# The efficient yield curve: Evidence from Brazil

**Abstract**

This paper examines the efficient yield curve estimation, exploiting the relationship between yield curve and macroeconomic variables by focusing on an emerging market case: Brazil. Differently from the classic literature on the Brazilian yield curve, where the Diebold & Li (2006) model is estimated only through the two-step method, the model herein is also put in the state-space form, and the parameters are simultaneously estimated using the Kalman filter. The results obtained showed that the Kalman filter is the most suitable method for the model estimation.
*Keywords:* Term Structure, Yield Curve, Kalman Filter

JEL C53, E43, E47

## 1. Introduction

The predictability of asset returns has attracted considerable attention of financial economists; while returns are predictable a subject of ongoing debates remains. The yield curve has proved to be a leading indicator for economic activity and inflation. It has also a massive influence over the development of macroeconomic scenarios, which are composed by companies, institutions and investors.

Understanding the behavior of the term structure of interest rate is important to macroeconomists, financial economists and fixed income managers, such understanding has motivated several studies and theoretical advances in the modeling of the yield curve.

One of the most popular approaches to forecasting the yield curve is the Nelson & Siegel (1987) (hereafter DNS) model, proposed by Diebold & Li (2006). The attractiveness for factor models of the Nelson Siegel type is due to its parsimony and empirical performance. Models of this type can capture most of the behavior on the term structure of interest rate by only three factors. Existing evidence suggests that these specifications are remarkably well suited both fitting the term structure and forecasting its movements.

Interpolation models were developed by McCulloch (1971) who interpolated the discount function rather than the yields or the asset prices in a direct manner; and by Vasicek (1977), who adjusted exponential splines to the discount curve, obtaining smoother adjustments for the longest area of the curve.

Following this same line of research, Vicente & Tabak (2007) compared a Gaussian affine model to Diebold and Li model for Brazilian data and concluded that the latter model is slightly superior in terms of yield curve forecasts. Vereda *et al.* (2008) employed a VAR approach to forecast the term structure of interest rates and found that incorporating macro variables can improve forecasting performance, especially for longer-term forecasts. In Caldeira *et al.* (2010) the Nelson Siegel model is cast in state-space form, and the parameters are simultaneously and efficiently estimated using the Kalman filter.

The term structure of interest rates and macroeconomics variables researches are recent. In a seminal paper, Ang & Piazessi (2003) describe the joint dynamics of bond yields and macroeconomic variables (inflation and growth), concluding that macro factors primarily explain movements at the short end and middle of the yield curve. Thus Hordal *et al.* (2006) estimate a tractable model of macroeconomic (inflation, output gap and short-term policy interest rate) and yield curve dynamics. Through a German data, they find that the macroeconomic factors affect the term structure of interest rates in different ways while inflation and output shock mostly affects the curvature of the yield curve, monetary policy shocks have a marked impact on the slope of the yield curve.

Using an affine term structure model, Bernanke *et al.* (2004) show that model which only uses macroeconomic variables, predicts the yields reasonably well at all maturities, while Smith & Taylor (2009) proposed a derived formula that links the coefficients of the monetary policy rule for the short term interest rate to the coefficients of the implied affine equations for long-term interest rates, and showed that an increase in the coefficients in the monetary policy rule leads to an upward shift in the coefficients of all maturities.

Finally, Kaya (2013) contribute to literature, studying the relationship between the yield curve and macroeconomic factors, they find that the relationship is significantly affected by the change in monetary policy which is associated with the implementation of inflation targeting (IT) regime. Studies in the literature have generally focused on the developed countries, particularly on US, and this very important literature has remained scarce for the emerging market cases.

The purpose of this paper is twofold: i. Use the three-factor model for the term structure as proposed by Nelson & Siegel (1987) reinterpreting the factors as level, slope and curvature of the yield curve just as in Diebold & Li (2006), in order to make the estimation. To estimate the models, two approachesa Are applied: first, the classic two-step method; second, the state-space approach.

This paper contributes to the literature in two ways. First, to compared different methods for estimate Brazilian yield curve and find evidences that Kalman Filter is the most suitable method for the estimation. Second, a recent data-set that includes the last few years, in which yields have had significant oscillations in fact of financial and economic crisis, is employed.

The reminder of this paper is structured as follows. Section 2 provides a description of the structure of Diebold-Li model for the yield curve and its state-space form. Section 3 describes the data-set employed in the analysis. Section 4 presents the dynamic Nelson Siegle Model results. Section 5 concludes the paper.

## 2. Yield Curve Methodology

Before describing the structure of the model, it is necessary to define discount curve, forward curve and yield curve, as well as their interrelations. The term structure of interest rates is represented by a set of spot rates for different maturities. Each point corresponds to a yield $y_i(\tau)$ associated with maturity $\tau$ , obtained from a security traded on the market.

At any point of time $t$, there will be a collection of zero-coupon bonds that differ only in terms of maturity. However, in a given moment, there may not be a bond available to all desired maturities as bonds are not negotiated for all possible maturities.

One of the most basic constructions describing the term structure of the interest rate, from which other curves are often derived, is the discount function. Let $P_t(\tau)$ be the price of a zero-coupon bond at time $t$, which pays \$1 at maturity $\tau$. Supposedly, every zero-coupon bond is default-free and has strictly positive prices. Thus, the discount function is defined by:

$$P_t(\tau) = e^{\tau y_t(\tau)} \tag{1}$$

The yield $y_t(\tau)$ at which the bond is discounted is the internal rate of return of the zero-coupon bond, at time $t$, and with maturity $\tau$ , expressed as:

$$y_t(\tau) = \frac{-ln(P_t(\tau))}{\tau} \tag{2}$$

The forward rate at time $t$ applied to the time interval between $\tau_1$ and $\tau_2$ , is defined by:

$$f_t(\tau_1\tau_2) = \frac{1}{\tau_1 - \tau_2}\int_{T1}^{T2} y_t(x)dx \tag{3}$$

The same argument applies to forward rates for k-periods. The forward rate can be interpreted as the marginal rate of return necessary to maintain a bond for an additional period. The limit of expression when $\tau_1$ draws closer to $\tau_2$, denoted by $f_t(\tau)$, is the instantaneous forward rate:

$$f_t(\tau) = \frac{-P_t(\tau)}{P_t(\tau)} \tag{4}$$

The instantaneous forward rate curve, $f_t(\tau)$ , provides the decay rate of discount function $P_t(\tau)$ in each point $\tau$. The yield curve $y_t(\tau)$ is the average decay rate for the interval between 0 and $\tau$ , expressed by:

$$y_t(\tau) = \frac{1}{\tau}\int_0^T f_t(x)dx \tag{5}$$

The function $f_t(\tau)$ of forward rates describes the (instantaneous) rate of an investment return which is maintained for a very short time interval. The instantaneous forward rate curve is a very important theoretical construct, even though its value for a single maturity is of little practical interest, due to the high transaction cost associated with a contract between two points in the future if these two points are too close to each another. Only the mean of $f_t(\tau)$ for a future time interval is of practical interest.

At any point at time t, there will be a set of bonds with different maturities, $\tau$ , and different payment flows, which may be used to estimate the yield curves, discount curves and forward curves, which are not observable in practice. There are different approaches to the construction of yield curves. McCulloch (1971) and Vasicek (1977) build yield curves using estimated smooth discount curves and converting them into rates at relevant maturities. The method put forward by McCulloch (1971) discounts function interpolation. The advantage of this method is that the estimation model has only has linear parameters, its disadvantage is the resulting erratic curves for longer maturities, i.e., the adjusted discount curve diverges for longer maturities instead of converging to zero. Vasicek (1977) use of exponential splines to adjust the discount function, which would eliminate the divergence problem for longer maturities.

Statistical models were also used to estimate the term structure of interest rate as in Nelson & Siegel (1987), Swenson (1994), among others. These models proved to be quite useful in the analysis and pricing of fixed income securities, and special attention should be paid to the work carried out by Nelson & Siegel (1987), medium- and long-term factors began to be interpreted as slope, curvature and level factors.

Famma & Bliss (1987) proposed a method for the construction of the term structure using forward rates estimated for the observed maturities. The method consists in sequentially building the forward rates necessary to successively price bonds with longer maturities, known as the un-smoothed forward rates proposed by Fama and Bliss. The yield curve resulting from this procedure is a (discontinuous) function with jumps relative to the maturity of the bond being traded.

### *2.1. Diebold and Li Model*

Empirical results suggest that the Diebold & Li (2006) model has a good forecasting power if compared with an affine term structure model and the random walk benchmark, especially for the short-term interest rates. Therefore, it provides a good stating point for emerging markets research.

The Diebold & Li (2006) method follows the Nelson & Siegel (1987) exponential components framework to distill the entire yield curve, period by period, into a three-dimensional parameter that dynamically evolves. The corresponding yield curve is:

$$y_t(\tau) = \beta_{1t} + \beta_{2t}(\frac{1 - e^{-\lambda\tau}}{\lambda\tau}) + \beta_{3t}(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}) \tag{6}$$

where $t$ is the date and $\tau$ the maturity. The parameter $\lambda$ controls the exponential decay rate and the maturity where $\beta_{3t}$ achieves its maximum curvature loading. Small values of $\lambda$ produce slow decay and can better fit the curve at long maturities, while large values of $\lambda$ produce fast decay and can better fit the curve at short maturities. According to Diebold & Li (2006), we adopted a constant given by the value that maximizes the loading on the medium-term factor, as shown in , setting the value of 0.0609 corresponding to a maximum curvature loading between bonds of 2- and 3- year maturities.

The terms $\beta_{1t}$, $\beta_{2t}$ and $\beta_{3t}$ are interpreted as three latent dynamic factors. The $\beta_{1t}$ loading is a constant of value 1, that does not decay to 0 at the limit; hence, it is viewed as a long-term factor. Thus, $\beta_{2t}$ seems a function that starts at 1 and decays monotonically and quickly to 0; which can be viewed as a short-term factor. Finally, $\beta_{3t}$ starts on value 0, increasing and decaying to 0; hence it can be viewed as a medium-term factor. These three factors are commonly interpreted as level, slope and curvature by the effect they have on the yield curve, respectively.

In a recent paper, Lima *et al.* (2006) study different models for the forecasting of interest rates in Brazil. They compare the forecasting accuracy of VAR and VEC models with that of naive forecasts form a simple random walk model. The authors conclude that VAR/VEC models are not able to produce forecasts which are superior to the random walk benchmark.

*2.2. Estimation and forecasting using the state-space form*

When the state-space form is used, two approaches can be employed to estimate the latent factors and the parameters. The initial approach proposed by Diebold & Li (2006) is based on two steps and, therefore, is inefficient, disregarding the uncertainty that it is inherent to the first-step estimates in the subsequent step. In the first stage, the measurement equation is estimated using cross-sectional data, in which the estimates for the factors are obtained for each time period. Assuming that the decay parameter is constant, the measurement equation becomes linear and can be estimated by ordinary least squares. In the second stage, the time dynamics of the parameters is specified and estimated as an AR(1) or VAR(1) process.

Diebold & Li (2006) showed that it is possible to estimate this model by maximum likelihood in a single step by using the Kalman filter, providing efficient estimates for the parameters and smoothed estimates for the unobservable factors.

This approach is not only adopted in Diebold & Li (2006), but also in Koopman & Mallee (2007), among others. The procedure utilizes the Kalman filter to build the likelihood function, which is then maximized in order to obtain parameter estimates. We consider the Nelson Siegel model as a linear Gaussian state space model.

Our measurement equation models the time-series process of the yields according to latent factors, assuming the form:

$$\begin{bmatrix} y_t(m_1) \\ y_t(m_2) \\ \vdots \\ y_t(m_t) \end{bmatrix} = \begin{bmatrix} 1 & \frac{1-e^{\lambda m_1}}{\lambda m_1} & \frac{1-e^{\lambda m_1}}{\lambda m_1} - e^{m_1} \\ 1 & \frac{1-e^{\lambda m_2}}{\lambda m_2} & \frac{1-e^{\lambda m_2}}{\lambda m_2} - e^{m_2} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{\lambda m_N}}{\lambda m_N} & \frac{1-e^{\lambda m_N}}{\lambda m_N} - e^{m_N} \end{bmatrix} \begin{bmatrix} L_t \\ S_t \\ C_t \end{bmatrix} + \begin{bmatrix} \epsilon_t(m_1) \\ \epsilon_t(m_2) \\ \vdots \\ \epsilon_t(m_N) \end{bmatrix} \tag{7}$$

which can be expressed in matrix notation as above:

$$y_t = \Lambda(\lambda)F_t + \epsilon_t \tag{8}$$

where $\epsilon_t \sim MN(0, \sum_\epsilon), t = 1, ..., T$. Thus, $y_t$ represents a $NX1$ vector of yields, $\Lambda(\lambda)$ is a $NX3$ factor loading matrix, $F_t$ is a $3X1$ latent factor vector, and $\epsilon_t$ is a $NX1$ yield disturbance vector. The diagonal structure of $\sum_\epsilon$ implies that measurement error across maturities of $y_t$ are uncorrelated and that it is a fairly standard assumption in the literature. The transition equation can be expressed by the vector autoregressive process:

$$\begin{bmatrix} L_t - \mu_{Lt} \\ S_t - \mu_{St} \\ C_t - \mu_{Ct} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} L_{t-1} - \mu_{Lt} \\ S_{t-1} - \mu_{St} \\ C_{t-1} - \mu_{Ct} \end{bmatrix} + \begin{bmatrix} \eta_{Lt} \\ \eta_{St} \\ \eta_{Ct} \end{bmatrix} \tag{9}$$

which can be expressed as a matrix notation:

$$F_t = (I - A)\mu + AF_{t-1} + \eta_t \tag{10}$$

Where $\eta_n \sim MN(0, \sum_\eta), t = 1, ..., T$. Thus, $\mu$ is a $3X1$ mean vector, $A$ is a $3X3$ coefficient matrix, $\sum_\eta$ is a $3X1$ factor disturbance matrix. The assumption that the deviations to the NS factors are uncorrelated is not standard in the literature. But Diebold & Li (2006) conclude that the off-diagonal elements are marginally significant and the point estimates and standard errors of the A matrix are little changed when estimating $\sum_\eta$.

We are able to use Kalman filter to estimate latent factors since the DNS state space model is linear in latent factors. The Kalman filter procedure is carried out recursively during $t = 1, ..., T$ with initial values for the latent factors and their variances being the unconditional mean and unconditional variance, respectively. Purchasing $F_{t|t}$ as the minimum mean square linear estimator (MMSLE) of $F_t$ and $v_{t|t}$ as the MSE matrix. So, $f_{1|0} = \mu$ and $v_{1|0} = (I - A)^{-1} \sum_\eta$. With observation $y_t$ and initial values of $f_{1|0}$ and $v_{1|0}$, the Kalman filter updates the values for $f_{t|t}$ and $v_{t|t}$ through:

$$f_{t|t} = f_{t|t-1} + K_t e_{t|t-1} \tag{11}$$

$$v_{t|t} = v_{t|t-1} + K_t \Lambda(\lambda) v_{t|t-1} \tag{12}$$

Where $e_{t|t-1} = y_t \Lambda(\lambda) f_{t|t-1}$ is the predicted error, $ev_{t|t-1} = \Lambda(\lambda) v_{t|t-1} \Lambda(\lambda)' + \sum_\epsilon$ is the predicted error variance matrix and $K_t = v_{t|t-1} \Lambda(\lambda)' ev_{t|t-1}^{-1}$ is the Kalman gain matrix. The next period MMSLE of the latent factors and associated variance matrix conditional on yields are governed by the prediciton equations:

$$f_{t|t-1} = (I - A)\mu + AF_{t-1|t-1} \tag{13}$$

$$v_{t|t-1} = Av_{t-1|t-1} A' + \sum_\eta \tag{14}$$

Denote $\theta$ as the system parameter vector. The parameters to be estimated via numerical maximum likelihood estimation are $\theta_{DNS} = [A_{ij}, \sum_{\eta ij}, \sum_{\epsilon ij}, \mu, \lambda]$. So, we could represent the likelihood function as:

$$l(\theta) = -\frac{NT}{2}log2\pi - \frac{1}{2}\sum_{t=1}^{T} log|ev_t| - \frac{1}{2}\sum_{t=1}^{T} e_t'(ev_t)^{-1}e_t \tag{15}$$

As a result, $l(\theta)$ can be evaluated by Kalman filter through a quasi-Newton optimization method for the purposes of maximization without inverting the Hessian matrix (more details around Kalman filter estimation can be obtained on Durbin & Koopman (2001), Anderson & Moore (1979) and Simon (2006)).

The values of $f_{t|t}$ and $v_{t|t}$ for the last iteration of the Kalman filter are used as initial values in the recursive algorithm to obtain smoothed values of the unobserved factors. Iterating the following equations, we obtain the smoothed estimates:

$$f_{t|T} = f_{t|t} + v_{t|t}\Lambda(\lambda)'v_{t+1}^{-1}(f_{t+1|T} - \Lambda(\lambda)f_{t|t} - \mu) \tag{16}$$

$$v_{t|T} = v_{t|t} + v_{t|t}\Lambda(\lambda)'v_{t+1}^{-1}(v_{t+1|T} - f_{t+1|t})v_{t+1}^{-1}{}'\Lambda(\lambda)v_{t|t} \tag{17}$$

These smoothed estimates provide a more accurate inference on $f_t$ because it uses more information from the system than the filtered estimates.

The maximum likelihood estimator obtained thereby is preferable to the two-step method, as in the latter the estimation of parameters in the second stage does not take into consideration the uncertainty over the values of the estimated factors in the first stage, producing inefficient parameter estimates. The joint estimation of the measurement and state equations on the other hand does not have such problem and yields efficient estimates for the parameters. Another advantage of likelihood estimation is the joint estimation of the decay parameter which, in the two-step method, has to be calibrated according to some measure. Almeida *et al.* (2007) show that different rules for the calibration of the decay parameter yield different results for the out-of-sample forecast of the term structure of interest rate, indicating that the two-step estimation method lacks robustness. Moreover, the Kalman smoother allows to obtain smoothed estimates for the latent variables, which take the whole sample information into account in order to infer on the time series of the unobserved factors.

## 3. Data Set

Brazilian economic data sets are relatively comprehensive when it comes to inflation measures and market expectations. As a consequence of a high-inflationary history, Brazilian market participants were adapted to use a variety of price indexes and price expectations indexes, some of which are available at the weekly or even daily frequencies. Overall, this means that the Brazilian data provide lots of useful high-frequency information about yield curve behavior.

The data set analyzed consists of Brazilian Interbank Deposit Futures Contract (DI1). The source of the data is the Brazilian Mercantile and Futures Exchange (BM&FBovespa), which is the entity that offers DI-futuro contracts and determines the maturities with authorized contracts. The DI-futuro contract with maturity $\tau$ is a zero-coupon future contract in which the underlying asset is the DI-futuro interest rate accrued on a daily basis, capitalized between trading period t and $\tau$. The DI-futuro rate is the average daily rate of Brazilian interbank deposits (borowing/lending), calculated by the Clearinghouse for Custody and Settlements (CETIP) for all business days. The DI-futuro rate, which is published on a daily basis, is expressed in annually compounded terms, based on 252 business days.

The data were observed between January.2008 and March.2018, and represent the most liquid ID contracts negotiated during the analyzed period. Table 1 presents statistics for Brazilian yield curve. For each time series we report the mean, standard deviation, minimum and maximum.

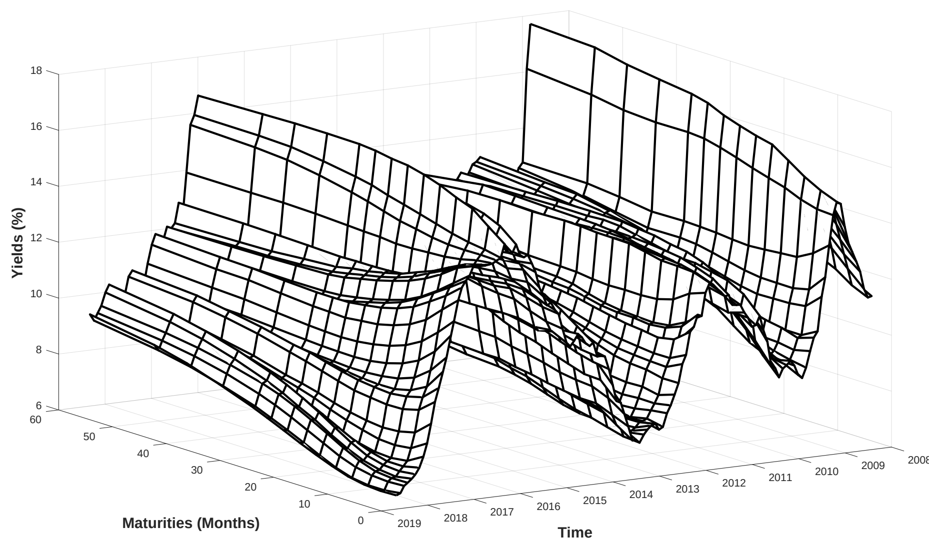
Table 1: Descriptive Statistics, Yield Curves (2008:01 to 2018:03)

| **Descriptive Statistics** | | | | |
|---|---|---|---|---|
| **Maturity** | **Mean** | **Std Dv** | **Minimum** | **Maximum** |
| **3** | 0.1088 | 0.0233 | 0.0630 | 0.1476 |
| **6** | 0.1094 | 0.0238 | 0.0622 | 0.1519 |
| **9** | 0.1102 | 0.0238 | 0.0622 | 0.1565 |
| **12** | 0.1112 | 0.0237 | 0.0630 | 0.1587 |
| **15** | 0.1124 | 0.0233 | 0.0649 | 0.1613 |
| **18** | 0.1135 | 0.0228 | 0.0675 | 0.1632 |
| **21** | 0.1145 | 0.0221 | 0.0702 | 0.1647 |
| **24** | 0.1155 | 0.0215 | 0.0729 | 0.1659 |
| **27** | 0.1163 | 0.0210 | 0.0753 | 0.1686 |
| **30** | 0.1169 | 0.0205 | 0.0778 | 0.1700 |
| **36** | 0.1178 | 0.0197 | 0.0814 | 0.1716 |
| **42** | 0.1186 | 0.0191 | 0.0834 | 0.1732 |
| **48** | 0.1192 | 0.0186 | 0.0846 | 0.1758 |
| **60** | 0.1197 | 0.0178 | 0.0864 | 0.1770 |

It should be noted that the rates for shorter maturities are more volatile, which is a stylized fact in the interest rate literature. Another point to note is that, even in the case of a sample period that is not very long, there is a wide variation between the minimum and maximum rates for all maturities, reflecting the behavior of monetary policy in the period - even more when it comes to high maturities.

The next Figure shows the behavior of the interest curve for the analyzed period. It is important to highlight that this interest curve was obtained considering fixed maturities. Note the various formats assumed by the term structure of the interest curve, in periods in which it has positive slopes, negative slopes and periods with more than one inversion. Thus the sample seems adequate to compare the performance of different adjustment methodologies.

Figure 1: Term Structure of the Interest Rate (January.2008 to March.2018)




Thus, in context to explore the relationship between the interest curve and the macroeconomic variables, a variable matrix was set up, based on Kaya (2013) composed by: inflation, output gap, capacity utilization, exchange rate and policy rate.

Inflation rate was calculated as $\pi_t = (logIPCA_t - logIPCA_{t-12})$, where IPCA denotes de Consumer Price Index and IPCA series are collected from IPEA. To calculate the output gap, it was first calculated the PIB growth as $gr_t = (logPIB_t - logPIB_{t-12})$, and so, detrend the $gr_t$ by using Hodric-Prescott (HP) filter. Adopting this process, we ensure that output gap measures at time $t$ do not rely on unavailable information at that point. Utilization capacity was calculated as $uc_t = (logUCI_t - logUCI_{t-12})$, where UCI series are collected from IPEA and are seasonally adjusted.

For the nominal exchange rate, $e_t$ Real was used against the US Dollar. For the policy rate, $r_t$ the overnight interest rate of Central Bank of Brazil, SELIC, was used. The exchange rates and the price indices are collected from IPEA.

The relationship between yield curve and macroeconomic variables is not stable, as is possible to verify in many works (Bansal & Zhou, 2002)

With the subprime crisis in 2008, Brazil and several countries were affected, financially damaging leveraged sectors and modifying the financing profile of the others. In 2014, the Brazilian economy started a deep economic crisis and only after 2017 Brazil shyly resume GDP growth in order to get out of the deepest financial crisis in the country, related to the shocking levels of GDP and unemployment.

A simple approach to take into account a possible structural break is to estimate any given model in each period. Thus, the sample period is divided as January.1998-December.2013 (pre-2014 period) and January.2014-March.2018 (post-2014 period) for the macroeconomic and yield curve relationship investigation.

## 4. Dynamic Nelson Siegle Model

In Section 2, the model of Diebold & Li (2006) was fit out in state-space form, with a transition equation, which models the dynamics of the factors, and a linear measurement equation that relates the observed yields to the state vector. The parameters were estimated simultaneously by maximum likelihood using the Kalman filter, which is an efficient estimator and which eliminates the problem related to how to calibrate the decay parameter. The yields used resume on future ID rates between January 2008 and March 2018, totaling 1722 observations (123 observations for each one of the 14 maturities).

Differently the two-step method, in the Kalman filter estimation, the parameters are estimated in a single step. So, $\lambda$ governs the decay rate of factor loadings of both the level and curvature are estimated with other parameters and not determined a priori. The initial values of the parameters for Kalman filter were obtained from the estimation using the two-step method. Figure 2 shows the factor loadings for level, slope and curvature, obtained from the estimated $\lambda$.

With an estimated $\lambda$ equal to 0.85, the factor loading of the curvature assumes maximum value for maturities between 13 and 23 months. The main argument in favor of Diebold-Li three-factor model is its capacity to yield forecasts successfully. Although it is not the best model when the fit of the term structure of interest rate is the major goal, the model put forward by Diebold & Li (2006) can replicate the several shapes taken by the yield curves.

Figure 2: Factor Loading $\lambda = 0.85$

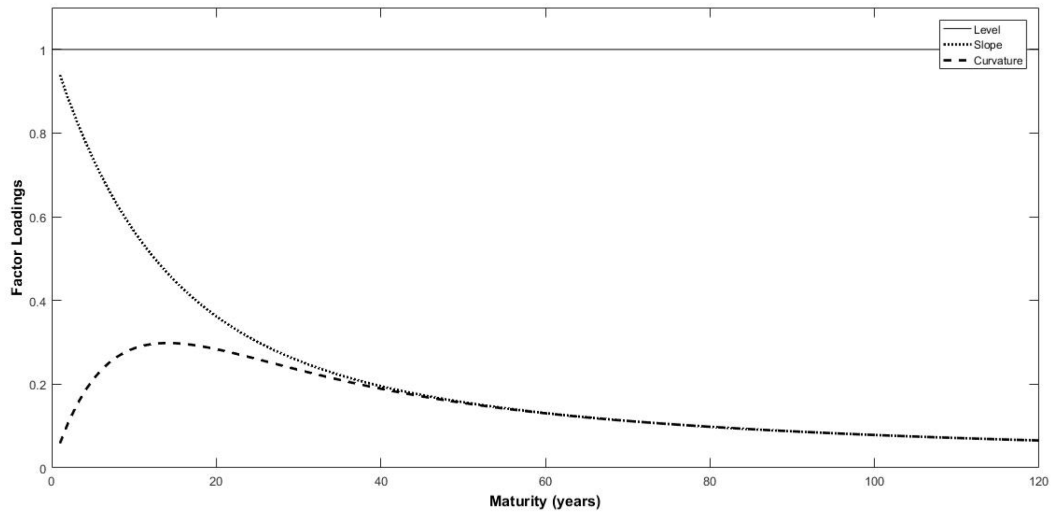
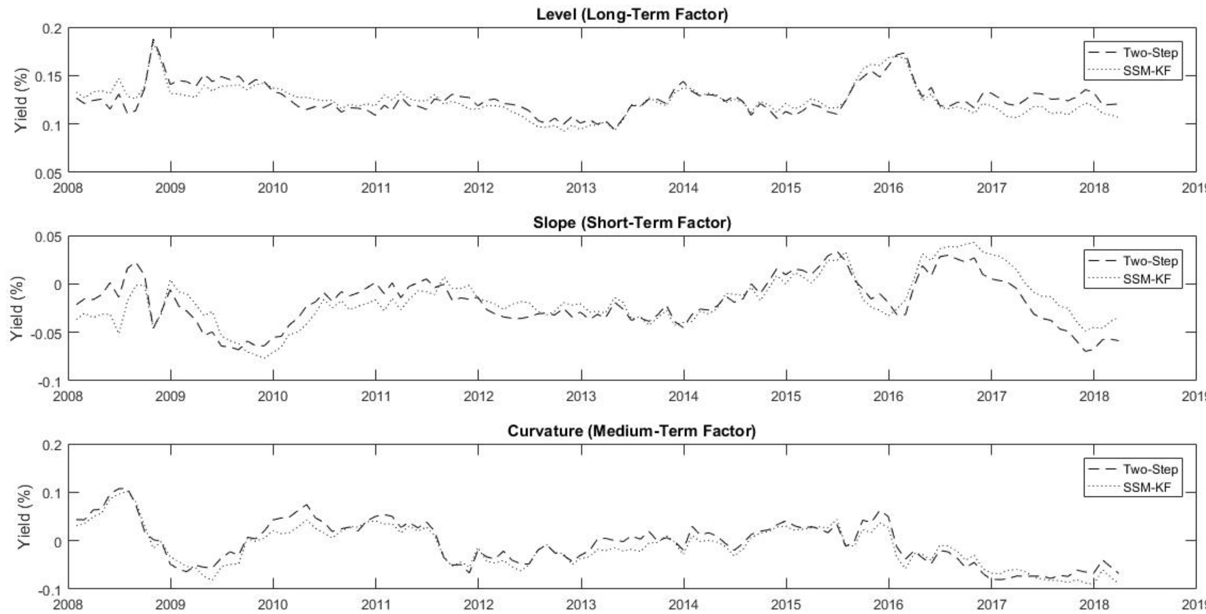



Figure 3 shows the yield curve performance, divided by the factors (level, slope and curvature), comparing the Two-Step and Kalman Filter paths. Note that the model estimated with three factors fits well to a wide variety of shapes of the yield curve: positively sloped, negatively sloped and with different curvature shapes. Note that the level of the yield curve exhibited a more volatile behavior after November 2008, when the financial crisis had a stronger impact on the assets traded in the Brazilian market. Another volatile behavior is observed during 2016-2017, when a instability crisis caused a significant impact on Brazilian assets market. The 2016-2017 volatile can be observed on slope and curvature, even blander. In general, both methods have similar trajectories. However, we can notice that in periods with curve changes,the Kalman filter has a smoother estimate, or in other words, a more efficient estimation.

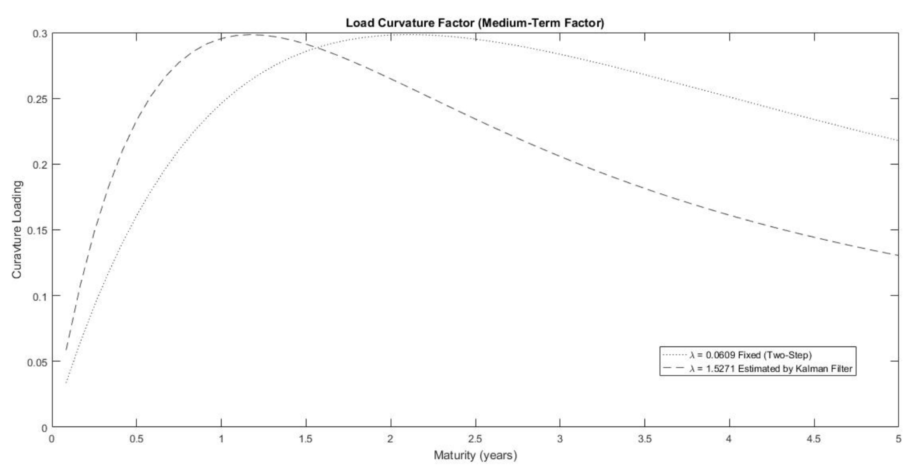
An advantage of parametric models, besides being more parsimonious, lies in the economic interpretation of the parameters, as is the case of Nelson & Siegel (1987) model interpreted by Diebold & Li (2006) in which the three factors are interpreted as level, slope and curvature of the interest curve. This economic intuition of the parameters allows a more effective immunization of fixed income portfolios, for example, being possible to neutralize the portfolio not only to parallel movements of the yield curve, but also to changes in the inclination and curvature. In this regard, see Caldeira (2011).

Figure 3: Factors Estimates: Two-Step and Kalman Filter Methods




In Figure 4 we conclude that the $\lambda$ estimated by Kalman Filter appears with a smoother trajectory, showing more adherence in the estimation of the yield curve and proving that models in state-space form can estimate on a most efficiently way than classic methods like two steps.

Figure 4: **Load Curvature Fator: $\lambda$ Two-Step and $\lambda$ Kalman Filter**



The curvature loading factor is greater in the high regime than in the low regime for the one through 12-months maturities and therefore influences the yields of those maturities more than in the low regime. For longer maturities in the high regime, the curvature loading factor decays quickly and is less of a factor in yield determination than in the low regime.

Figure 5 presents the real yield curve (observed) and the adjusted yield curve through the two different analyzed methods: Diebold & Li (2006) model estimated in two-step, and in a single step estimation using the Kalman filter. The quality of the adjustment of each methodology is analyzed in different situations, considering periods with positive inclination, inverted curve and negative inclination. This allows us to verify if some estimation method presents a better performance for certain yield curve shapes.

In June.2008, the rates for low maturities were loading elevation expectations, since the intermediate maturities the yield curve presented downward inclination, assuming a well defined curve for maturities on 20-months ahead. Even the two-step, as the Kalman filter, showed a similar adjustment.

The curve of January.2009, greatly differs from the format shown in June.2008, but in the last case, the interest rates for the shorter maturities are in higher and the curve does not show a sharp inversion of inclination. The models introduce a better adjustment for maturities between 20 and 50 months.

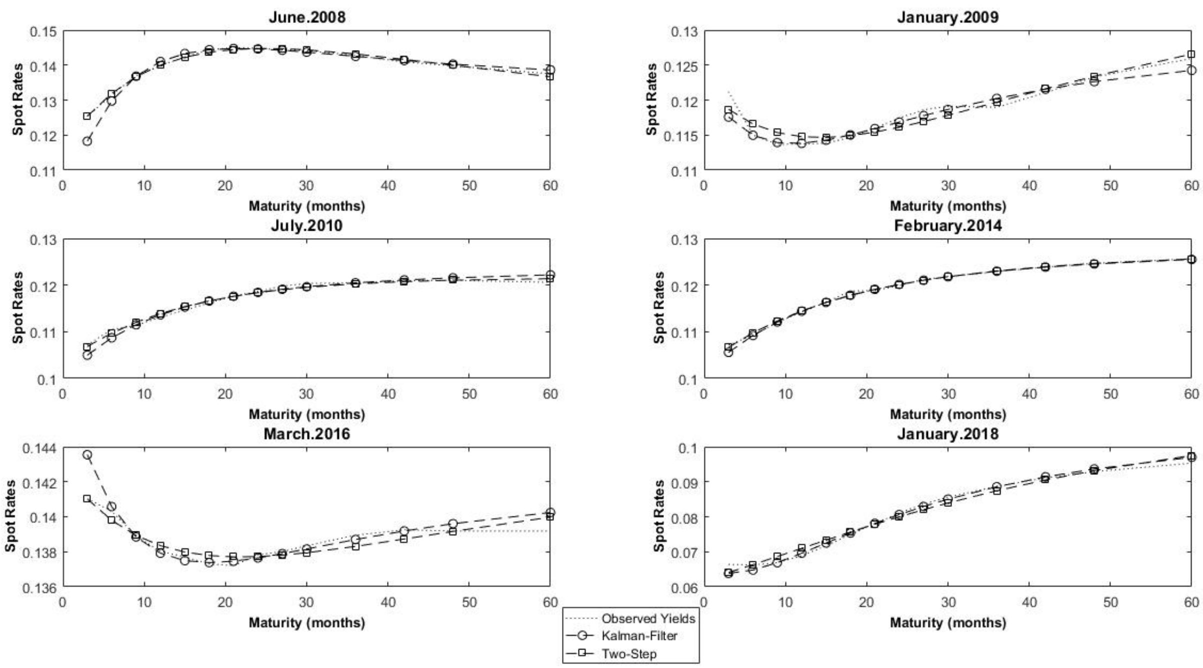
The next shows the real and adjusted interest curve for July.2010, period in which interest rates for the shorter maturities were lower than those previously analyzed, reflecting the monetary policy on elevation of interest rates implemented by the monetary authority in the period. Note that the curve clearly shows a positive slope, reflecting expectation regarding the possible start on high cycle of the basic interest rate. It can be observed that in this case, in which the curve presented positive inclination from the shortest maturities without inversions, every model shows a good performance in relation to the adjustment of the interest curve in all maturities.

The same occurs for the curve in February.2014, on a period when the interest rates for the shorter maturities were lower, reflecting the monetary policy of elevation. The curve displays a positive slope, reflecting expectation of a rise in interest rates. Over again, both models present a good adjustment for all maturities.

The curve of March.2016 differs greatly from the February.2014 and July.2010 curves, where the lowest maturities show elevate interest rates, with an expectation of lowering. Both models introduce a better adjustment for maturities between 10 an 30 months; without maintaining accuracy in periods where there is a greater trend of high or low interest rates.

The last curve refers to January.2018, period with lower interest rates for shorter maturities, reflecting the monetary policy on elevations of interest rates in the face of the troubled economic scenario. Note that the curve shows a positive slope (even more pronounced than the curves of the dates February.2014 and July.2010), reflecting the market expectation of a high cycle.

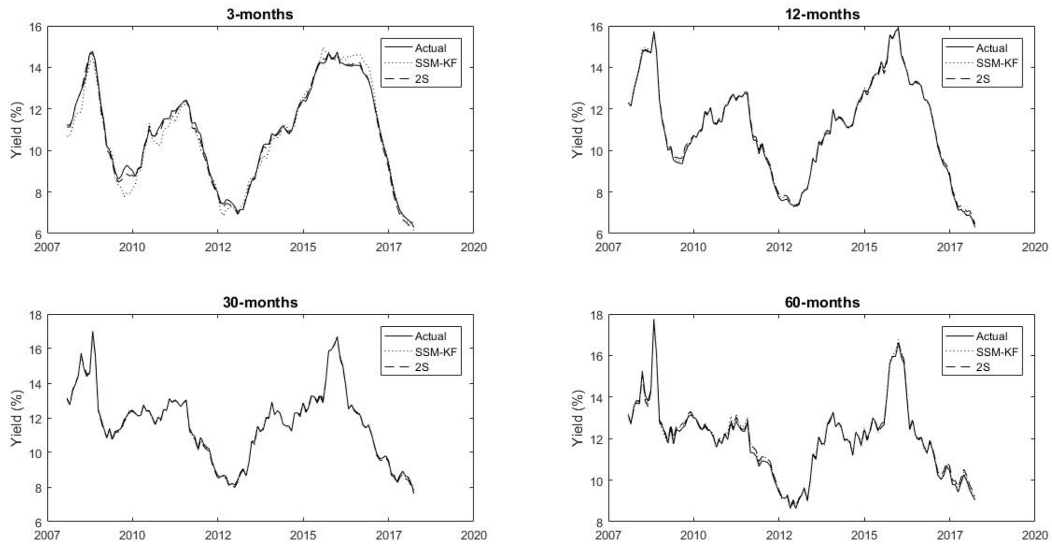
Figure 5: Observed and Adjusted Yields [Specific Dates Behavior]




Finally, Figure 6 brings the adjusted and interpolated yield curve for four different maturities. The fit of the models to the real data reinforces previous observations that, in the majority of cases, both methods used present good performance to adjust the yield curve; and in some cases the Kalman Filter exhibits superior performance. Even at times when the curve does not show strong inversions of curvature and inclination, the models also present a satisfactory adjustment.

In Table 2 we present statistics for both the two-step and Kalman filter, and calculate the mean, standard deviation and RMSE (Root Mean Squared Error). We perceive that the two models analyzed capture a substantial part of the dynamics of the interest curve. The Kalman filter RMSE performs better for almost all maturities, except for the tips (3, 6 and 48 months), proving its accuracy when compared with two-step. The worst performance is observed for the shortest maturity which is more susceptible to Selic rate fluctuations. In Caldeira *et al.* (2010) the model estimated by Kalman filter outperforms the model estimated in two-steps in all maturities; showing a adherence in conclusions of the studies.

Figure 6: Observed and Adjusted Yields [Specific Maturities Behavior]




We can verify that the Kalman filter presents smaller variation in its means for all maturities along, when compared with two steps method; being one more fact in favor of the estimated method in the form of state space. About the standard deviation, we arrive at the same conclusion, demonstrating that the Kalman filter generally performs better.

Table 2: Comparative Statistics: Two-Step x Kalman Filter

| **Descriptive Statistics** | | | | | | |
|---|---|---|---|---|---|---|
| | **Kalman Filter** | | | **Two-Step** | | |
| **Maturity** | **Mean** | **Std Dev** | **RMSE** | **Mean** | **Std Dev** | **RMSE** |
| **3** | 0.1169 | 0.3978 | 0.4131 | 0.0618 | 0.1387 | 0.1514 |
| **6** | 0.0524 | 0.1367 | 0.1459 | -0.0065 | 0.0877 | 0.0876 |
| **9** | 0.0012 | 0.0225 | 0.0224 | -0.0478 | 0.1046 | 0.1146 |
| **12** | -0.0199 | 0.0577 | 0.0600 | -0.0536 | 0.0985 | 0.1180 |
| **15** | -0.0196 | 0.0577 | 0.0608 | -0.0372 | 0.0764 | 0.0847 |
| **18** | -0.0106 | 0.0381 | 0.0394 | -0.0139 | 0.0539 | 0.0555 |
| **21** | -0.0008 | 0.0234 | 0.0232 | 0.0077 | 0.0443 | 0.0448 |
| **24** | 0.0103 | 0.0326 | 0.0341 | 0.0274 | 0.0588 | 0.0646 |
| **27** | 0.0170 | 0.0383 | 0.0417 | 0.0398 | 0.0703 | 0.0800 |
| **30** | 0.0105 | 0.0429 | 0.0440 | 0.0366 | 0.0784 | 0.0862 |
| **36** | -0.0013 | 0.0368 | 0.0366 | 0.0250 | 0.0737 | 0.0776 |
| **42** | -0.0017 | 0.0336 | 0.0335 | 0.0193 | 0.0632 | 0.0659 |
| **48** | -0.0109 | 0.0480 | 0.0491 | 0.0016 | 0.0408 | 0.0407 |
| **60** | -0.0530 | 0.1150 | 0.1262 | -0.0604 | 0.1163 | 0.1307 |

## 5. Concluding Remarks

This paper presents the main concepts related to structure of the yield curve and of bonds with and without coupon, with zero-coupon bonds, which are employed in the construction of the yield curve. It was also presented the interrelations between the various yield curves estimates.

In the present research, was estimated Diebold & Li (2006) model for Brazilian data by the inefficient two-step method and into a state-space form estimated by maximum likelihood using Kalman filter. The maximum likelihood estimation allows for the joint estimation of all parameters of the model, preventing the a priori selection of the decay parameter.

The results indicate that the model estimated by maximum likelihood performs better than the model estimated by two-step method for almost all forecasting horizons. This results show the flexibility of the model to adjust itself to a wide variety of yield curve shapes, and that the estimation by the Kalman filter is better than its counterparts estimated by the two-step method.

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