Two and Three Stages Least Squares as Aitken estimators

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Abstract

This note generalizes the existing derivation of Two and Three Stages Least Squares estimators as Aitken Generalised Least Squares estimators.

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Two and Three Stages Least Squares as Aitken estimators

Aitken Generalized Least Squares estimators [1] are well-known and frequently used in econometric estimation, likewise Two and Three Stages Least Squares estimators. The present note generalizes the existing derivation of Two and Three Stages Least Squares estimators as Aitken Generalized Least Squares estimators. Almost fifty years ago, Drettakis [2] revealed a special case of our present generalization.

Using the standard notation, let us write the ith structural equation of a simultaneous linear system as

$$y_i = Z_i \delta_i + u_i \quad , \tag{1}$$

and the corresponding system of all stochastic equations as

$$y = Z\delta + u \quad . \tag{2}$$

Under the assumption that

$$E(u_i) = 0 \quad E\left(u_i u_j\right) = \sigma_{ij}^2 I$$

or in matrix notation

$$E(u) = 0 \quad E(uu') = \Sigma \otimes I \quad ,$$

the Two Stages Least Squares estimate of δ_i is

$$\hat{\delta}_i = [Z_i' X(X'X)^{-1} X' Z_i]^{-1} Z_i' X(X'X)^{-1} X' y_i \quad , \tag{3}$$

and the Three Stages Least Squares estimate of δ is

$$\hat{\delta} = \{Z'[\Sigma^{-1} \otimes X(X'X)^{-1}X']\}^{-1} Z_i^{-1} Z'[\Sigma^{-1} \otimes X(X'X)^{-1}X']y \quad , \tag{4}$$

where X is the matrix of all predetermined variables in the system and $\Sigma = \left\{\sigma_{ij}^2\right\}$ is a positive definite symmetric matrix.

In order to show that (3) and (4) can be interpreted as Aitken Generalised Least Squares estimators of (1) and (2) respectively, we must use a matrix, say L, of non stochastic elements in the following way: First, premultiply (1) by L' and (2) by

$I \otimes L'$, and secondly, apply the Generalized Least Squares method to the resulting specifications. In both cases we only need to select L such that

$$L(L'L)^{-1}L' = (X'X)^{-1}X' \tag{5}$$

On the assumption that the variables in X are strictly exogenous, econometric textbooks favor the choice of $L=X$, whereas Drettakis (1973) suggested using $L=X(X'X)^{-½}$ which, in turn, produces the required result.

In fact, there exists an infinite number of L matrices that are suitable and satisfy (5). Let us define

$$L = X(X'X)^c \quad , \tag{6}$$

where c is any number, real or complex.

By straightforward matrix calculations we can see that (5) is satisfied when L is of the form given in (6). As we can see, the textbook choice is obtained by setting $c=0$ and Drettakis' choice by setting $c=½$.

References

[1] A. C. Aitken, "On Least-squares and Linear Combinations of Observations", *Proceedings of the Royal Society of Edinburgh*, vol. 55, 1936, pp. 42–48.

[2] E. Drettakis, "An Expository Note on the Derivation of the Two Stage and the Three Stages Least Squares Estimators", *Bulletin of Economic Research*, vol. 25, 1973, pp. 146-147.