}$, $\langle y,x\rangle = \langle x,y\rangle^*$.
4. $\forall x \in \mathcal{M}$, $\langle x,x\rangle \geq 0$ *and if* $\langle x,x\rangle = 0$ *then* $x = 0$.

If $\mathcal{M}$ is a Banach space under the norm $||x||_{\mathcal{M}} = ||\langle x,x\rangle||_{\mathcal{A}}^{\frac{1}{2}}$, where $\|\cdot\|_{\mathcal{A}}$ is the norm in $\mathcal{A}$, then $\mathcal{M}$ is called a Hilbert $\mathcal{A}$-module (or Hilbert C^*-module over $\mathcal{A}$). Otherwise, by using the completeness of the C^*-algebra $\mathcal{A}$, the $\mathcal{A}$-valued inner product and the action of $\mathcal{A}$ on $\mathcal{M}$ can be extended to form the completion $\widehat{\mathcal{M}}$ of $\mathcal{M}$ which becomes a Hilbert $\mathcal{A}$-module.

Hereafter are some simple examples of Hilbert C^*-modules.

1. Every complex Hilbert space is a Hilbert C^*-module over $\mathbb{C}$.
2. Every C^*-algebra $\mathcal{A}$ is a Hilbert module over $\mathcal{A}$. The $\mathcal{A}$-valued inner product is given by $\langle a,b\rangle = a^*b$, $\forall\ a,b \in \mathcal{A}$. A concrete example is $\mathcal{B}(H)$ the set of bounded operators on a Hilbert space H.

The norm $||\cdot||_{\mathcal{M}}$ satisfies the following properties.

1. $\forall x \in \mathcal{M}$, $\forall a \in \mathcal{A}$, $||x \cdot a||_{\mathcal{M}} \leq ||x||_{\mathcal{M}}||a||_{\mathcal{A}}$.

2. $\forall x, y \in \mathcal{M}$, $||\langle x, y\rangle||_{\mathcal{A}} \leq ||x||_{\mathcal{M}}||y||_{\mathcal{M}}$.

In the category of Hilbert $\mathcal{A}$-modules the isomorphisms are defined as follows.

Definition 2.2 *Two Hilbert $\mathcal{A}$-modules $\mathcal{M}_1$ and $\mathcal{M}_2$, with respective $\mathcal{A}$-valued inner products $\langle \cdot, \cdot\rangle_1$ and $\langle \cdot, \cdot\rangle_2$ are isomorphic if there exists a bijective bounded $\mathcal{A}$-linear mapping $L : \mathcal{M}_1 \to \mathcal{M}_2$ such that the identity $\langle L(x), L(y)\rangle_2 = \langle x, y\rangle_1$ holds for all $x, y \in \mathcal{M}_1$.*

In the sequel, we call the $\mathcal{A}$-valued inner product simply an $\mathcal{A}$-product and we make the convention that all $\mathcal{A}$-modules will be right modules. Interested readers can consult [8], [11], [7] and references therein for more details on Hilbert C^*-modules.

3 Representation theory of groups

The representations of groups play a central role in noncommutative harmonic analysis. In this section we recall some elements of the representation theory that we may need with emphasis on the compact groups case.
Let G be a group and H be a vector space. A representation of G on H is a homomorphism U: $t \mapsto U_t$ from G into $GL(H)$ the group of invertible operators in H. The space H is called the representation space of U. If G and H are topological spaces then U is said to be a continuous representation if the map $G \times H \to H$, $(t, \xi) \mapsto U_t\xi$ is continuous. Moreover when H is a Hilbert space, if for any $t \in G$, U_t is a unitary operator of H then U is called a unitary representation. In this case, the representation U is continuous if only if the map $t \mapsto U_t\xi$ defined from G to H is continuous for all $\xi \in H$. The representation U is said to be of finite dimension if its representation space H is of finite dimension.
Two representations U and V of a group G with representation spaces H and K respectively are said to be equivalent if there exists an isomorphism $T : H \to K$ that intertwines U and V, that is $\forall t \in G$, $T \circ U_t = V_t \circ T$.
A subvector space L of H is said to be invariant by the representation U if $\forall t \in G$, $\forall \xi \in L$, $U_t\xi \in L$. Any representation admits at least two trivial invariant subspaces: $\{0\}$ and H. The representation U is said to be irreducible if it does not admit a non trivial invariant subspace, otherwise U is said to be decomposable. The set of all equivalent classes of unitary irreducible representations of G is called the unitary dual of G and will be denoted Σ.
If L is invariant then one can define in an obvious way a represention of G on L called a subrepresentation of U. A representation U of G with representation space H is the direct sum of representations U_i of G on H_i if every H_i is an invariant subspace of H, H is the direct sum of H_i and each U_i is a

subrepresentation of U. One writes $U = \oplus_i U_i$.
The representations of a compact group behave nicely. Their main properties are gathered in the following proposition.

Proposition 3.1 *Let G be a compact group. Then*

1. *Every unitary representation of G admits a subrepresentation of finite dimension.*
2. *Every irreducible unitary representation of G is finite dimensional.*
3. *Every unitary representation of G is the direct sum of irreducible unitary representations.*

For more details on representation theory we refer to [6] and [9].

4 Fourier transform on compact groups

This section draws a lot from [2], [4], [5] and [12]. In what follows, G is a compact group and Σ denotes its dual objet, the set of all equivalence classes of unitary irreducible representations of G. We denote by U^σ an element of the class $\sigma \in \Sigma$, by H_σ its Hilbert representation space and by d_σ the dimension of H_σ. Let $(\xi_1^\sigma, \cdots, \xi_{d_\sigma}^\sigma)$ be a basis of H_σ. The matrix elements of U^σ related to the above basis are defined by $u_{ij}^\sigma(t) = \langle U_t^\sigma \xi_j^\sigma, \xi_i^\sigma \rangle$ for all $i, j \in \{1, \cdots, d_\sigma\}$ and $t \in G$. The contragredient of the representation U^σ is the representation denoted by $\overline{U^\sigma}$ whose matrix elements are the complex conjugate of those of U^σ. We recall the orthogonality relations due to Schur:

$$\int_G \overline{u_{ij}^\sigma(t)} u_{kl}^\sigma(t) d\lambda(t) = \frac{1}{d_\sigma} \delta_i^k \delta_j^l$$

where the integration is taken against the normalized Haar measure λ of G and δ_i^j is the Kronecker's delta [9, section 27].

Let us denote by $L_1(G, \mathcal{M})$ the space of Bochner-integrable $\mathcal{M}$-valued maps on G. For $f \in L_1(G, \mathcal{M})$, the Fourier transform $\widehat{f}$ of f is given by

$$\widehat{f}(\sigma)(\xi, \eta) = \int_G \langle \overline{U_t^\sigma} \xi, \eta \rangle_{H_\sigma} f(t) d\lambda(t),\ \xi, \eta \in H_\sigma.$$

In this paper we are mostly interested in the space $L_2(G, \mathcal{M})$ of Bochner-square-integrable $\mathcal{M}$-valued maps on G. Since the Haar measure on G is finite, we have $L_2(G, \mathcal{M}) \subset L_1(G, \mathcal{M})$ so the above Fourier transform formula is valid for functions in $L_2(G, \mathcal{M})$. Each $\widehat{f}(\sigma)$ is interpreted as a sesquilinear mapping from $H_\sigma \times H_\sigma$ in $\mathcal{M}$. Now set $\mathscr{S}(\Sigma, \mathcal{M}) = \prod_{\sigma \in \Sigma} \mathscr{S}(H_\sigma; \mathcal{M})$ where $\mathscr{S}(H_\sigma; \mathcal{M})$ is the space of all sesquilinear maps from $H_\sigma \times H_\sigma$ to $\mathcal{M}$.

We consider $\mathscr{S}_2(\Sigma,\mathcal{M}) = \left\{ \phi \in \mathscr{S}(\Sigma,\mathcal{M}) \,:\, \sum_{\sigma\in\Sigma} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} ||\phi(\sigma)(\xi_j^\sigma,\xi_i^\sigma)||_{\mathcal{M}}^2 < \infty \right\}$.

The authors in [3] proved that the Fourier transform is a norm preserving isomorphism from $L_2(G,\mathcal{M})$ on $\mathscr{S}_2(\Sigma;\mathcal{M})$ when they are respectively endowed with the norms

$$||f||_L = \left(\int_G ||f(t)||_{\mathcal{M}}^2 d\lambda(t)\right)^{\frac{1}{2}} \tag{1}$$

and

$$||\phi||_S = \left(\sum_{\sigma\in\Sigma} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} ||\phi(\sigma)(\xi_j^\sigma,\xi_i^\sigma)||_{\mathcal{M}}^2\right)^{\frac{1}{2}}. \tag{2}$$

Moreover they proved the following reconstruction formula which is valid for all $f \in L_2(G,\mathcal{M})$:

$$f = \sum_{\sigma\in\Sigma} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \hat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)u_{ij}^\sigma. \tag{3}$$

5 Main results

Proposition 5.1 *The mapping $\mathcal{A}\times L_2(G,\mathcal{M}) \to L_2(G,\mathcal{M})$, $(a,f)\mapsto f\cdot a$ with*

$$(f\cdot a)(t) = f(t)a \;\; \textit{for all } t\in G \tag{4}$$

is an action of $\mathcal{A}$ on $L_2(G,\mathcal{M})$.

Proof Let $f\in L_2(G,\mathcal{M})$ and $a\in\mathcal{A}$, we have $f\cdot a\in L_2(G,\mathcal{M})$ since

$$\begin{aligned}
\int_G ||f\cdot a(t)||_{\mathcal{M}}^2 d\lambda(t) &= \int_G ||f(t)a||_{\mathcal{M}}^2 d\lambda(t)\\
&= \int_G ||\langle f(t)a, f(t)a\rangle||_{\mathcal{A}} d\lambda(t)\\
&= \int_G ||a^*\langle f(t), f(t)\rangle a||_{\mathcal{A}} d\lambda(t)\\
&\leq \int_G ||a^*||_{\mathcal{A}}||f(t)||_{\mathcal{M}}^2||a||_{\mathcal{A}} d\lambda(t)\\
&= \int_G ||f(t)||_{\mathcal{M}}^2||a||_{\mathcal{A}}^2 d\lambda(t)\\
&= \left(\int_G ||f(t)||_{\mathcal{M}}^2 d\lambda(t)\right)||a||_{\mathcal{A}}^2 < \infty.
\end{aligned}$$

Moreover for $t\in G$ and $a,b\in\mathcal{A}$ we have

$$((f\cdot a)\cdot b)(t) = (f\cdot a)(t)b = (f(t)a)b = f(t)(ab) = (f\cdot(ab))(t).$$

Therefore $(f \cdot a) \cdot b = f \cdot (ab)$. □

Let f and g be in $L_2(G, \mathcal{M})$. We set

$$\langle f, g \rangle_L = \int_G \langle f(t), g(t) \rangle d\lambda(t). \tag{5}$$

Proposition 5.2 *The mapping* $L_2(G, \mathcal{M}) \times L_2(G, \mathcal{M}) \to \mathcal{A}$, $(f, g) \mapsto \langle f, g \rangle_L$ *is an* $\mathcal{A}$*-product on* $L_2(G, \mathcal{M})$.

Proof Let $f, g \in L_2(G, \mathcal{M})$ and $a \in \mathcal{A}$.

1. Then $||f(\cdot)||_{\mathcal{M}}$ and $||g(\cdot)||_{\mathcal{M}}$ belong to $L_2(G)$, so $||f(\cdot)||_{\mathcal{M}} ||g(\cdot)||_{\mathcal{M}} \in L_1(G)$. We have $\int_G ||\langle f(t), g(t) \rangle||_{\mathcal{A}} d\lambda(t) \leq \int_G ||f(t)||_{\mathcal{M}} ||g(t)||_{\mathcal{M}} d\lambda(t) < \infty$. Hence $\langle \cdot, \cdot \rangle_L$ is well-defined.

2. Let $\alpha, \beta \in \mathbb{C}$. We have

$$\langle f, \alpha g + \beta h \rangle_L = \int_G \langle f(t), \alpha g(t) + \beta h(t) \rangle d\lambda(t) = \alpha \langle f, g \rangle_L + \beta \langle f, h \rangle_L.$$

3. We have

$$\langle f, g \cdot a \rangle_L = \int_G \langle f(t), g(t)a \rangle d\lambda(t) = \int_G \langle f(t), g(t) \rangle a d\lambda(t) = \langle f, g \rangle_L a.$$

4. We have

$$\begin{aligned} \langle f, g \rangle_L &= \int_G \langle f(t), g(t) \rangle d\lambda(t) = \int_G \langle g(t), f(t) \rangle^* d\lambda(t) \\ &= \left(\int_G \langle g(t), f(t) \rangle d\lambda(t) \right)^* = \langle g, f \rangle_L^*. \end{aligned}$$

5. We have $\langle f, f \rangle_L = \int_G \langle f(t), f(t) \rangle d\lambda(t) \geq 0$ as $\forall t \in G$, $\langle f(t), f(t) \rangle \geq 0$. Furthermore let $f \in L_2(G, \mathcal{M})$ such that $\int_G \langle f(t), f(t) \rangle d\lambda(t) = 0$, then $t \mapsto \langle f(t), f(t) \rangle$ is null λ-a.e., hence $f = 0$ λ-a.e.. So $\langle f, f \rangle_L = 0$ gives $f = 0$ since $f \in L_2(G, \mathcal{M})$.

□

As a consequence of Proposition 5.1 and Proposition 5.2 we have:

Corollary 5.3 *The space $L_2(G,\mathcal{M})$ is a pre-Hilbert $\mathcal{A}$-module under the action of $\mathcal{A}$ on $L_2(G,\mathcal{M})$ defined by $f\cdot a = f(\cdot)a$ for all $f\in L_2(G,\mathcal{M})$, $a\in\mathcal{A}$ and the $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle_L$.*

Proof We will only show that $L_2(G,\mathcal{M})$ is a complex vector space. Let f, $g\in L_2(G,\mathcal{M})$, we have

$$\begin{aligned}\int_G ||(f+g)(t)||^2_{\mathcal{M}}d\lambda(t) &= \int_G ||\langle f(t)+g(t), f(t)+g(t)\rangle||_{\mathcal{A}}d\lambda(t)\\ &\leqslant \int_G ||\langle f(t),f(t)\rangle||_{\mathcal{A}}d\lambda(t)+\int_G ||\langle f(t),g(t)\rangle||_{\mathcal{A}}d\lambda(t)\\ &\quad+\int_G ||\langle g(t),f(t)\rangle||_{\mathcal{A}}d\lambda(t)+\int_G ||\langle g(t),g(t)\rangle||_{\mathcal{A}}d\lambda(t).\end{aligned}$$

The integral $\int_G ||\langle f(t),f(t)\rangle||_{\mathcal{A}}d\lambda(t) = \int_G ||f(t)||^2_{\mathcal{M}}d\lambda(t)$ is finite. It is the same to $\int_G ||\langle g(t),g(t)\rangle||_{\mathcal{A}}d\lambda(t)$.
By Hölder's inequality,

$$\begin{aligned}\int_G ||\langle f(t),g(t)\rangle||_{\mathcal{A}}d\lambda(t) &\leqslant \int_G ||f(t)||_{\mathcal{M}}||g(t)||_{\mathcal{M}}d\lambda(t)\\ &\leqslant \left(\int_G ||f(t)||^2_{\mathcal{M}}d\lambda(t)\right)^{\frac{1}{2}}\left(\int_G ||g(t)||^2_{\mathcal{M}}d\lambda(t)\right)^{\frac{1}{2}} < +\infty.\end{aligned}$$

Similary $\int_G ||\langle g(t),f(t)\rangle||_{\mathcal{A}}d\lambda(t) < +\infty$.
Hence $\int_G ||(f+g)(t)||^2_{\mathcal{M}}d\lambda(t) < +\infty$ and $f+g\in L_2(G,\mathcal{M})$.
Moreover for all $f\in L_2(G,\mathcal{M})$ and $\alpha\in\mathbb{C}$, $\alpha f\in L_2(G,\mathcal{M})$. □

The discrete analogues of the above results are proved for $\mathscr{S}_2(\Sigma,\mathcal{M})$.

Proposition 5.4 *The mapping $\mathcal{A}\times\mathscr{S}_2(\Sigma,\mathcal{M})\mapsto\mathscr{S}_2(\Sigma,\mathcal{M})$, $(a,\phi)\mapsto\phi\cdot a$ with*

$$(\phi\cdot a)(\sigma)(\xi,\eta) = (\phi(\sigma)(\xi,\eta))a,$$

for all $\sigma\in\Sigma$ and $\xi,\eta\in H_\sigma$, is an action of $\mathcal{A}$ on $\mathscr{S}_2(\Sigma,\mathcal{M})$.

Proof Let $\phi\in\mathscr{S}_2(\Sigma,\mathcal{M})$ and $a\in\mathcal{A}$, we have:

$$\begin{aligned}
||\phi\cdot a||_S^2 &= \sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j=1}^{d_\sigma} ||(\phi\cdot a)(\sigma)(\xi_j^\sigma,\xi_i^\sigma)||_{\mathcal{M}}^2\\
&= \sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j=1}^{d_\sigma} ||\phi(\sigma)(\xi_j^\sigma,\xi_i^\sigma)a||_{\mathcal{M}}^2\\
&\leq \sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j=1}^{d_\sigma} ||\phi(\sigma)(\xi_j^\sigma,\xi_i^\sigma)||_{\mathcal{M}}^2 ||a||_{\mathcal{A}}^2 < \infty.
\end{aligned}$$

For $\sigma\in\Sigma$, $a,b\in\mathcal{A}$ and $\eta,\xi\in H_\sigma$, we have

$$\begin{aligned}
[(\phi\cdot a)\cdot b](\sigma)(\xi,\eta) &= [(\phi\cdot a)(\sigma)(\xi,\eta)]b = (\phi(\sigma)(\xi,\eta)a)b = \phi(\sigma)(\xi,\eta)(ab)\\
&= (\phi\cdot(ab))(\sigma)(\xi,\eta).
\end{aligned}$$

□

For $\phi,\psi\in\mathscr{S}_2(\Sigma,\mathcal{M})$, we set

$$\langle\phi,\psi\rangle_S = \sum_{\sigma\in\Sigma} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \langle\phi(\sigma)(\xi_i^\sigma,\xi_j^\sigma),\psi(\sigma)(\xi_i^\sigma,\xi_j^\sigma)\rangle. \tag{6}$$

Proposition 5.5 *The mapping $\mathscr{S}_2(\Sigma,\mathcal{M})\times\mathscr{S}_2(\Sigma,\mathcal{M})\to\mathcal{A}$, $(\phi,\psi)\mapsto\langle\phi,\psi\rangle_S$ is an $\mathcal{A}$-product on $\mathscr{S}_2(\Sigma,\mathcal{M})$.*

Proof Let $\phi,\psi,\varphi\in\mathscr{S}_2(\Sigma,\mathcal{M})$, $\alpha,\beta\in\mathbb{C}$ and $a\in\mathcal{A}$.

1. The equality $\langle\phi,\alpha\psi+\beta\varphi\rangle_S = \alpha\langle\phi,\psi\rangle_S+\beta\langle\phi,\varphi\rangle_S$ is trivial.

2.

$$\begin{aligned}
\langle\phi,\varphi\cdot a\rangle_S &= \sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j=1}^{d_\sigma} \langle\phi(\sigma)(\xi_i^\sigma,\xi_j^\sigma),\varphi(\sigma)(\xi_i^\sigma,\xi_j^\sigma)a\rangle\\
&= \left(\sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j=1}^{d_\sigma} \langle\phi(\sigma)(\xi_i^\sigma,\xi_j^\sigma),\varphi(\sigma)(\xi_i^\sigma,\xi_j^\sigma)\rangle\right) a = \langle\phi,\varphi\rangle_S a.
\end{aligned}$$

3.

$$\begin{aligned}
\langle\phi,\varphi\rangle_S &= \sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j=1}^{d_\sigma} \langle\phi(\sigma)(\xi_i^\sigma,\xi_j^\sigma),\varphi(\sigma)(\xi_i^\sigma,\xi_j^\sigma)\rangle\\
&= \sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j=1}^{d_\sigma} \langle\varphi(\sigma)(\xi_i^\sigma,\xi_j^\sigma),\phi(\sigma)(\xi_i^\sigma,\xi_j^\sigma)\rangle^*\\
&= \left(\sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j=1}^{d_\sigma} \langle\varphi(\sigma)(\xi_i^\sigma,\xi_j^\sigma),\phi(\sigma)(\xi_i^\sigma,\xi_j^\sigma)\rangle\right)^* = \langle\varphi,\phi\rangle_S^*.
\end{aligned}$$

4. $\langle \phi, \phi \rangle_S = 0 \Rightarrow \phi = 0$. Indeed, if $\sum_{\sigma\in\Sigma} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \langle \phi(\sigma)(\xi_i^\sigma, \xi_j^\sigma), \phi(\sigma)(\xi_i^\sigma, \xi_j^\sigma)\rangle =$ 0 then $\langle \phi(\sigma)(\xi_i^\sigma, \xi_j^\sigma), \phi(\sigma)(\xi_i^\sigma, \xi_j^\sigma)\rangle = 0$ for $\sigma \in \Sigma$, $i, j \in \{1, 2, \cdots, d_\sigma\}$. But $\langle \phi(\sigma)(\xi_i^\sigma, \xi_j^\sigma), \phi(\sigma)(\xi_i^\sigma, \xi_j^\sigma)\rangle = 0$ for all $\sigma \in \Sigma$, $i, j \in \{1, 2, \cdots, d_\sigma\}$ implies $\phi(\sigma)(\xi_i^\sigma, \xi_j^\sigma) = 0$ for all $\sigma \in \Sigma$, $i, j \in \{1, 2, \cdots, d_\sigma\}$. So $\phi = 0$. □

We deduce from the two propositions above the following corollary.

Corollary 5.6 *The space $\mathscr{S}_2(\Sigma, \mathcal{M})$ is a pre-Hilbert module under the action of $\mathcal{A}$ on $\mathscr{S}_2(\Sigma, \mathcal{M})$ defined by $\phi \cdot a = \phi(\cdot)a$ and with the $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle_S$.*

The following Parseval type result holds.

Proposition 5.7 *Let $f, g \in L_2(G, \mathcal{M})$. We have $\langle f, g \rangle_L = \langle \hat{f}, \hat{g} \rangle_S$.*

Proof

Let $f, g \in L_2(G, \mathcal{M})$. We can write $f = \sum_{\sigma\in\Sigma}\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} d_\sigma a_{ij}^\sigma u_{ij}^\sigma$ and $g = \sum_{\sigma\in\Sigma}\sum_{k=1}^{d_\sigma}\sum_{l=1}^{d_\sigma} d_\sigma b_{kl}^\sigma u_{kl}^\sigma$ where $a_{ij}^\sigma = \hat{f}(\sigma)(\xi_j^\sigma, \xi_i^\sigma)$ and $b_{kl}^\sigma = \hat{g}(\sigma)(\xi_l^\sigma, \xi_k^\sigma)$. Then

$$\begin{aligned}
\langle f, g \rangle_L &= \langle \sum_{\sigma\in\Sigma}\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} d_\sigma a_{ij}^\sigma u_{ij}^\sigma, \sum_{\sigma\in\Sigma}\sum_{k=1}^{d_\sigma}\sum_{l=1}^{d_\sigma} d_\sigma b_{kl}^\sigma u_{kl}^\sigma \rangle_L \\
&= \sum_{\sigma\in\Sigma}\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma}\sum_{k=1}^{d_\sigma}\sum_{l=1}^{d_\sigma} d_\sigma^2 \langle a_{ij}^\sigma u_{ij}^\sigma, b_{kl}^\sigma u_{kl}^\sigma \rangle_L \\
&= \sum_{\sigma\in\Sigma}\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma}\sum_{k=1}^{d_\sigma}\sum_{l=1}^{d_\sigma} d_\sigma^2 \int_G \langle a_{ij}^\sigma u_{ij}^\sigma(t), b_{kl}^\sigma u_{kl}^\sigma(t) \rangle d\lambda(t) \\
&= \sum_{\sigma\in\Sigma}\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma}\sum_{k=1}^{d_\sigma}\sum_{l=1}^{d_\sigma} d_\sigma^2 \int_G \langle a_{ij}^\sigma, b_{kl}^\sigma \rangle \overline{u_{ij}^\sigma(t)} u_{kl}^\sigma(t) d\lambda(t) \\
&= \sum_{\sigma\in\Sigma}\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma}\sum_{k=1}^{d_\sigma}\sum_{l=1}^{d_\sigma} d_\sigma^2 \langle a_{ij}^\sigma, b_{kl}^\sigma \rangle \int_G \overline{u_{ij}^\sigma(t)} u_{kl}^\sigma(t) d\lambda(t) \\
&= \sum_{\sigma\in\Sigma}\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma}\sum_{k=1}^{d_\sigma}\sum_{l=1}^{d_\sigma} d_\sigma^2 \langle a_{ij}^\sigma, b_{kl}^\sigma \rangle \delta_i^k \delta_j^l \\
&= \sum_{\sigma\in\Sigma}\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} d_\sigma \langle a_{ij}^\sigma, b_{ij}^\sigma \rangle \\
&= \langle \hat{f}, \hat{g} \rangle_S.
\end{aligned}$$

□

It follows from the above proposition that the Fourier transform is an $\mathcal{A}$-product preserving operator.

Now let us set

$$||f||_{\mathscr{L}} = ||\langle f, f\rangle_L||_{\mathcal{A}}^{\frac{1}{2}},\ f \in L_2(G, \mathcal{M}) \tag{7}$$

and

$$||\phi||_{\mathscr{S}} = ||\langle \phi, \phi\rangle_S||_{\mathcal{A}}^{\frac{1}{2}},\ \phi \in \mathscr{S}_2(\Sigma, \mathcal{M}). \tag{8}$$

We deduce the following result as a consequence of Proposition 5.7.

Corollary 5.8 *The map $f \mapsto \hat{f}$ is a linear isometry from $(L_2(G, \mathcal{M}), ||\cdot||_{\mathscr{L}})$ into $(\mathscr{S}_2(\Sigma, \mathcal{M}), ||\cdot||_{\mathscr{S}})$.*

Set

$$|f|_L = \langle f, f\rangle_L^{\frac{1}{2}}. \tag{9}$$

Proposition 5.9 *For all $f, g \in L_2(G, \mathcal{M})$, we have*

$$|f-g|^2 + |f+g|_L^2 = 2(|f|_L^2 + |g|_L^2).$$

Proof Let $f, g \in L_2(G, \mathcal{M})$. We have

$$\begin{aligned}
|f+g|_L^2 + |f-g|_L^2 &= \langle f+g, f+g\rangle_L + \langle f-g, f-g\rangle_L \\
&= \langle f, f\rangle_L + \langle f, g\rangle_L + \langle g, f\rangle_L + \langle g, g\rangle_L \\
&\quad + \langle f, f\rangle_L - \langle f, g\rangle_L - \langle g, f\rangle_L + \langle g, g\rangle_L \\
&= 2\langle f, f\rangle_L + 2\langle g, g\rangle_L \\
&= 2(|f|_L^2 + |g|_L^2).
\end{aligned}$$

□

Proposition 5.10 *There exists $C_0 > 0$ such that for all $\sigma \in \Sigma$ and for all $i, j, k, l \in \{1, \cdots, d_\sigma\}$ we have $\int_G |\overline{u_{ij}^\sigma(t)} u_{kl}^\sigma(t)| d\lambda(t) \leq C_0$.*

Proof For all $\sigma \in \Sigma$ and $i, j \in \{1, \cdots, d_\sigma\}$ the functions $t \mapsto u_{ij}^\sigma(t)$ are continuous on G. They are bounded since G is a compact. Therefore there exists $C_0 > 0$ such that $\forall i, j, k, l \in \{1, \cdots, d_\sigma\}$, $\forall t \in G$, $|\overline{u_{ij}^\sigma(t)} u_{kl}^\sigma(t)| \leq C_0$. We obtain $\int_G |\overline{u_{ij}^\sigma(t)} u_{kl}^\sigma(t)| d\lambda(t) \leq \int_G C_0 d\lambda(t) = C_0 \lambda(G) = C_0$ since $\lambda(G) = 1$.
□

Proposition 5.11 $\forall \sigma \in \Sigma$, $\exists N_\sigma \in \mathbb{N}^*$,

$$\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma}\sum_{k=1}^{d_\sigma}\sum_{l=1}^{d_\sigma} ||\langle a_{ij}^\sigma, a_{kl}^\sigma\rangle||_{\mathcal{A}} \leq N_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} ||a_{ij}^\sigma||_{\mathcal{M}}^2.$$

Proof We have

$$\begin{aligned}\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma}\sum_{k=1}^{d_\sigma}\sum_{l=1}^{d_\sigma} ||\langle a_{ij}^\sigma, a_{kl}^\sigma\rangle||_{\mathcal{A}} &\leq \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma}\sum_{k=1}^{d_\sigma}\sum_{l=1}^{d_\sigma} ||a_{ij}^\sigma||_{\mathcal{M}} \times ||a_{kl}^\sigma||_{\mathcal{M}} \\ &\leq \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma}\sum_{k=1}^{d_\sigma}\sum_{l=1}^{d_\sigma} \max(||a_{ij}^\sigma||_{\mathcal{M}}^2, ||a_{kl}^\sigma||_{\mathcal{M}}^2).\end{aligned}$$

There are $n_{ij}^\sigma \in \{1, 2, \cdots, d_\sigma\}$ such that $\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma}\sum_{k=1}^{d_\sigma}\sum_{l=1}^{d_\sigma} \max(||a_{ij}^\sigma||_{\mathcal{M}}^2, ||a_{kl}^\sigma||_{\mathcal{M}}^2) = \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} n_{ij}^\sigma ||a_{ij}^\sigma||_{\mathcal{M}}^2$. Let us put $N_\sigma = \max_{i,j} n_{ij}^\sigma$. Hence

$$\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma}\sum_{k=1}^{d_\sigma}\sum_{l=1}^{d_\sigma} ||\langle a_{ij}^\sigma, a_{kl}^\sigma\rangle||_{\mathcal{A}} \leq \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} n_{ij}^\sigma ||a_{ij}^\sigma||_{\mathcal{M}}^2 \leq N_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} ||a_{ij}^\sigma||_{\mathcal{M}}^2.$$

$\square$

Proposition 5.12 *If for all $\sigma \in \Sigma$, $\sup_{\sigma\in\Sigma}(d_\sigma N_\sigma) < \infty$ then $\forall f \in L_2(G, \mathcal{M})$, there exists $C_1 > 0$ such that $||f||_L \leq C_1 ||\hat{f}||_S$.*

Proof Let $f \in L_2(G, \mathcal{M})$. One can write $f = \sum_{\sigma\in\Sigma}\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} d_\sigma a_{ij}^\sigma u_{ij}^\sigma$ where $a_{ij}^\sigma = \hat{f}(\sigma)(\xi_j^\sigma, \xi_i^\sigma)$. We have:

$$\begin{aligned}
||f||_L^2 &= \int_G ||f(t)||_{\mathcal{M}}^2 d\lambda(t) \\
&= \int_G ||\langle f(t), f(t)\rangle||_{\mathcal{A}} d\lambda(t) \\
&= \int_G ||\langle \sum_{\sigma\in\Sigma}\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} d_\sigma a_{ij}^\sigma u_{ij}^\sigma(t), \sum_{\sigma\in\Sigma}\sum_{k=1}^{d_\sigma}\sum_{l=1}^{d_\sigma} d_\sigma a_{kl}^\sigma u_{kl}^\sigma(t)\rangle||_{\mathcal{A}} d\lambda(t) \\
&= \int_G ||\sum_{\sigma\in\Sigma}\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma}\sum_{k=1}^{d_\sigma}\sum_{l=1}^{d_\sigma} d_\sigma^2 \langle a_{ij}^\sigma, a_{kl}^\sigma\rangle \overline{u_{ij}^\sigma(t)} u_{kl}^\sigma(t)||_{\mathcal{A}} d\lambda(t) \\
&\leq \int_G \sum_{\sigma\in\Sigma}\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma}\sum_{k=1}^{d_\sigma}\sum_{l=1}^{d_\sigma} d_\sigma^2 ||\langle a_{ij}^\sigma, a_{kl}^\sigma\rangle \overline{u_{ij}^\sigma(t)} u_{kl}^\sigma(t)||_{\mathcal{A}} d\lambda(t) \\
&= \sum_{\sigma\in\Sigma}\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma}\sum_{k=1}^{d_\sigma}\sum_{l=1}^{d_\sigma} d_\sigma^2 ||\langle a_{ij}^\sigma, a_{kl}^\sigma\rangle||_{\mathcal{A}} \int_G |\overline{u_{ij}^\sigma(t)} u_{kl}^\sigma(t)| d\lambda(t) \\
&\leq C_0 \sum_{\sigma\in\Sigma}\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma}\sum_{k=1}^{d_\sigma}\sum_{l=1}^{d_\sigma} d_\sigma^2 ||\langle a_{ij}^\sigma, a_{kl}^\sigma\rangle||_{\mathcal{A}} \\
&\leq C_0 \sum_{\sigma\in\Sigma}\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} N_\sigma d_\sigma^2 ||a_{ij}^\sigma||_{\mathcal{M}}^2 \\
&= C_0 \sum_{\sigma\in\Sigma} N_\sigma d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} d_\sigma ||a_{ij}^\sigma||_{\mathcal{M}}^2 \\
&\leq C_0 \sum_{\sigma\in\Sigma} \sup_{\sigma\in\Sigma}(d_\sigma N_\sigma) \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} d_\sigma ||a_{ij}^\sigma||_{\mathcal{M}}^2 \\
&= C_0 \sup_{\sigma\in\Sigma}(d_\sigma N_\sigma) \sum_{\sigma\in\Sigma}\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} d_\sigma ||a_{ij}^\sigma||_{\mathcal{M}}^2 \\
&= C_0 \sup_{\sigma\in\Sigma}(d_\sigma N_\sigma) ||\hat{f}||_S^2.
\end{aligned}$$

We set $C_1 = \sqrt{C_0 \sup_{\sigma\in\Sigma}(d_\sigma N_\sigma)}$. Hence $||f||_L \leq C_1 ||\hat{f}||_S$. □

Now we consider $L_2(G)\otimes\mathcal{M}$ the tensor product of $L_2(G)$ and $\mathcal{M}$. For a generic element $f\otimes x \in L_2(G)\otimes\mathcal{M}$ and $a\in\mathcal{A}$, the relation $(f\otimes x)\cdot a = f\otimes(xa)$ defines an action of $\mathcal{A}$ on $L_2(G)\otimes\mathcal{M}$. In fact, let $a, b\in A$ and $f\otimes x \in L_2(G)\otimes\mathcal{M}$. We have $[(f\otimes x)\cdot a]\cdot b = (f\otimes(xa))\cdot b = f\otimes(xab) = (f\otimes x)\cdot(ab)$. Let $f\otimes x$ and $g\otimes y$ be generic elements of $L_2(G)\otimes\mathcal{M}$. We set

$$\langle f\otimes x, g\otimes y\rangle_\otimes = \langle f, g\rangle_l \langle x, y\rangle \tag{10}$$

where $\langle f,g\rangle_l = \int_G f(t)\overline{g(t)}d\lambda(t)$.

Proposition 5.13 *The map defined from $(L_2(G)\otimes\mathcal{M})\times(L_2(G)\otimes\mathcal{M})$ into $\mathcal{A}$ by $(f\otimes x, g\otimes y)\mapsto\langle f\otimes x, g\otimes y\rangle_\otimes$ is an $\mathcal{A}$-product.*

Proof Let $f\otimes x$, $g\otimes y\in L_2(G)\otimes\mathcal{M}$ be generic elements, $a\in\mathcal{A}$ and $\lambda\in\mathbb{C}$. We have:

1. $\langle f\otimes x,(g\otimes y)\cdot a\rangle_\otimes = \langle f\otimes x,(g\otimes ya)\rangle_\otimes = \langle f,g\rangle_l\langle x,ya\rangle = \langle f,g\rangle_l\langle x,y\rangle a = \langle f\otimes x, g\otimes y\rangle_\otimes a$. Even $\langle (f\otimes x)\cdot a, g\otimes x\rangle_\otimes = \langle f\otimes(xa), g\otimes x\rangle_\otimes = \langle f,g\rangle_l\langle xa,y\rangle = \langle f,g\rangle_l a^*\langle x,y\rangle = a^*\langle f,g\rangle_l\langle x,y\rangle = a^*\langle f\otimes x, g\otimes y\rangle_\otimes$.
2. $\langle f\otimes x, g\otimes y\rangle_\otimes = \langle f,g\rangle_l\langle x,y\rangle = \overline{\langle g,f\rangle_l}\langle y,x\rangle^* = (\langle g,f\rangle_l\langle y,x\rangle)^* = \langle g\otimes y, f\otimes x\rangle_\otimes^*$.
3. $\langle\lambda f\otimes x, g\otimes y\rangle_\otimes = \langle\lambda f,g\rangle_l\langle x,y\rangle = \lambda\langle f,g\rangle_l\langle x,y\rangle = \lambda\langle f\otimes x, g\otimes y\rangle_\otimes$.
4. $\langle f\otimes x, f\otimes x\rangle_\otimes = \langle f,f\rangle_l\langle x,x\rangle\geq 0$. And, on the other hand, $\langle f\otimes x, f\otimes x\rangle_\otimes = 0$ implies $\langle f,f\rangle_l = 0$ or $\langle x,x\rangle = 0$. Hence $f\otimes x = 0$.

Let us denote by $L_2(G)\hat{\otimes}\mathcal{M}$ the completion of the pre-Hilbert $\mathcal{A}$-module $L_2(G)\otimes\mathcal{M}$ with respect to the norm:

$$||f\otimes x||_\otimes = ||\langle f\otimes x, f\otimes x\rangle_\otimes||_{\mathcal{A}}^{\frac{1}{2}} \tag{11}$$

where $f\otimes x$ is a generic element in $L_2(G)\otimes\mathcal{M}$.

We denote by $\mathbb{L}_2(G,\mathcal{M})$ the completion of $L_2(G,\mathcal{M})$ under the norm $||\cdot||_{\mathscr{L}}$ defined in (7).

The following proposition extends to module-valued functions on compact groups the Proposition 4.2.1.1 in [1] concerning C^*-algebra-valued functions in mesure spaces. Its proof follows highly [1].

Proposition 5.14 *The Hilbert $\mathcal{A}$-modules $L_2(G)\hat{\otimes}\mathcal{M}$ and $\mathbb{L}_2(G,\mathcal{M})$ are isomorphic.*

Proof For $f\in L_2(G)$ and $x\in\mathcal{M}$, $f(\cdot)x\in L_2(G,\mathcal{M})$ since there is a sequence $f_n(\cdot)x$ of countably valued functions which converges to $f(\cdot)x$ almost everywhere and

$$\begin{aligned}||f(\cdot)x||_L^2 &= \int_G ||f(t)x||_{\mathcal{M}}^2 d\lambda(t)\\ &= \int_G |f(t)|^2||x||_{\mathcal{M}}^2 d\lambda(t)\\ &= ||f||_2^2||x||_{\mathcal{M}}^2 < \infty,\end{aligned}$$

where $||\cdot||_2$ is the L_2-norm of $L_2(G)$. Since the mapping $(f,x)\mapsto f(\cdot)x$ is bilinear, the mapping $U: L_2(G)\otimes\mathcal{M}\to L_2(G,\mathcal{M})$ defined by $U(f\otimes x)=f(\cdot)x$ for any $f\in L_2(G)$ and $x\in\mathcal{M}$ is linear and well defined. Let us prove that U preserves the action of $\mathcal{A}$ on $L_2(G)\otimes\mathcal{M}$. Let $f\otimes x\in L_2(G)\otimes\mathcal{M}$ and $a\in\mathcal{A}$, $U((f\otimes x)\cdot a)=U(f\cdot(xa))=f(\cdot)xa=[f(\cdot)x]a=U(f\otimes x)\cdot a$. On the other hand let $f\otimes x$, $g\otimes y\in L_2(G)\otimes\mathcal{M}$ be generic elements. We have

$$\begin{aligned}
\langle U(f\otimes x),U(g\otimes y)\rangle_L &= \langle f(\cdot)x,g(\cdot)y\rangle_L\\
&= \int_G\langle f(t)x,g(t)y\rangle d\lambda(t)\\
&= \int_G f(t)\langle x,y\rangle\overline{g(t)}d\lambda(t)\\
&= \int_G f(t)\overline{g(t)}\langle x,y\rangle d\lambda(t)\\
&= \left(\int_G f(t)\overline{g(t)}d\lambda(t)\right)\langle x,y\rangle\\
&= \langle f,g\rangle_l\langle x,y\rangle\\
&= \langle f\otimes x,g\otimes y\rangle_\otimes.
\end{aligned}$$

Hence, U preserves the inner product. It is an isometry. So U is injective and continuous since U is an $\mathcal{A}$-linear operator of $L_2(G)\otimes\mathcal{M}$ in $L_2(G,\mathcal{M})$. Now let us show that the range of U is dense in $\mathbb{L}_2(G,\mathcal{M})$. Let $F\in\mathbb{L}_2(G,\mathcal{M})$ be a simple function. For every $t\in G$, $F(t)=\sum_{i=1}^n x_i\chi_{E_i}(t)=\sum_{i=1}^n\chi_{E_i}(t)x_i$ where (E_i) are disjoint measurable subsets of G and χ_{E_i} is the characteristic function of E_i. For all $1\le i\le n$, $\chi_{E_i}\in L_2(G)$ and $\chi_{E_i}(t)x_i=U(\chi_{E_i}\otimes x_i)(t)$. Therefore $F(t)=\sum_{i=1}^n U(\chi_{E_i}\otimes x_i)(t)=U(\sum_{i=1}^n\chi_{E_i}\otimes x_i)(t)$ and F is in the range of U. Hence the set of simple functions is a subset of the range of U which itself is included in $L_2(G,\mathcal{M})$. The range of U is dense in $\mathbb{L}_2(G,\mathcal{M})$ since the $\mathcal{M}$-valued simple functions are dense in $\mathbb{L}_2(G,\mathcal{M})$. Therefore, U can be extended to a unitary operator defined from $L_2(G)\hat{\otimes}\mathcal{M}$ to $\mathbb{L}_2(G,\mathcal{M})$. Consequently, the proposition is obtained. $\square$

References

[1] Ariyani, *The Generalized continuous wavelet transform on Hilbert Modules*, PhD thesis, University of New South Wales, Sydney, Australia, 2008.

[2] V. S. K. Assiamoua, $L_1(G,A)$-*multipliers*, Acta Sci. Math., 53 (1989), 309 -318.

[3] V. S. K. Assiamoua, A. Olubummo, *A Fourier-Stieltjes transforms of vector-valued measures on compact groups.* Acta Sci. Math. (Szeged), Vol. 53, 1989, 301-307.

[4] J. Diestel, J. J. Uhl, *Vector measures*, Math. Surveys 15, Amer. Math. Soc., Providence, 1977.

[5] N. Dinculeanu, *Vector measures*, Noordhoff International Publishing (1974), Internationl Publishing Company B. V., Leyden, The Netherlands.

[6] G. B. Folland, *A course in Abstract Harmonic Analysis*, CRC press, 1995.

[7] M. Frank, *Isomorphisms of Hilbert C^*-modules and $*$-isomorphisms of related operator C^*-algebras*, Math. Scand. 80(1997), 313-319.

[8] V. I. Istrăţescu, *Inner Product Structures*, D. Reidel Publishing Company, Dordrecht, Holland (1987).

[9] E. Hewitt, K. A. Ross, *Abstract Harmonic Analysis. Vol. I and II*, Springer-Verlag, Berlin, 1970.

[10] I. Kaplansky, *Projections in Banach algebras*, Annals of Mathematics 53 (2), (1951) 235-249.

[11] E. C. Lance, *Hilbert C*-Modules, A toolkit for operator algebraists*, Cambridge University Press (1995).

[12] Y. Mensah, *Facts about the Fourier-Stieltjes transform of vector measures on compact groups*, Int. J. Analysis Appl., Vol. 2, Issue 1, (2013) 19-25.

M. Todjro
Department of Mathematics, University of Lomé, P.O.Box 1515, Lomé, Togo
email: todjrom7@gmail.com

Y. Mensah
Department of Mathematics, University of Lomé, P.O.Box 1515, Lomé, Togo
and International Chair in Mathematical Physics and Applications-Unesco Chair, University of Abomey-Calavi, Cotonou, Benin
email: mensahyaogan2@gmail.com

V. S. K. Assiamoua
Department of Mathematics, University of Lomé, P.O.Box 1515, Lomé, Togo
email: kofassiam@yahoo.fr