Using Power Lindley for Generating Family of Distributions

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ABSTRACT

In this paper, a new flexible lifetime class of probability distributions is introduced based on the power Lindley (PL) distribution and the T-X family of distributions called the odd power Lindley (OPL-G) distribution. The proposed class can generate as many continuous lifetimes distributions, which can be used for modeling lifetime data in many fields. Some special models of the proposed class are discussed. Several properties of the proposed class are studied, such as density, survival function, hazard rate function, limiting behavior, quantile function, moments, and distribution of order statistics. The method of maximum likelihood estimation will be used to estimate the parameters of this new class of distributions. Asymptotic properties of the MLEs and a simulation are introduced to verify the performance of the parameter estimates. We finally choose a model of the proposed class and fit it to three real data sets in order to demonstrate the flexibility and potential of the proposed class.

- **Keywords:** Odd power Lindley-G distributions, power Lindley distribution, lifetime distribution, maximum likelihood estimation

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1. Introduction

The failure behavior of any system varies from one system to another due to the nature of the system. and hence, it can be considered as a random variable. Therefore, it is logical to try to find an appropriate statistical model for system failure. In other applications, we may be interested in the survival of a system, which characterized by its hazard rate, e.g., the number of shops runout of business. We point out that "survival analysis" is the scientific name for both failure rate and hazard rate. It should also be noted that failure rate and hazard rate have the same mathematical function.

Modeling survival data depends on the behavior of the hazard (failure), which can be monotone (non-increasing and non-decreasing) or non-monotone (bathtub and upside-down bathtub, or unimodal). Many statistical models have been proposed for modeling of survival data based on the behavior of the failure or the hazard of the system. These models include exponential, gamma, Weibull, and log normal distributions. A one-parameter distribution was introduced by Lindley [15] as an alternative model for data with a non-monotone hazard rate shape and became the well-known Lindley distribution. Ghitany et al. [11] studied the properties and applications of the Lindley distribution and found that this distribution may perform better in modeling than in exponential distribution. The power Lindley (PL) distribution with its inference was proposed by Ghitany et al. [10] and extended by Alkarni [3]. We should point out that the PL distribution often shows potential for fitting data and compete Weibull distribution.

The benefit of introducing a family of distributions is to provide high flexibility in modeling lifetime data. Many generators of distributions have been proposed recently and listed by Gomes-Silva et al. [12], Tahir et al. [21] and Abouelmagd et al. [1].

In this paper, we focus on establishing a new family of distributions using the PL distribution as he generator distribution with the idea of the T-X family of distribution as defined by Alzaatreh et al. [4]. The PL distribution was chosen owing to its flexibility and simplicity since it has two parameters and models survival data, compared with the Weibull distribution; see Alkarni [2]. The proposed class generalized the one introduced by Gomes-Silva et al. [12] into a more powerful and flexible class of distributions.

The remainder of this paper is organized as follows. In Section 2, we introduce the odd power Lindley-G (OPL-G) family of distributions. In section 3, we discuss the general properties of the OPL-G distribution, such as the probability density function (PDF) and its behavior, hazard rate function, reliability function, moments, moments generating function, quantile, and distribution of order statistics. The estimation of the OPL-G parameters is investigated in Section 4 using the method of maximum likelihood estimation (MLE) and a large sample inference with EM algorithm of solving the nonlinear equations in the MLE. In section 5, some special sub classes and models of the OPL-G family are introduced. In Section 6, a simulation is introduced in order to check the reliability of using MLE estimates in the inference. In Section 7, a chosen model of the family is fitted to three real datasets and compared to some existing models in order to illustrate the applicability and flexibility of the OPL-G distribution. Finally, some concluding remarks are addressed in Section 8.

2. The Odd Power Lindley-G Family of Distributions

The T-X family of distribution, as it was defined by Alzaatreh et al. [4] by its cumulative distribution function, is given by

$$F(x) = \int_{a}^{W[G(x)]} r(t)dt, \qquad (2.1)$$

where $r(t)$ is the PDF of a lifetime random variable and $W[G(x)]$ is a cumulative distribution function (CDF) of a random variable X. The PDF corresponding to (2.1) is given by

$$f(x)=\frac{d}{dx}W[G(x)].r(W[G(x)]). \qquad (2.2)$$

The OPL-G is then defined by letting $W[G(x)]=\frac{G(x;\boldsymbol{\theta})}{1-G(x;\boldsymbol{\theta})}=H(x;\boldsymbol{\theta})$, and the generator $r(t)$ is taken to be $r(t)=\frac{\alpha\beta^2}{\beta+1}(1+t^{\alpha})t^{\alpha-1}e^{-\beta t^{\alpha}}, \alpha,\beta,t>0$, where $G(x;\boldsymbol{\theta})$ is a baseline CDF that depends on a parameter vector $\boldsymbol{\theta}=(\theta_1,...,\theta_q)$. Using (2.1), the OPL-G CDF is given by

$$F(x;\beta,\alpha,\boldsymbol{\theta})=1-\left[1+\frac{\beta}{\beta+1}\left[H(x;\boldsymbol{\theta})\right]^{\alpha}\right]e^{-\beta[H(x;\boldsymbol{\theta})]^{\alpha}};\beta,\alpha,\boldsymbol{\theta},x>0. \qquad (2.3)$$

For each choice of G, and hence, $H(x;\boldsymbol{\theta})$, we have a new continuous life time distribution. Note that for $\alpha=1$, the OPL-G is reduced to the odd Lindley-G (OL-G) family of distributions proposed by Gomes-Silva et al. [12].

An interpretation of the OPL-G family of distributions can be given as follows. Let Y be a lifetime random variable having a certain continuous G distribution. The odds ratio that an individual (or component) following the lifetime Y will die (failure) at time x is $H(x;\boldsymbol{\theta})$. Consider that the variability of this odds of death is represented by the random variable X and assume that it follows the PL distribution with scale parameter β and shape parameter α. Then,

$P(Y\le x)=P\left(X\le H(x;\boldsymbol{\theta})\right)=F(x;\beta,\alpha,\boldsymbol{\theta})$,

Which is given by (2.3). Table 1 lists $H(x;\boldsymbol{\theta})$ for some well-known lifetime distributions and their corresponding parameter vector $\boldsymbol{\theta}$.

Table 1: Useful $H(x;\boldsymbol{\theta})$ for some continuous lifetime distributions

Distribution	$H(x;\boldsymbol{\theta})$	$\boldsymbol{\theta}$
Uniform	$x/(\theta-x)$	θ
Exponential	$e^{\lambda x}-1$	λ
Weibull	$e^{(\lambda x)^{\gamma}}-1$	(λ,γ)
Frechet	$(e^{(\lambda x)^{\gamma}}-1)^{-1}$	(λ,γ)
Half-logistic	$(e^{x}-1)/2$	-
Power function	$[(\theta k)^{-k}-1]^{-1}$	(θ,k)
Pareto	$(x/\theta)^{k}-1$	(θ,k)
Burr XII	$[1+(x/s)^{c}]^{k}-1$	(s,k,c)
Log-logistic	$[1+(x/s)^{c}]-1$	(s,c)
Lomax	$[1+(x/s)]^{k}-1$	(s,k)
Gumbel	$\{\exp[\exp(-(x-m)/\sigma)]-1\}^{-1}$	(μ,σ)
Kumaraswamy	$(1-x^{\alpha})^{-\beta}-1$	(α,β)

Normal	$\Phi((x-\mu)/\sigma)/(1-\Phi((x-\mu)/\sigma))$	(μ,σ)

3. General properties

3.1 Probability density function

The general form of the PDF corresponding to (2.3) is given by

$$f(x;\beta,\alpha,\boldsymbol{\theta})=\frac{\alpha\beta^2}{\beta+1}\left[1+\left[H(x;\boldsymbol{\theta})\right]^{\alpha}\right]\left(H(x;\boldsymbol{\theta})\right)^{\alpha-1}h(x;\boldsymbol{\theta})e^{-\beta\left[H(x;\boldsymbol{\theta})\right]^{\alpha}};\beta,\alpha,\boldsymbol{\theta},x>0.$$

(2.4)

where $h(x;\boldsymbol{\theta})=\frac{d}{dx}H(x;\boldsymbol{\theta})=\frac{g(x;\boldsymbol{\theta})}{(1-G(x;\boldsymbol{\theta}))^2}$. Hereafter, we refer to a random variable X with PDF and CDF in (2.3) and (2.4), respectively, as $X\sim$ OPL-G $(\beta,\alpha,\boldsymbol{\theta})$.

The shape of the PDF depends on the function $H(x;\boldsymbol{\theta})$ and can be described analytically.

3.2 Hazard rate and reliability functions

The probability that any cause survives for some time x is called the reliability (survival) of the cause of delay and is given by

$$R(x)=P(X>x)=1-F(x).$$

The hazard rate function (HRF) also known as the failure rate function is generally defined as

$$\tau(x)=\frac{f(x)}{1-F(x)}$$

By using (2.3) and (2.4), the HRF is given by

$$\tau(x)=\frac{\alpha\beta^2\left[1+\left[H(x;\boldsymbol{\theta})\right]^{\alpha}\right]\left(H(x;\boldsymbol{\theta})\right)^{\alpha-1}h(x)}{\beta+1+\beta\left[H(x;\boldsymbol{\theta})\right]^{\alpha}};\beta,\alpha,\boldsymbol{\theta},x>0. \qquad (2.5)$$

The shape of the HRF depends on the function $H(x;\boldsymbol{\theta})$ and can be described analytically.

3.3 Quantile function and order statistics

The quantile function is useful for generating data from the OPL-G distribution for simulation. Let X be a random variable with CDF as shown in (2.3). The quantile function, i.e., $Q_X(p),$ is the root of the equation $F_X(Q_X(\mathrm{p}))=\mathrm{p}, \mathrm{p}\in(0,1)$.

Substituting this equation in (2.3), we have

$$\beta+1+\beta[H(Q_X(p))]^{\alpha}e^{-\beta[H(Q_X(p))]^{\alpha}}=(\beta+1)(1-p)$$

Multiplying both sides by $-e^{-(\beta+1)}$, we have the Lambert equation,

$$-\beta-1-\beta[H(Q_X(p))]^{\alpha}e^{-\beta-1-\beta[H(Q_X(p))]^{\alpha}}=-(\beta+1)(1-p)e^{-(\beta+1)}.$$

Hence, we have the negative Lambert function W of the real argument $-(\beta+1)(1-p)e^{-(\beta+1)}$, i.e.,

$$W_{-1}\left[(\beta+1)(p-1)e^{-(\beta+1)}\right]=-\beta-1-\beta[H(Q_X(p))]^{\alpha}.$$

Solving the above equation for $Q_X(p),$ we have the quantile function for PL-G distribution given by

$$Q_X(p)=H^{-1}\left\{\left[-1-\frac{1}{\beta}-\frac{1}{\beta}W_{-1}\left((\beta+1)(p-1)e^{-(\beta+1)}\right)\right]^{\frac{1}{\alpha}}\right\}. \qquad (3.3.1)$$

Order statistics are among the most fundamental tools in non-parametric statistics and inference. These can be used to tackle estimation problems and hypothesis tests in many ways. The PDF of the k^{th} order statistics from a random sample $X_1,\ldots,X_n$ from the PL-G is given by

$$f_{k:n}(x)=\frac{n!}{(k-1)!(n-k)!}f_X(x)[F_X(x)]^{k-1}[1-F_X(x)]^{n-k},$$

$$=\frac{n!}{(k-1)!(n-k)!}f_X(x)\sum_{i=0}^{n-k}\binom{n-k}{i}(-1)^i[F_X(x)]^{k+i-1}. \qquad (3.3.2)$$

The associate CDF can be obtained as

$$F_{k:n}(x)=\frac{n!}{(k-1)!(k-i)!}\sum_{i=0}^{n-k}\frac{\binom{n-k}{i}(-1)^i}{k+i}[F_X(x)]^{k+i}. \qquad (3.3.3)$$

3.4 Moments and generating function

Many of the necessary characteristics of a distribution can be obtained from ordinary moments. The OPL-G moments can be obtained numerically in any of the modern statistical packages, such as R. These packages contain the required mathematical functions for computing moments.

Alternatively, some structural properties of the new family, such as the ordinary and incomplete moments and generating function, can be determined from well-established properties of the Exp-G distributions. The properties of Exp-G distributions have been studied by many authors in recent years, see Mudholkar and Srivastava [16] and Mudholkar et al. [17] for exponentiated Weibull, Gupta et al. [13] for exponentiated Pareto, Gupta and Kundu [14] for exponentiated exponential, Nadarajah [18] for exponentiated Gumbel, Shirke and Kakade [20] for exponentiated log-normal and Nadarajah, and Gupta [19] for exponentiated gamma distributions.

4. Estimation and inference

Let $x_{1,}x_2,...,x_n$ be a random sample with size n obtained from the OPL-G distribution with parameters β, α and $\boldsymbol{\theta}$. Let $\Theta=(\beta,\alpha,\boldsymbol{\theta})^T$ be the $p\times 1$ unknown parameter vector. The log likelihood function of the PL-G distribution is given by

$$l_n = l_n(\Theta,x) = n\log\alpha + 2n\log\beta - n\log(\beta+1) + \sum_{i=1}^{n}\log\left[1+\left[H(x_i;\boldsymbol{\theta})\right]^{\alpha}\right] + (\alpha-1)\sum_{i=1}^{n}\log H(x_i;\boldsymbol{\theta})$$
$$+\sum_{i=1}^{n}\log h(x_i;\boldsymbol{\theta}) - \beta\sum_{i=1}^{n}\left[H(x_i;\boldsymbol{\theta})\right]^{\alpha}.$$

The associate score function is $U_n(\Theta)=(\partial l_n/\partial\beta, \partial l_n/\partial\alpha, \partial l_n/\partial\boldsymbol{\theta},)^T$ where the elements of $U_n(\Theta)$ are given by

$$\frac{\partial l_n}{\partial\beta} = \frac{2n}{\beta} - \frac{n}{\beta+1} - \sum_{i=1}^{n}\left[H(x_i;\boldsymbol{\theta})\right]^{\alpha},$$

$$\frac{\partial l_n}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\frac{\left[H(x_i;\boldsymbol{\theta})\right]^{\alpha}\log[H(x_i;\boldsymbol{\theta})]}{1+\left[H(x_i;\boldsymbol{\theta})\right]^{\alpha}} + \sum_{i=1}^{n}\log[H(x_i;\boldsymbol{\theta})],$$
$$-\beta\sum_{i=1}^{n}\left[H(x_i;\boldsymbol{\theta})\right]^{\alpha}\log[H(x_i;\boldsymbol{\theta})],$$

$$\frac{\partial l_n}{\partial\theta_i} = \sum_{i=1}^{n}\frac{\left[H(x_i;\boldsymbol{\theta})\right]^{\alpha-1}}{1+\left[H(x_i;\boldsymbol{\theta})\right]^{\alpha}}\frac{\partial H(x_i;\boldsymbol{\theta})}{\partial\theta_i} + (\alpha-1)\sum_{i=1}^{n}\frac{1}{H(x_i;\boldsymbol{\theta})}\frac{\partial H(x_i;\boldsymbol{\theta})}{\partial\theta_i}$$
$$+\sum_{i=1}^{n}\frac{1}{h(x_i;\boldsymbol{\theta})}\frac{\partial h(x_i;\boldsymbol{\theta})}{\partial\theta_i} - \alpha\beta\sum_{i=1}^{n}\left[H(x_i;\boldsymbol{\theta})\right]^{\alpha}\frac{\partial H(x_i;\boldsymbol{\theta})}{\partial\theta_i},$$
$$i=1,\ldots,q,$$

respectively.

The maximum likelihood estimates of Θ can be obtained as a solution of $U_n(\Theta)=0$ by any numerical method, such as Newton–Raphson in R. Fisher information matrix is a $p\times p$ matrix consisting of the second partial derivatives of $U_n(\Theta)$ and is given by

$$I_n(\Theta)=\begin{bmatrix} I_{\beta\beta} & I_{\beta\alpha} & I_{\beta\theta} \\ I_{\alpha\beta} & I_{\alpha\alpha} & I_{\alpha\theta} \\ I_{\theta\beta} & I_{\theta\alpha} & I_{\theta\theta} \end{bmatrix},$$

where the elements of $I_n(\Theta)$ are the second partial derivatives of $U_n(\Theta)$. These elements can be obtained from R or MATLAB to obtain a confidence interval for the estimates. Under the standard regular conditions for the large sample approximation in Cox and Hinkley [9], which was fulfilled for the proposed model, the distribution of Θ is approximately $N_p(\Theta, J_n(\Theta)^{-1})$, where $J_n(\Theta)=\mathrm{E}[\mathrm{I}_n(\Theta)]$. Whenever the parameters are in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\Theta-\Theta)$ is $N_p(0, J(\Theta)^{-1})$, where $J(\Theta)^{-1}=\lim_{n\to\infty} n^{-1} I_n(\Theta)$ is the unit information matrix and p is the number of parameters of the distribution. The asymptotic multivariate normal $N_p(\Theta, I_n(\Theta)^{-1})$ distribution of Θ can be used to approximate the confidence interval for the parameters, the hazard rate, and the survival functions. An $100(1-\gamma)$ asymptotic confidence interval for parameter Θ_i is given by

$$(\Theta_i - Z_{\frac{\gamma}{2}}\sqrt{I^{ii}}, \Theta_i + Z_{\frac{\gamma}{2}}\sqrt{I^{ii}}),$$

where I^{ii} is the (i,i) diagonal element of $I_n(\Theta)^{-1}$ for $i=1,\ldots,p$, and $Z_{\frac{\gamma}{2}}$ is the quantile $1-\frac{\gamma}{2}$ of the standard normal distribution.

5. Special subclasses

In this section we introduce some subclasses of the OPL-G class of distributions by choosing several choices of $H(x;\theta)$.

5.1 Odd power Lindley exponential distribution

Taking $H(x)=e^{\lambda x}-1; \lambda>0,$ then by (2.3) and (2.4), we obtain the CDF and PDF of the odd power Lindley exponential (OPLE) distribution as

$$F(x;\beta,\alpha,\lambda)=1-\left[1+\frac{\beta}{\beta+1}(e^{\lambda x}-1)^{\alpha}\right]e^{-\beta(e^{\lambda x}-1)^{\alpha}};\beta,\alpha,\lambda,x>0, \text{ and}$$

$$f(x;\beta,\alpha,\lambda)=\frac{\alpha\lambda\beta^2}{\beta+1}\left[1+(e^{\lambda x}-1)^{\alpha}\right](e^{\lambda x}-1)^{\alpha-1}e^{\lambda x-\beta(e^{\lambda x}-1)^{\alpha}};\beta,\alpha,\lambda,x>0,$$

respectively.

The OPLE HRF is obtained by direct substitution in (2.5) as follows:

$$\tau(x)=\frac{\alpha\lambda\beta^{2}\left[1+(e^{\lambda x}-1)^{\alpha}\right](e^{\lambda x}-1)^{\alpha-1}e^{\lambda x}}{\beta+1+\beta(e^{\lambda x}-1)^{\alpha}};\beta,\alpha,\lambda,x>0.$$

Plots of the PDF and HRF of the OPLE distribution for some selected parameter values are shown in Figure 1.

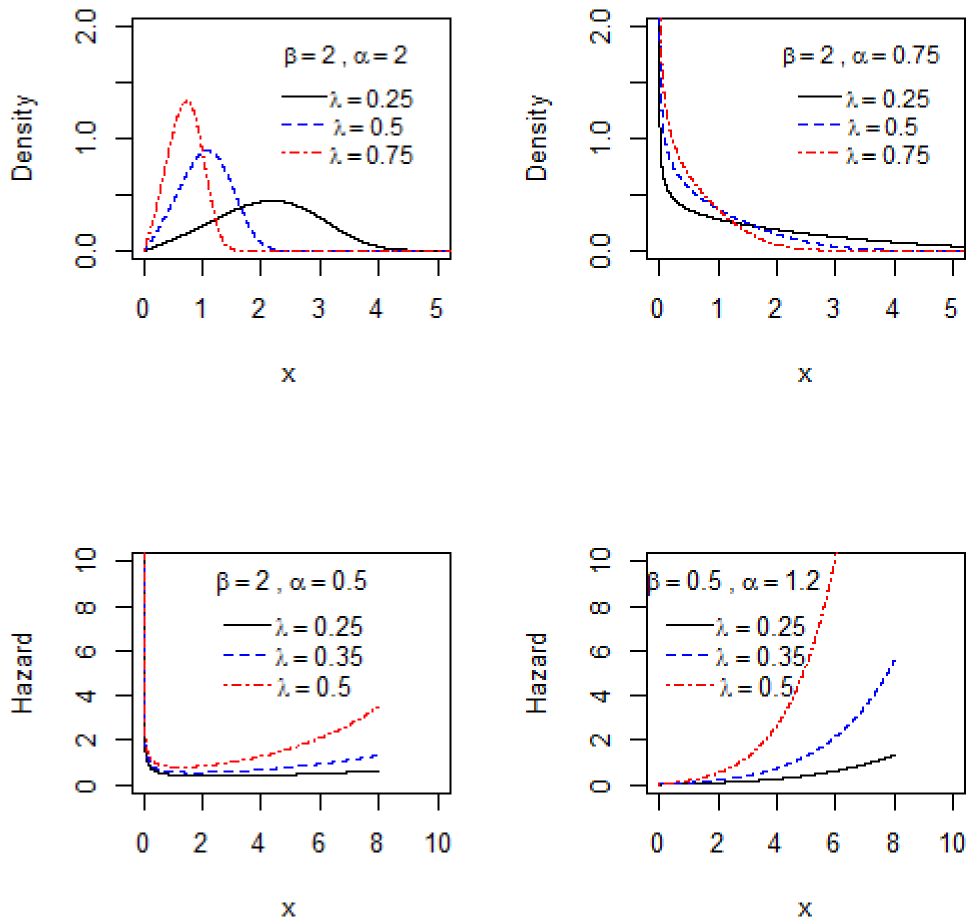

Figure 1: Plots of the density function of the OPLE distribution for different values of λ, β and α.

The OPLE distribution contains several special sub models. When $\alpha=1$, we have the odd Lindley exponential (OLE) distribution. When $\lambda=\log(x+1)/x$, we have the power Lindley distribution introduced by Ghitany et al. [10]. For $\alpha=1$ and $\lambda=\log(x+1)/x$, we have the Lindley distribution [15]. As can be seen from Figure 1, the PDF monotonically increases with modal of ∞ as $x\to 0$ whenever $\alpha<1$. The shape of the

PDF is upside down bathtub or is increasing and deceasing for $\alpha \geq 1$. We also note that $f(x)=0$ as $x \to 0$ whenever $\alpha \geq 1$.

5.2 Odd power Lindley Pareto distribution

Taking $H(x)=(x/\lambda)^k-1; \lambda, k>0,$ then by (2.3) and (2.4), we obtain the CDF and PDF of the OPLP distribution as

$$F(x;\beta,\alpha,\lambda,k)=1-\left[1+\frac{\beta}{\beta+1}((x/\lambda)^k-1)^\alpha\right]e^{-\beta((x/\lambda)^k-1)^\alpha}; \beta,\alpha,\lambda,k,x>0,$$

$$f(x;\beta,\alpha,\lambda,k)=\frac{\alpha\beta^2}{\lambda^{k-1}(\beta+1)}\left[1+((x/\lambda)^k-1)^\alpha\right]((x/\lambda)^k-1)^{\alpha-1}x^{k-1}e^{-\beta((x/\lambda)^k-1)^\alpha};$$
$$\beta,\alpha,\lambda,k,x>0.$$

respectively.

The HRF of the OPLP distribution is obtained by direct substitution in (2.5) as follows

$$\tau(x)=\frac{\alpha k\beta^2}{\lambda^k(\beta+1)}\frac{\left[1+((x/\lambda)^k-1)^\alpha\right]((x/\lambda)^k-1)^{\alpha-1}x^{k-1}}{\beta+1+\beta((x/\lambda)^k-1)^\alpha}; \beta,\alpha,\lambda,k,x>0.$$

Plots of the PDF and HRF of the OPLP distribution for some selected parameters values are shown in Figure 2.

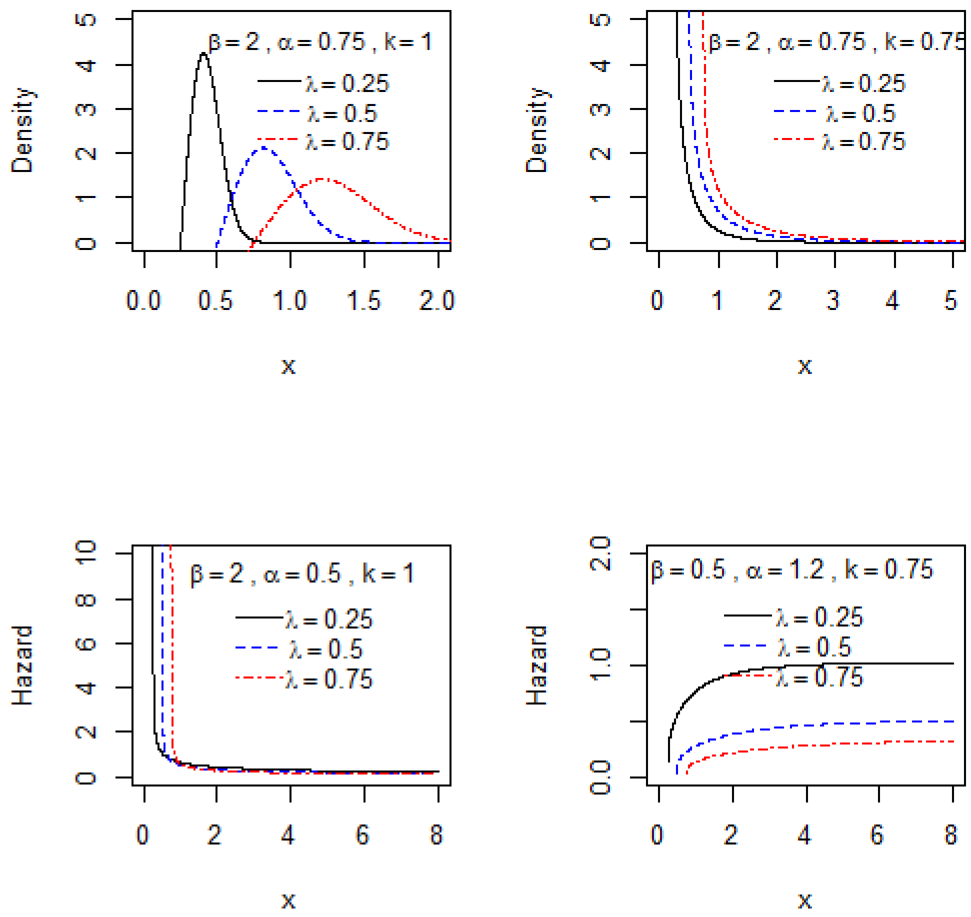

Figure 2: Plots of the density function of the OPLP distribution for different values of λ, β and k.

The OPLP distribution contains several special sub models. When $\alpha = 1$, we have the odd Lindley Pareto distribution proposed by Zeghdoudi and Lazri [22]. When $\lambda = x/(x+1)^{1/k}$, we have the power Lindley distribution introduced by Ghitany et al. [10]. For $\alpha = 1$ and $\lambda = x/(x+1)^{1/k}$, we have the Lindley distribution [15]. As can be seen from Figure 2, the PDF monotonically decreases with modal of ∞ as $x \to 0$ whenever $\alpha < 1$ and $k < 1$. The shape of the PDF is upside down bathtub for $\alpha \geq 1$ and $k \leq 1$. We also note that $f(x) = 0$ as $x \to \infty$.

5.3 Odd power Lindley half-logistic distribution

Taking $H(x) = (e^x - 1)/2$, then by (2.3) and (2.4), we obtain the CDF and PDF of the odd Power Lindley half-logistic (OPLHL) distribution as

$$F(x;\beta,\alpha)=1-\left[1+\frac{\beta}{\beta+1}((e^{x}-1)/2)^{\alpha}\right]e^{-\beta((e^{x}-1)/2)^{\alpha}};\beta,\alpha,x>0,$$

$$f(x;\beta,\alpha)=\frac{\alpha\beta^{2}}{2^{\alpha}(\beta+1)}\left[1+((e^{x}-1)/2)^{\alpha}\right](e^{x}-1)^{\alpha-1}e^{x-\beta((e^{x}-1)/2)^{\alpha}};\beta,\alpha,x>0.$$

The HRF of the OPLHL is obtained by direct substitution in (2.5) as follows

$$\tau(x)=\frac{\alpha\beta^{2}}{2^{\alpha}}\frac{\left[1+((e^{x}-1)/2)^{\alpha}\right](e^{x}-1)^{\alpha-1}e^{x}}{\beta+1+\beta((e^{x}-1)/2)^{\alpha}};\beta,\alpha,x>0.$$

Plots of the PDF and HRF of the OPLHL distribution for some selected parameters values are shown in Figure 3.

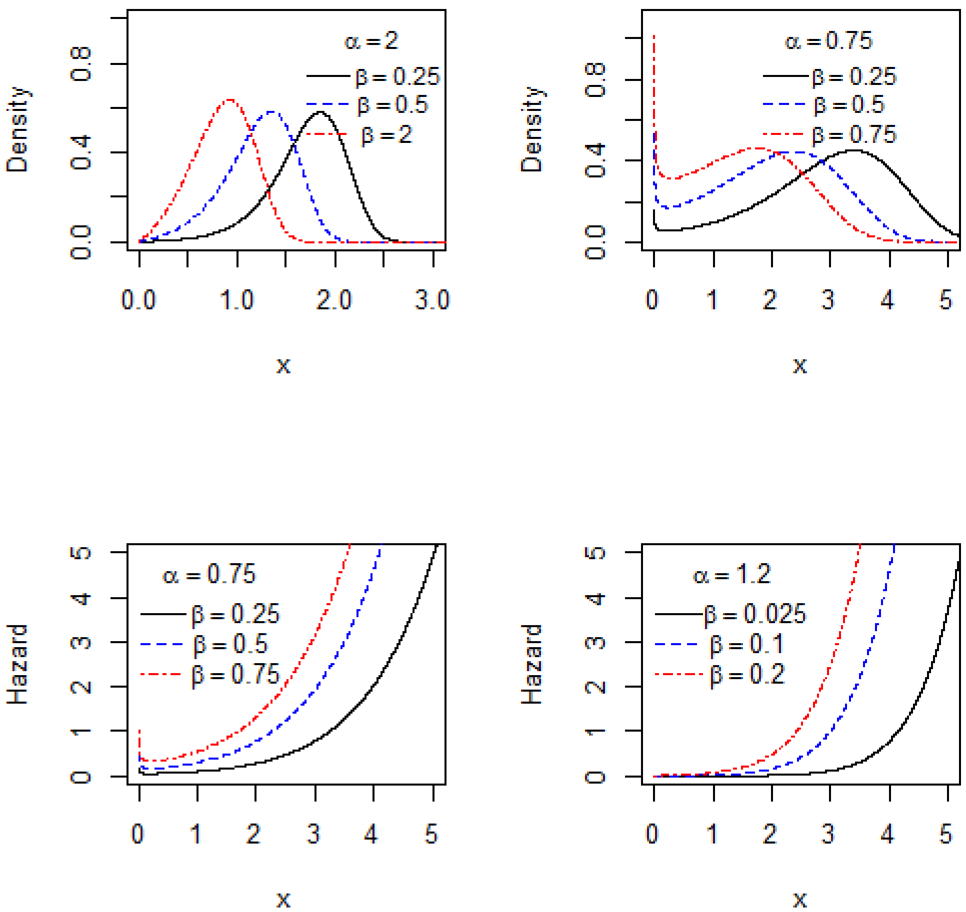

Figure 3: Plots of the density function of the OPLHL distribution for different values of

β and α.

The OPLHL distribution contains several special sub models. When $\alpha = 1,$ we have the Lindley half-logistic distribution. As can be seen in Figure 3, the monotonically increases and decreases with modal of ∞ as $x \to 0$ whenever $\alpha \geq 1$. The shape of the PDF is upside down bathtub for $\alpha < 1$. We also note that $f(x) = 0$ as $x \to \infty$.

5.4 Odd power Lindley Weibull distribution

Taking $H(x) = e^{(\lambda x)^{\gamma}} - 1; \lambda, \gamma > 0,$ then by (2.3) and (2.4), we obtain the CDF and PDF of the odd Power Lindley Weibull (OPLW) distribution as

$$F(x;\beta,\alpha,\lambda,\gamma) = 1 - \left[1 + \frac{\beta}{\beta+1}(e^{(\lambda x)^{\gamma}} - 1)^{\alpha}\right] e^{-\beta(e^{(\lambda x)^{\gamma}} - 1)^{\alpha}}; \beta,\alpha,\lambda,\gamma,x > 0,$$

$$f(x;\beta,\alpha,\lambda,\gamma) = \frac{\alpha\gamma\lambda^{\gamma}\beta^{2}}{\beta+1}\left[1 + (e^{(\lambda x)^{\gamma}} - 1)^{\alpha}\right]\left(e^{(\lambda x)^{\gamma}} - 1\right)^{\alpha-1} x^{\gamma-1} e^{(\lambda x)^{\gamma} - \beta(e^{(\lambda x)^{\gamma}} - 1)^{\alpha}}; \beta,\alpha,\lambda,\gamma,x > 0.$$

respectively.

Plots of the PDF of the OPLW distribution for some selected parameters values are shown in Figure 4.

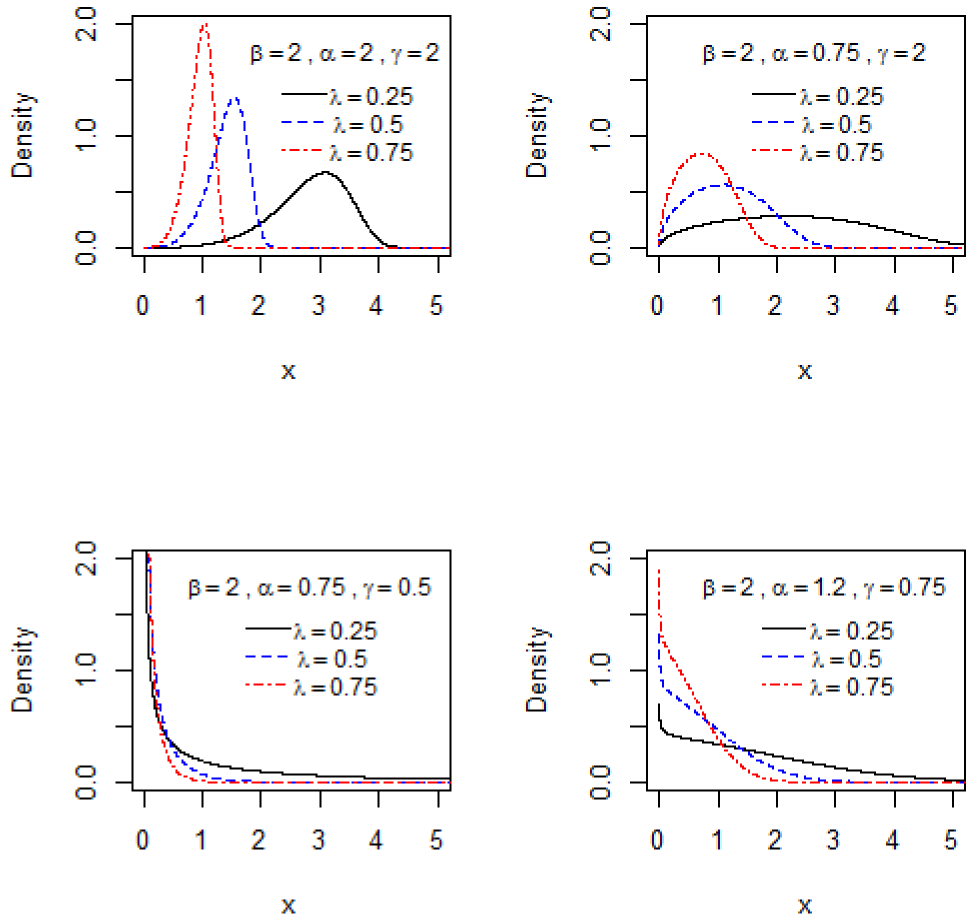

Figure 4: Plots of the density function of the OPLW distribution for different values of λ, β, α and γ.

The OPLW distribution contains several special sub models. When $\alpha = 1$, we have the odd Lindley Weibull distribution proposed by Gomes–Silva [12]. When $\gamma = 1$, we have the OPLE. As can be seen in Figure 4, the PDF monotonically decreases with modal of ∞ as $x \to 0$ whenever $\gamma < 1$. The shape of the PDF is upside down bathtub for $\gamma \geq 1$. We also note that $f(x) = 0$ as $x \to \infty$.

The HRF of the OPLW distribution is obtained by direct substitution in (2.5) as follows

$$\tau(x) = \frac{\alpha\gamma\lambda^{\gamma}\beta^{2}\left[1+(e^{(\lambda x)^{\gamma}}-1)^{\alpha}\right]\left(e^{(\lambda x)^{\gamma}}-1\right)^{\alpha-1}x^{\gamma-1}e^{(\lambda x)^{\gamma}}}{\beta+1+\beta(e^{(\lambda x)^{\gamma}}-1)^{\alpha}};\beta,\alpha,\lambda,\gamma,x>0.$$

Plots of the HRF of the OPLW distribution for some selected parameters values are shown in Figure 5.

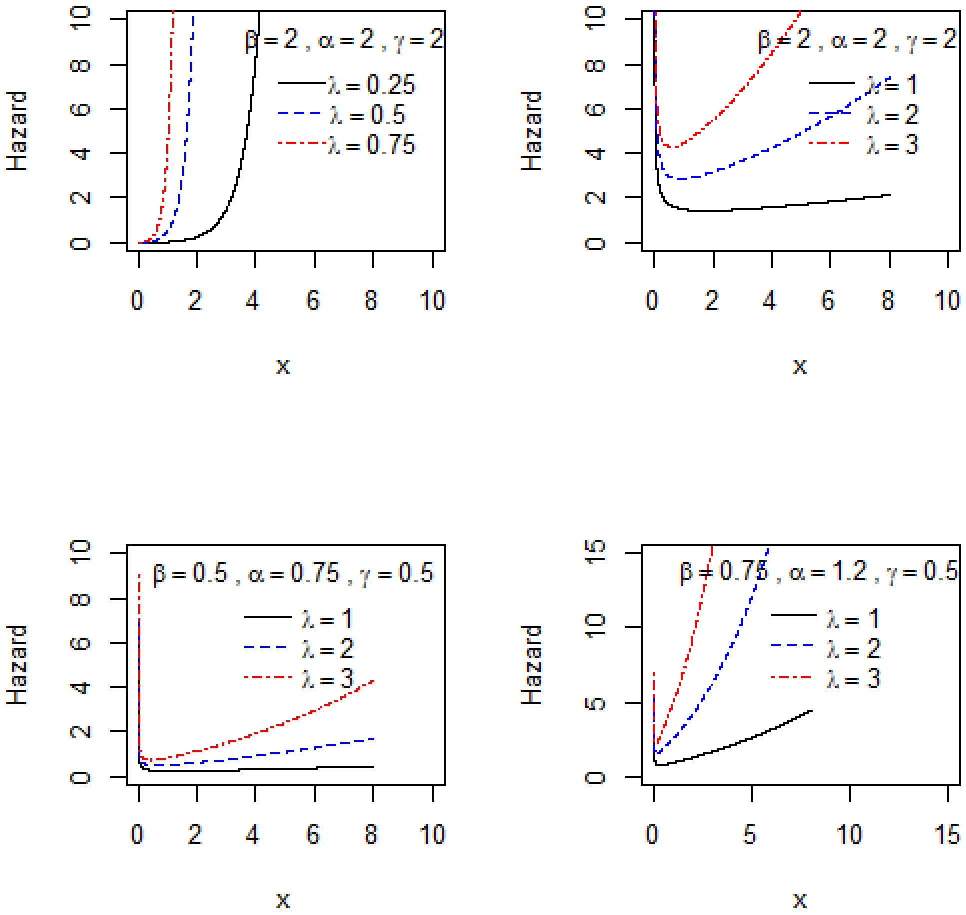

Figure 5: Plots of the hazard rate function of the OPLW distribution for different values of λ, β, α and γ.

As can be seen in Figure 5, the HRF monotonically increases with modal of ∞ as $x \to 0$ whenever $\lambda < 1$, and decreases and increases whenever $\lambda \geq 1$ with modal of ∞ as $x \to 0$.

6. Simulation study

In this section, the performances of the MLE's estimators are discussed using their average bias (AB), root mean squared error (RMSE), coverage probability (CP) of 95% confidence intervals of the parameters, and average width (AW) of 95% confidence intervals of the parameters.

Table 2 shows the comparative behavior of AB, RMSE, CP, and AW. We generated 5000 random samples of different sizes for two sets of parameters using the following lemma.

Lemma 6.1. Let U be a standard uniform variable between zero and one. Then, the random variable

$$X = \frac{1}{\lambda}\sqrt[\gamma]{\log\left[\sqrt[\alpha]{-1-\frac{1}{\beta}-\frac{1}{\beta}W_{-1}\left((\beta+1)(U-1)e^{-(\beta+1)}\right)}+1\right]},$$

is said to come from the OPLW distribution with parameters λ, β, α and γ.

For each samples of size $n = 500, 1000, 1500, 2000, 2500$, and 3000 combined with two sets of parameters:

$(\lambda = 0.5, \beta = 1.2, \alpha = 1.2, \gamma = 0.5)$ and $(\lambda = 0.75, \beta = 0.75, \gamma = 2, \alpha = 2)$, for simulation on the basis of 5000 samples generated by using Lemma 6.1. It can be seen that, as the sample size increases, the RMSE and bias decrease toward zero. Moreover, the average confidence width decreases as the sample size increases, and the coverage probabilities of the confidence interval are quite close to the nominal 95% level. We conclude that the MLE's estimate and their asymptotic results can be used in inference applications such as hypothesis and confidence intervals.

Table 2: AB, RMSE, CP, and AW for varying $n, \lambda, \beta, \gamma$ and α.

		$\lambda = 0.5, \beta = 1.2, \alpha = 1.2, \gamma = 0.5$				$\lambda = 0.75, \beta = 0.75, \gamma = 2, \alpha = 2$			
Par.	n	AB	RMSE	CP	AW	AB	RMSE	CP	AW
λ	500	0.1068	0.424	0.844	1.989	0.0736	0.182	0.986	0.842
	1000	0.07731	0.333	0.868	1.308	0.0447	0.140	0.9804	0.566
	1500	0.0630	0.273	0.882	1.010	0.0284	0.112	0.966	0.432
	2000	0.0523	0.230	0.901	0.838	0.0198	0.097	0.958	0.357
	2500	0.0361	0.192	0.903	0.691	0.0174	0.087	0.947	0.315
	3000	0.0312	0.172	0.905	0.620	0.0118	0.073	0.940	0.270
β	500	0.2260	0.730	0.918	2.873	0.0407	0.517	0.878	2.668
	1000	0.1388	0.552	0.908	2.077	0.0505	0.456	0.896	2.171
	1500	0.0849	0.451	0.921	1.695	0.0702	0.422	0.913	1.896
	2000	0.0507	0.386	0.923	1.477	0.0811	0.399	0.926	1.690
	2500	0.0534	0.348	0.929	1.323	0.0598	0.365	0.927	1.501
	3000	0.0405	0.317	0.928	1.213	0.0673	0.348	0.931	1.383
γ	500	0.1378	0.332	0.954	1.177	0.3870	1.082	0.975	4.066

	1000	0.077	0.216	0.943	0.755	0.2062	0.745	0.956	2.818
	1500	0.0475	0.165	0.935	0.579	0.1711	0.609	0.953	2.329
	2000	0.0285	0.133	0.934	0.483	0.1568	0.538	0.947	2.028
	2500	0.0282	0.118	0.938	0.431	0.1076	0.470	0.944	1.748
	3000	0.0204	0.104	0.938	0.388	0.1085	0.428	0.947	1.597
α	500	-0.0650	0.380	0.866	1.561	-0.2452	0.628	0.846	2.985
	1000	-0.0650	0.281	0.896	1.104	-0.1351	0.499	0.889	2.255
	1500	-0.0403	0.231	0.915	0.900	-0.1061	0.429	0.903	1.850
	2000	-0.0196	0.120	0.926	0.787	-0.0946	0.376	0.910	1.584
	2500	-0.0247	0.181	0.921	0.701	-0.0642	0.344	0.921	1.421
	3000	-0.0154	0.163	0.931	0.643	-0.0622	0.310	0.924	1.277

7. Applications

In this section, we fit the $OPLW\ (\lambda,\beta,\alpha,\gamma)$ distribution to three real data sets and compare it with some of the other distributions, such as the Weibull–Weibull (WW) distribution proposed by Bourguignon et al. [8], power Lindley (PL) distribution introduced by Ghitany et al. [10], and the Weibull (W) distribution. The densities accordingly are given by

$$f_{WW}(x;\alpha,\beta,\lambda,\gamma)=\alpha\beta\lambda\gamma x^{\gamma-1}\left[e^{\lambda x^{\gamma}}-1\right]^{\beta-1}e^{-\alpha\left[e^{\lambda x^{\gamma}}-1\right]^{\beta}+\lambda x^{\gamma}},\alpha,\beta,\lambda,\gamma,x>0,$$

$$f_{PL}(x;\beta,\alpha)=\frac{\alpha\beta^{2}}{\beta+1}(1+x^{\alpha})x^{\alpha-1}e^{-\beta x^{\alpha}},\beta,\alpha,x>0,$$

$$f_{W}(x;\lambda,\kappa)=\frac{k}{\lambda}\left(\frac{k}{\lambda}\right)^{k-1}e^{-\left(\frac{k}{\lambda}\right)^{k}},\lambda,k,x>0,$$

The first data set represents the vinyl chloride obtained from clean up gradient monitoring wells in mg/l, which was examined and analyzed by Bhaumik et al. [7]. The values of this data set are 5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8, 0.8, 0.4, 0.6, 0.9, 0.4, 2, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1, 0.2, 0.1, 0.1, 1.8, 0.9, 2, 4, 6.8, 1.2, 0.4, 0.2.

The second data set represents the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20 mm, see Bader & Priest [5]. Its values are given as:

1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585.

The third data set represents the waiting times (in minutes) before service of 100 bank customers, which was examined and analyzed by Ghitany et al. [10]. The values of this data set are:

0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5.

For each distribution and for each data set, we derive the maximum likelihood estimates (MLEs), the maximized log likelihood (Log L), the Kolmogorov–Smirnov statistics (K–S) with its respective p-value, the Akaike Information Criterion (AIC), and the Bayesian Information Criterion (BIC). The K–S test is valid to test the goodness of fit of underlying distributions to the failure data, as shown by Bagheri et al. [6]. The results of fitting the data are presented in Table 3, 4, and 5. The fitted densities and the empirical distribution versus the fitted CDFs for the data set are shown in Figures 6,7 and 8. They indicate that the OPLW distribution fits the data better than the other distributions, except that the PL distribution was almost the same for the second data set. The K–S test statistic takes the smallest value with the largest value of its corresponding p-value for the OPLW distribution. Moreover, this conclusion is confirmed from the log likelihood for all the fitted models.

Table 3: Parameter estimates, KS statistics, P-value, log likelihood, AIC, and BIC for vinyl chloride.

Dist.	MLE	K-S	p-value	-log(L)	AIC	BIC
OPLW	$\hat{\lambda}=4.1830$ $\hat{\gamma}=0.0315$ $\hat{\beta}=0.0004$ $\hat{\alpha}=13.3718$	0.0789	0.9728	55.07	118.1	124.2
WW	$\hat{\lambda}=0.7389$ $\hat{\gamma}=0.0144$ $\hat{\beta}=49.3021$ $\hat{\alpha}=0.0064$	0.0929	0.905	55.5	119	125.1
PL	$\hat{\beta}=0.9139$ $\hat{\alpha}=0.8832$	0.0923	0.9086	55.6	115.2	118.3
W	$\hat{\lambda}=1.8879$ $\hat{k}=1.0102$	0.0889	0.9288	55.4	114.8	117.9

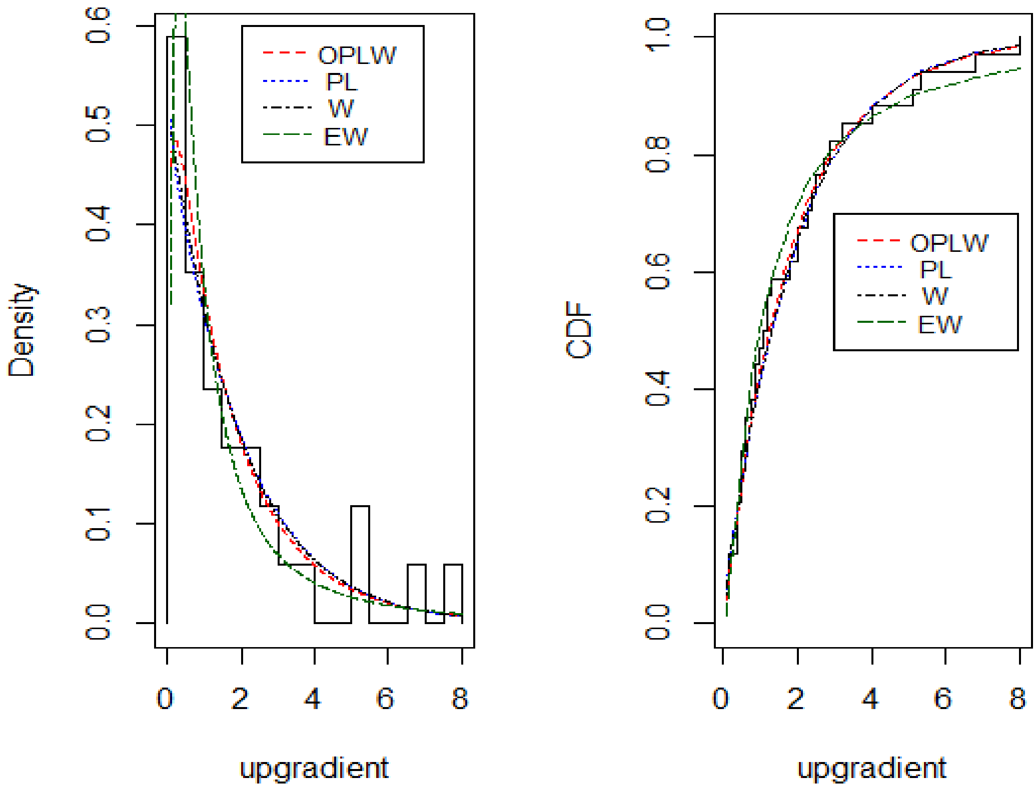

Figure 6: Plots of fitted models of the vinyl chloride data.

Table 4: Parameter estimates, KS statistic, P-value, log likelihood, AIC, and BIC for carbon fiber tensile strength.

Dist.	MLE	K-S	p-value	-log(L)	AIC	BIC
OPLW	$\hat{\lambda}=0.5683$ $\hat{\gamma}=0.1694$ $\hat{\beta}=0.0003$ $\hat{\alpha}=13.5703$	0.0454	0.9977	48.9	105.8	114.7

WW	$\hat{\lambda}=0.6895$ $\hat{\gamma}=0.0760$ $\hat{\beta}=51.1272$ $\hat{\alpha}=0.0072$	0.0576	0.9661	49.7	107.4	116.3
PL	$\hat{\beta}=0.0497$ $\hat{\alpha}=3.8678$	0.0443	0.9984	49.1	102.2	106.7
W	$\hat{\lambda}=2.6509$ $\hat{k}=5.5049$	0.05626	0.9725	49.6	103.2	107.7

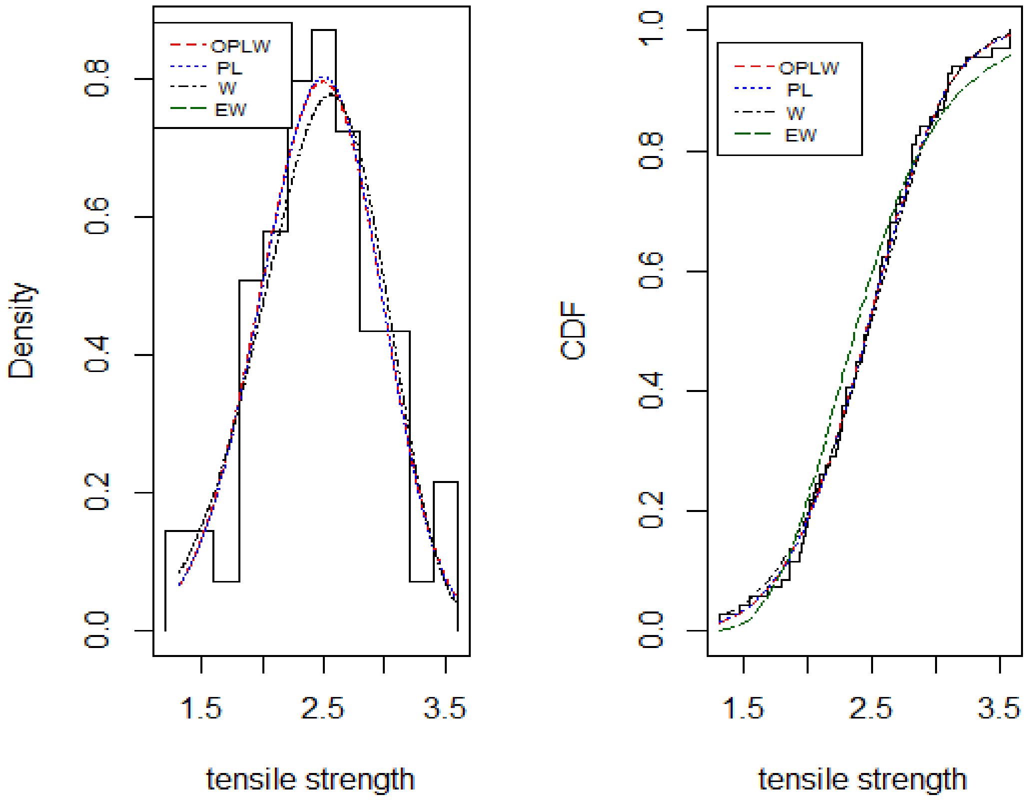

Figure 7: Plots of fitted models of the carbon fiber tensile strength data.

Table 5: Parameter estimates, KS statistics, P-value, log likelihood, AIC, and BIC for the waiting times.

Dist.	MLE	K-S	p-value	-log(L)	AIC	BIC
OPLW	$\hat{\lambda}=0.5956$ $\hat{\gamma}=0.0484$ $\hat{\beta}=0.0004$ $\hat{\alpha}=12.5014$	0.0428	0.9912	317.5	642.9	653.3
WW	$\hat{\lambda}=0.7245$ $\hat{\gamma}=0.0298$ $\hat{\beta}=33.9681$ $\hat{\alpha}=0.0039$	0 0.0594	0.8726	318.9	645.8	664.2
PL	$\hat{\beta}=0.1530$ $\hat{\alpha}=1.0832$	0.0516	0.9525	318.3	640.6	645.8
W	$\hat{\lambda}=10.9553$ $\hat{k}=1.4585$	0.0573	0.8975	318.7	641.5	646.7

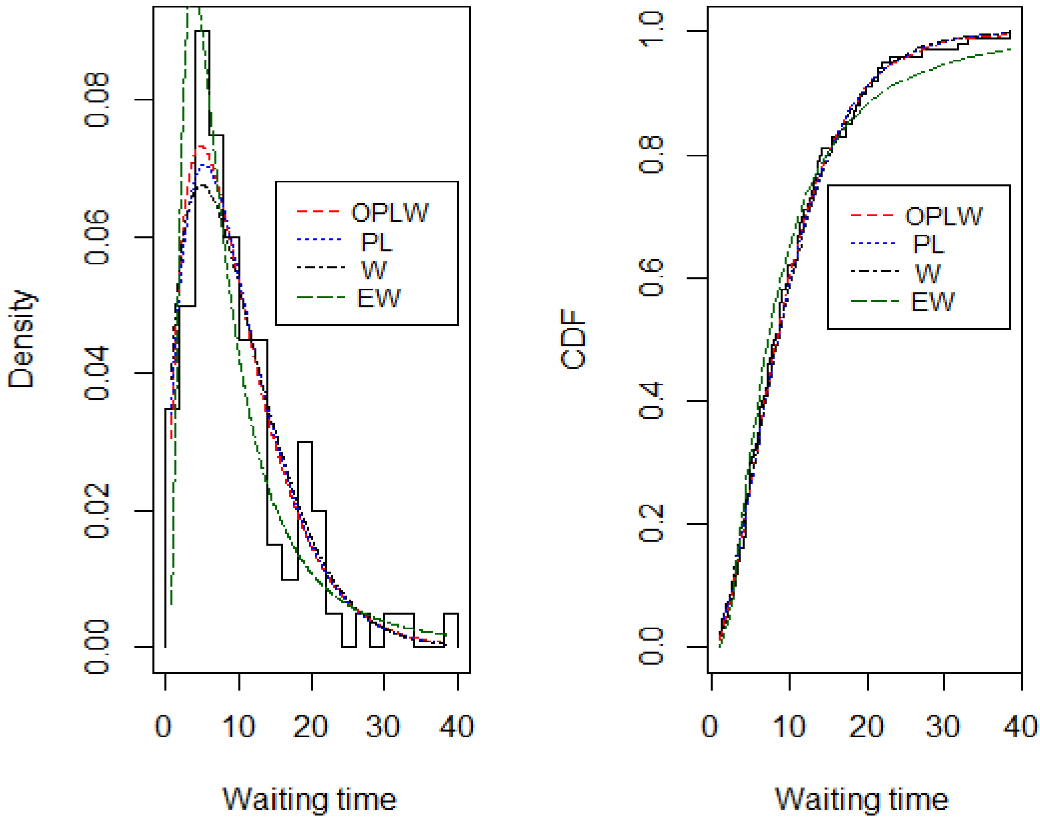

Figure 8: Plots of fitted models of the waiting time data.

8. Concluding remarks

The purpose of this paper was to define a new family of lifetime distributions called the odd power Lindley-G (OPL-G) distributions, which generates many lifetime mixture distributions. The properties of the OPL-G family of distributions were derived in flexible and useful forms, including density, survival function, hazard rate function, quantile function, distribution of order statistics, and maximum likelihood estimates. Several models were introduced as special cases of the proposed class. The odd power Lindley Weibull (OPLW) distribution was introduced as an example of the proposed class. Simulation of the OPLW was carried out to check the reliability and performance of the MLE estimates. The OPLW distribution was applied to three data sets in order to show the flexibility and advantages of the proposed class of distributions, and the results were compared with those obtained using some existing distributions.

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Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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