WIRTINGER'S INTEGRAL INEQUALITY ON TIME SCALE

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Abstract. In this paper, we establish a Wirtinger-type inequality on an arbitrary time scale. We give, as special cases of the time scales, new Wirtinger-type inequality in the continuous and discrete cases, respectively.

1. Introduction

A time scale, (we denote it by the symbol $\mathbb{T}$) is an arbitrary nonempty closed subset of the real numbers. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in T : s > t\}$. If $t < \sup T$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf T$ and $\rho(t) = t$, then t is called left-dense. Graininess function $\mu : T \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$ (see [2], [3], [6]).

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathbb{C}_{rd} = \mathbb{C}_{rd}(\mathbb{T}) = \mathbb{C}_{rd}(\mathbb{T}, \mathbb{R})$. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by $\mathbb{C}^1_{rd} = \mathbb{C}^1_{rd}(\mathbb{T}) = \mathbb{C}^1_{rd}(\mathbb{T}, \mathbb{R})$. We define the time scale interval $[a, b]_{\mathbb{T}}$ by $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$.

In 2000, Hilscher [8] proved a Wirtinger-type inequality on time scales in the form:

Theorem 1.1. *(Discrete Wirtinger Inequality, [?]) If M be positive and strictly monotone such that M^{Δ} exists and is rd-continuous, then*

$$\int_a^b \left|M^{\Delta}(t)\right| y^2(\sigma(t))\,\Delta t \le \Psi^2 \int_a^b \frac{M(t)M(\sigma(t))}{|M^{\Delta}(t)|}\left(y^{\Delta}(t)\right)^2 \Delta t \tag{1.1}$$

for any y with $y(a) = y(b) = 0$ and such that y^{Δ} exists and is rd-continuous, where

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(1.2)
$$\Psi=\left(\sup_{t\in[a,b]\cap\mathbb{T}}\frac{M(t)}{M(\sigma(t))}\right)^{\frac{1}{2}}+\left[\left(\sup_{t\in[a,b]\cap\mathbb{T}}\frac{\mu(t)\left|M^{\Delta}(t)\right|}{M(\sigma(t))}\right)+\left(\sup_{t\in[a,b]\cap\mathbb{T}}\frac{M(t)}{M(\sigma(t))}\right)\right]^{\frac{1}{2}}.$$

In [4] authors extended the following theorem:

Theorem 1.2. *([4]) Suppose $\gamma\geq 1$ is an odd integer. For a positive $M\in C^1_{rd}(\mathfrak{T})$ satisfying either $M^{\Delta}>0$ or $M^{\Delta}<0$ on $\mathfrak{T}$, we have*

$$\int_a^b \frac{M^{\gamma}(t)M(\sigma(t))}{\left|M^{\Delta}(t)\right|^{\gamma}}\left(y^{\Delta}(t)\right)^{\gamma+1}\Delta t\geq\frac{1}{\Psi^{\gamma+1}(\alpha,\beta,\gamma)}\int_a^b\left|M^{\Delta}(t)\right|y^{\gamma+1}(t)\,\Delta t \tag{1.3}$$

for any $y\in C^1_{rd}(\mathfrak{T})$ with $y(a)=y(b)=0$, where $\Psi(\alpha,\beta,\gamma)$ is the largest root of

$$x^{\gamma+1}-2^{\gamma-1}(\gamma+1)\,\alpha x^{\gamma}-2^{\gamma-1}\beta=0, \tag{1.4}$$

whereby

$$\alpha:=\sup_{t\in\mathfrak{T}^k}\left(\frac{M(\sigma(t))}{M(t)}\right)^{\frac{\gamma}{\gamma+1}},\quad \beta:=\sup_{t\in\mathfrak{T}^k}\left(\frac{\mu(t)\left|M^{\Delta}(t)\right|}{M(t)}\right)^{\gamma}.$$

2. Main results

Let us prove the following theorem:

Theorem 2.1. *Let $M\in\mathbb{C}^1_{rd}\left([a,b]_{\mathbb{T}}\right)^k$ be positive and strictly monotone such that satisfying either $M^{\Delta}>0$ or $M^{\Delta}<0$ on $\left([a,b]_{\mathbb{T}}\right)^k$. Then, for some integer $\eta\geq 1$ we have*

(2.1)
$$\int_a^b\left|M^{\Delta}(t)\right|y^{\eta+1}(\sigma(t))\,\Delta t\leq\Lambda^{\eta+1}(\omega,\xi_r,\psi)\int_a^b\frac{M^{\eta}(t)M(\sigma(t))}{\left|M^{\Delta}(t)\right|^{\eta}}\left(y^{\Delta}(t)\right)^{\eta+1}\Delta t$$

for any $y\in\mathbb{C}^1_{rd}\left([a,b]_{\mathbb{T}}\right)^k$, with $y(a)=y(b)=0$, where $\Lambda(\omega,\xi_r,\psi)$ is the largest root of equality

$$x^{\eta+1}=2^{\eta}\omega x^{\eta}+\sum_{r=1}^{\eta-1}2^{\eta-(r+1)}\xi_r x^r+2^{\eta-1}\psi, \tag{2.2}$$

whereby

$$\begin{aligned}\omega&=\sup_{t\in\left([a,b]_{\mathbb{T}}\right)^k}\left(\frac{M^{\sigma}}{M}\right)^{\frac{\eta}{\eta+1}},\quad \psi=\sup_{t\in\left([a,b]_{\mathbb{T}}\right)^k}\left(\frac{\mu^{\frac{1}{\eta}}\left|M^{\Delta}\right|}{M}\right)^{\eta},\\ \xi_r&=\sup_{t\in\left([a,b]_{\mathbb{T}}\right)^k}\left(\frac{\mu^{\frac{\eta+1}{r}}M^{\sigma}\left|M^{\Delta}\right|^{\frac{\eta(\eta-(r-1))}{r}-1}}{M^{\frac{\eta(\eta-(r-1))}{r}}}\right)^{\frac{\eta}{\eta+1}},\quad r=1,\ldots,\eta-1.\end{aligned} \tag{2.3}$$

We denote by

$$(2.4)\quad A=\int_a^b\left|M^{\Delta}(t)\right|y^{\eta+1}(\sigma(t))\,\Delta t,\quad B=\int_a^b\frac{M^{\eta}(t)M(\sigma(t))}{|M^{\Delta}(t)|^{\eta}}\left(y^{\Delta}(t)\right)^{\eta+1}\Delta t.$$

Using the integration by parts, whereby $y(a)=y(b)=0$, left side of inequality (2.1) become

$$\begin{aligned}
A&=\int_a^b\left|M^{\Delta}(t)\right|y^{\eta+1}(t)\,\Delta t=\pm\int_a^b M^{\Delta}(t)\,y^{\eta+1}(t)\,\Delta t\\
&=\pm\left\{\left[M(t)\,y^{\eta+1}(t)\right]_a^b-\int_a^b M^{\sigma}(t)\left(y^{\eta+1}(t)\right)^{\Delta}\Delta t\right\}\\
&\le\int_a^b M^{\sigma}(t)\left|y^{\eta+1}\right|^{\Delta}(t)\,\Delta t=\int_a^b M^{\sigma}\left|\sum_{r=0}^{\eta}y^{r}\left(y^{\sigma}\right)^{\eta-r}\right|\left|y^{\Delta}\right|\Delta t\\
&=\int_a^b M^{\sigma}\left|\left(y^{\sigma}\right)^{\eta}+y\left(y^{\sigma}\right)^{\eta-1}+y^{2}\left(y^{\sigma}\right)^{\eta-2}+...+y^{\eta-1}\left(y^{\sigma}\right)+y^{\eta}\right|\left|y^{\Delta}\right|\Delta t\\
&=\int_a^b M^{\sigma}\left|\left(y+\mu y^{\Delta}\right)^{\eta}+y\left(y+\mu y^{\Delta}\right)^{\eta-1}+...+y^{\eta-1}\left(y+\mu y^{\Delta}\right)+y^{\eta}\right|\left|y^{\Delta}\right|\Delta t\\
&\le\int_a^b \begin{matrix}M^{\sigma}\{2^{\eta-1}|y|^{\eta}\left|y^{\Delta}\right|+2^{\eta-1}\mu\left|y^{\Delta}\right|^{\eta+1}+2^{\eta-2}|y|^{\eta}\left|y^{\Delta}\right|+2^{\eta-2}\mu|y|\left|y^{\Delta}\right|^{\eta}+...+\\ +|y|^{\eta}\left|y^{\Delta}\right|+\mu|y|^{\eta-1}\left|y^{\Delta}\right|^{2}+|y|^{\eta}\left|y^{\Delta}\right|\}\Delta t\end{matrix}\\
&=\int_a^b \begin{matrix}\{2^{\eta}M^{\sigma}|y|^{\eta}\left|y^{\Delta}\right|+2^{\eta-2}M^{\sigma}\mu|y|\left|y^{\Delta}\right|^{\eta}+2^{\eta-3}M^{\sigma}\mu|y|^{2}\left|y^{\Delta}\right|^{\eta-1}+\\ ...+M^{\sigma}\mu|y|^{\eta-1}\left|y^{\Delta}\right|^{2}+2^{\eta-1}M^{\sigma}\mu\left|y^{\Delta}\right|^{\eta+1}\}\Delta t\end{matrix}\\
&=2^{\eta}\int_a^b\left(\frac{M^{\eta}M^{\sigma}}{|M^{\Delta}|^{\eta}}\left|y^{\Delta}\right|^{\eta+1}\right)^{\frac{1}{\eta+1}}\left(\frac{M^{\eta}M^{\sigma}}{M}|y|^{\eta+1}\right)^{\frac{\eta}{\eta+1}}\Delta t+\\
&2^{\eta-2}\int_a^b\left(\frac{M^{\eta}M^{\sigma}}{|M^{\Delta}|^{\eta}}\left|y^{\Delta}\right|^{\eta+1}\right)^{\frac{\eta}{\eta+1}}\left(\frac{\mu^{\eta+1}M^{\sigma}\left|M^{\Delta}\right|^{\eta^{2}-1}\left|M^{\Delta}\right|}{M^{\eta^{2}}}|y|^{\eta+1}\right)^{\frac{1}{\eta+1}}\Delta t\\
&+2^{\eta-3}\int_a^b\left(\frac{M^{\eta}M^{\sigma}}{|M^{\Delta}|^{\eta}}\left|y^{\Delta}\right|^{\eta+1}\right)^{\frac{\eta-1}{\eta+1}}\left(\frac{\mu^{\frac{\eta+1}{2}}M^{\sigma}\left|M^{\Delta}\right|^{\frac{\eta(\eta-1)}{2}-1}\left|M^{\Delta}\right|}{M^{\frac{\eta(\eta-1)}{2}}}|y|^{\eta+1}\right)^{\frac{2}{\eta+1}}\Delta t+...\\
&+2\int_a^b\left(\frac{M^{\eta}M^{\sigma}}{|M^{\Delta}|^{\eta}}\left|y^{\Delta}\right|^{\eta+1}\right)^{\frac{3}{\eta+1}}\left(\frac{\mu^{\frac{\eta+1}{\eta-2}}M^{\sigma}\left|M^{\Delta}\right|^{\frac{3\eta}{\eta-2}-1}\left|M^{\Delta}\right|}{M^{\frac{3\eta}{\eta-2}}}|y|^{\eta+1}\right)^{\frac{\eta-2}{\eta+1}}\Delta t\\
&+\int_a^b\left(\frac{M^{\eta}M^{\sigma}}{|M^{\Delta}|^{\eta}}\left|y^{\Delta}\right|^{\eta+1}\right)^{\frac{2}{\eta+1}}\left(\frac{\mu^{\frac{\eta+1}{\eta-1}}M^{\sigma}\left|M^{\Delta}\right|^{\frac{2\eta}{\eta-1}-1}\left|M^{\Delta}\right|}{M^{\frac{2\eta}{\eta-1}}}|y|^{\eta+1}\right)^{\frac{\eta-1}{\eta+1}}\Delta t\\
&+2^{\eta-1}\int_a^b\left(\frac{M^{\eta}M^{\sigma}}{|M^{\Delta}|^{\eta}}\left|y^{\Delta}\right|^{\eta+1}\right)\left(\frac{\mu\left|M^{\Delta}\right|^{\eta}}{M^{\eta}}\right)\Delta t.
\end{aligned}$$

Applying Hölder inequality on each summand of the above inequality, except the last one, it follows

$$\begin{aligned}
A&\le 2^{\eta}\left\{\int_a^b\left(\frac{M^{\eta}M^{\sigma}}{|M^{\Delta}|^{\eta}}\left|y^{\Delta}\right|^{\eta+1}\right)\Delta t\right\}^{\frac{1}{\eta+1}}\left\{\int_a^b\left(\frac{M^{\eta}M^{\sigma}}{M}|y|^{\eta+1}\right)\Delta t\right\}^{\frac{\eta}{\eta+1}}\\
&+2^{\eta-2}\left\{\int_a^b\left(\frac{M^{\eta}M^{\sigma}}{|M^{\Delta}|^{\eta}}\left|y^{\Delta}\right|^{\eta+1}\right)\Delta t\right\}^{\frac{\eta}{\eta+1}}\left\{\int_a^b\left(\frac{\mu^{\eta+1}M^{\sigma}\left|M^{\Delta}\right|^{\eta^{2}-1}\left|M^{\Delta}\right|}{M^{\eta^{2}}}|y|^{\eta+1}\right)\Delta t\right\}^{\frac{1}{\eta+1}}
\end{aligned}$$

(2.5)

$$+\ldots+\left\{\int\limits_a^b\left(\frac{M^\eta M^\sigma}{|M^\Delta|^\eta}\left|y^\Delta\right|^{\eta+1}\right)\Delta t\right\}^{\frac{2}{\eta+1}}\left\{\int\limits_a^b\left(\frac{\mu^{\frac{\eta+1}{\eta-1}}M^\sigma\left|M^\Delta\right|^{\frac{2\eta}{\eta-1}-1}\left|M^\Delta\right|}{M^{\frac{2\eta}{\eta-1}}}\left|y\right|^{\eta+1}\right)\Delta t\right\}^{\frac{\eta-1}{\eta+1}}$$
$$+2^{\eta-1}\int\limits_a^b\left(\frac{M^\eta M^\sigma}{|M^\Delta|^\eta}\left|y^\Delta\right|^{\eta+1}\right)\left(\frac{\mu\left|M^\Delta\right|^\eta}{M^\eta}\right)\Delta t$$
$$=2^\eta\omega B^{\frac{1}{\eta+1}}A^{\frac{\eta}{\eta+1}}+2^{\eta-2}\xi_1B^{\frac{\eta}{\eta+1}}A^{\frac{1}{\eta+1}}+2^{\eta-3}\xi_2B^{\frac{\eta-1}{\eta+1}}A^{\frac{2}{\eta+1}}+\ldots$$
$$+2\xi_{\eta-2}B^{\frac{3}{\eta+1}}A^{\frac{\eta-2}{\eta+1}}+\xi_{\eta-1}B^{\frac{2}{\eta+1}}A^{\frac{\eta-1}{\eta+1}}+2^{\eta-1}\psi B,$$

i.e.

$$A\le 2^\eta\omega B^{\frac{1}{\eta+1}}A^{\frac{\eta}{\eta+1}}+\sum_{r=1}^{\eta-1}2^{\eta-(r+1)}\xi_rB^{\frac{\eta-(r-1)}{\eta+1}}A^{\frac{r}{\eta+1}}+2^{\eta-1}\psi B. \tag{2.6}$$

After some calculations one obtains it holds the following inequality

$$\left(\tfrac{A}{B}\right)^{\frac{1}{\eta+1}}\le 2^\eta\omega+2^{\eta-2}\xi_1\left(\tfrac{B}{A}\right)^{\frac{\eta-1}{\eta+1}}+2^{\eta-3}\xi_2\left(\tfrac{B}{A}\right)^{\frac{\eta-2}{\eta+1}}+\ldots$$
$$+2\xi_{\eta-2}\left(\tfrac{B}{A}\right)^{\frac{2}{\eta+1}}+\xi_{\eta-1}\left(\tfrac{B}{A}\right)^{\frac{1}{\eta+1}}+2^{\eta-1}\psi\left(\tfrac{B}{A}\right)^{\frac{\eta}{\eta+1}},$$

$$\left(\frac{A}{B}\right)^{\frac{1}{\eta+1}}\le 2^\eta\omega+\sum_{r=1}^{\eta-1}2^{\eta-(r+1)}\xi_r\left(\frac{B}{A}\right)^{\frac{\eta-r}{\eta+1}}+2^{\eta-1}\psi\left(\frac{B}{A}\right)^{\frac{\eta}{\eta+1}}.$$

By introducing $C=\left(\frac{A}{B}\right)^{\frac{1}{\eta+1}}$, we get

$$C\le 2^\eta\omega+\sum_{r=1}^{\eta-1}2^{\eta-(r+1)}\xi_rC^{r-\eta}+2^{\eta-1}\psi\left(\frac{B}{A}\right)^{-\eta},$$

i.e.

$$C^{\eta+1}\le 2^\eta\omega C^\eta+\sum_{r=1}^{\eta-1}2^{\eta-(r+1)}\xi_rC^r+2^{\eta-1}\psi, \tag{2.7}$$

whence follows the desired inequality,

$$A\le\Lambda^{\eta+1}(\omega,\xi_r,\gamma)\le B.$$

3. Application

Corollary 3.1. *In the case of $\mathbb{T}=\mathbb{R}$, the inequality (1.3) reduces to*

$$\int\limits_a^b\left|M'(t)\right|y^{\eta+1}(t)\,dt\le(2^\eta)^{\eta+1}\int\limits_a^b\frac{M^{\eta+1}(t)}{|M'(t)|^\eta}\left(y'(t)\right)^{\eta+1}dt. \tag{3.1}$$

Proof: In the case of $\mathbb{T}=\mathbb{R}$ it is $f^\Delta(t)=f'(t)$, $\sigma(t)=t$ and $\mu(t)=0$, so $\omega=1$, $\xi_r=0$ and $\psi=0$. By substitute this values in the equalities (2.2) we obtain $x^{\eta+1}=2^\eta x^\eta$. i.e. $x^\eta(x-2^\eta)=0$. Since $\int\limits_a^b f(t)\,\Delta t=\int\limits_a^b f(t)\,dt$, follows inequality (3.1).

Remark 3.2. Specially, in the case of $\eta = 1$, the largest root of the (1.3) is 2, so the inequality (1.3) becomes

$$\int_a^b |M'(t)|\, y^2((t))\, dt \le 4 \int_a^b \frac{M^2(t)}{|M'(t)|} (y'(t))^2\, dt, \tag{3.2}$$

what was proved in [6].

Corollary 3.3. *Let $\mathbb{T} = h\mathbb{Z}$. For a positive sequence $\{M_n\}_{0\le n\le N+1}$ satisfying either $\Delta M > 0$ or $\Delta M < 0$ on $[0, N] \cap h\mathbb{Z}$, we have*

$$\sum_{n=0}^{N} |\Delta_h M_n|\, y_n^{\eta+1} \le \Omega^\eta(\omega, \xi_r, \psi) \sum_{n=0}^{N} \frac{M_n^\eta M_{n+1}}{|\Delta_h M_n|^\eta} (\Delta_h y_n)^{\eta+1},$$

for any sequence $\{y_n\}_{0\le n\le N+1}$ with $y_0 = y_{N+1} = 0$, where $\Omega(\omega, \xi_r, \psi)$ is the smallest root of the inequality

$$(1 + 2\omega)\, 2^{\eta-1} x^\eta = \sum_{r=1}^{\eta-1} 2^{\eta-(r+1)} \xi_r x^r + 2^{\eta-1}\psi, \tag{3.3}$$

when

$$\begin{aligned} \omega &= \sup_{0\le n\le N} \left(\frac{M_{n+h}}{M_n}\right)^{\frac{\eta}{\eta+1}}, \\ \xi_r &= \sup_{0\le n\le N} \left(\frac{h^{\frac{\eta+1}{r}} M_{n+h} |\Delta_h M_n|^{\frac{\eta(\eta-(r-1))}{r}-1}}{M_n}\right)^{\frac{\eta}{\eta+1}}, \quad r = 1, ..., \eta - 1, \\ \psi &= \sup_{0\le n\le N} \left(\frac{h^{\frac{1}{\eta}} |\Delta_h M_n|}{M_n}\right)^\eta. \end{aligned} \tag{3.4}$$

Proof. Starting from the inequality

$$(1 + C)^{\eta+1} \le C^{\eta+1} + (\eta + 1)\, C^\eta + 2^{\eta-1} C^\eta$$

it is obtained

$$C^{\eta+1} \ge (1 + C)^{\eta+1} - (\eta + 1)\, C^\eta - 2^{\eta-1} C^\eta.$$

Involving this result in (1.2) proves it holds

$$(1 + C)^{\eta+1} - (\eta + 1)\, C^\eta - 2^{\eta-1} C^\eta - 2^\eta \omega C^\eta - \sum_{r=1}^{\eta-1} 2^{\eta-(r+1)} \xi_r C^r - 2^{\eta-1}\psi \le 0.$$

Since

$$(1 + C)^{\eta+1} \ge (\eta + 1)\, C^\eta,$$

last inequality becomes

$$(1 + 2\omega)\, 2^{\eta-1} C^\eta \ge \sum_{r=1}^{\eta-1} 2^{\eta-(r+1)} \xi_r C^r + 2^{\eta-1}\psi.$$

Since, for $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$ is $\sigma(t) = t + h$, $\mu(t) = h$, $f^\Delta(t) = \Delta_h f(t) = \frac{f(t+h)-f(t)}{h}$, $\int_a^b f(t)\,\Delta t = \sum_{t\in[0,N]\cap h\mathbb{Z}} \mu(t) f(t)$, so that

$$A = \sum_{n=0}^{N} |\Delta_h M_n| \, y_n^{\eta+1}, \qquad B = \sum_{n=0}^{N} \frac{M_n^{\eta} M_{n+1}}{|\Delta_h M_n|^{\eta}} \left(\Delta_h y_n\right)^{\eta+1},$$

whence follows the desired inequality.

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