The Topp Leone Marshall-Olkin-Weibull Poisson Distribution with Applications

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Abstract

A new power series distribution called the Topp-Leone Marshall-Olkin Weibull Poisson (TLMOWP) distribution is developed. We developed the statistical properties including the hazard rate, quantile and moment generating functions, moments, and Shannon and Rényi entropies. Maximum likelihood estimates of the model parameters were also derived. A simulation study to assess consistency of the maximum likelihood estimates was conducted. Real data examples are provided to demonstrate the usefulness of the proposed model in comparison with other competing non-nested models.

Keywords: Topp-Leone-Marshall-Olkin-G Distribution, Power Series Distribution, Poisson Distribution, Maximum Likelihood Estimation.

1 Introduction

There is increased demand for generalized distributions in the past two decades. Work on the generalized distributions gave birth to distributions with versatility and flexibility in relation to skewness and kurtosis. The additional shape and/or scale parameters give more weight to the tail ends of a distribution, thereby making it more applicable to heavily tailed data. Furthermore, generalized distributions are good for fitting data sets with non-monotonic hazard rates, compared to classical distributions which are applicable to monotonic hazard rates. Generalized distributions have wider applications in areas of science, engineering, commerce, medicine, and environmental science.

In 1955, Topp and Leone derived a probability distribution in order to model data that generated J-shaped failure data. They presented the Topp-Leone distribution as a special case of the triangular distribution, which has limited support on (0,1) whose distribution function is given by

$$F_{TL}(x;b) = [1-(1-x)^2]^b, \tag{1.1}$$

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where $0 < x < 1$ and $b > 0$. The Topp-Leone (TL) distribution serves an important role in modeling lifetime data in areas of science, insurance and finance. The distribution is a bounded J-shape distribution that has also motivated various researchers. Many researchers generalized the Topp-Leone distribution so as to introduce symmetry, skewness and kurtosis to the distribution. Generalizations of the Topp-Leone distribution available in the literature include the Topp-Leone-Marshall-Olkin-G family by Chipepa et al. (2020), Type II power Topp-Leone generated family by Bantan et al. (2020), Topp-Leone-Weibull by Rezaei et al. (2016), Topp-Leone generalized exponential by Sangsanit and Bodhisuwan (2016). Extensions of the Topp-Leone distribution exhibit flexibility in data modeling in relation to skewness, kurtosis and the hazard rate function. Also, the Topp-Leone generalized distributions are not restricted to the domain of (0,1).

Marshall and Olkin (1997) introduced the Marshall-Olkin-G family of distributions with distribution function

$$F_{MO}(x;\delta) = 1 - \frac{\delta\overline{G}(x;\xi)}{1-\overline{\delta}\overline{G}(x;\xi)}, \tag{1.2}$$

where $\overline{G}(x;\xi)$ is the survival function of the baseline distribution function, ξ is a vector of parameters from the baseline distribution, δ is a tilt parameter, and $\overline{\delta} = 1-\delta$. Some generalizations of the Marshall-Olkin distribution include work by Alizadeh et al. (2015), where they introduced the beta Marshall-Olkin-G (BMO-G), Lepetu et al. (2017) developed the Marshall-Olkin Log-logistic Extended Weibull (MOLLEW) family of distributions, Santos et al. (2014) developed the Marshall-Olkin extended Weibull family of distributions, Marshall-Olkin Kumaraswamy-G distribution by Chakraborty et al. (2017), Marshall-Olkin extended generalized Gompertz distribution by Lazhar et al. (2017), Marshall-Olkin extended Burr Type III distribution by Kumar et al. (2016) and Marshall-Olkin-Gompertz-G by Chipepa and Oluyede (2021).

From Johnson, Kotz, and Kemp (1992), a distribution with probability mass function

$$P(X=x) = \frac{a_x\theta^x}{\eta(\theta)}; \qquad x = 0,1,2,..., \quad \theta > 0, \tag{1.3}$$

where $a_i \geq 0$ and

$$\eta(\theta) = \sum_{x=0}^{\infty} a_x\theta^x$$

is the power series distribution associated with the function η and parameter θ. Let N be a zero truncated discrete random variable having a power series distribution, whose probability mass function (pmf) is given by

$$P(N=n) = \frac{a_n \theta^n}{C(\theta)}, n = 1, 2, 3, ..., \tag{1.4}$$

where $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$ is finite, $\theta > 0$ and $\{a_n\}_{n \geq 1}$ a sequence of positive real numbers. The power series family of distributions includes binomial, Poisson, geometric and logarithmic distributions from Johnson et al. (1994), see appendix for some useful quantities for the power series distributions. Several generalized distributions proposed in the literature involving the power series include the odd Weibull-Topp-Leone-G power series family of distributions by Oluyede et al. (2020), complementary extended Weibull-power series by Cordeiro and Silva (2014), the Burr XII power series by Silva and Cordeiro (2014), the exponentiated power generalized Weibull power series family of distributions by Aldahlan et al. (2020), the T?R $\{Y\}$ power series family of probability distributions by Osatohanmwen et al. (2020), extended Weibull-power series (EWPS) distribution by Silva et al. (2013), exponentiated generalized power series class of distributions by Oluyede et al. (2020), Weibull-power series distributions by Morais and Barreto-Souza (2011), complementary exponential power series by Flores et al. (2013) and the Burr-Weibull power series class of distributions by Oluyede et al. (2019).

Chipepa et al. (2020) aimed to find a model that could efficiently deal with heavily skewed and tailed data, data with kurtosis differing from the baseline distribution, and data that has non-monotonic hazard functions. Thus, they derived the Topp-Leone Marshall-Olkin-G (TLMOG) family of distributions by combining the Topp-Leone and Marshall-Olkin Distributions from equation (1.1) and equation (1.2) respectively. The density and distribution functions are given by

$$f_{TLMOG}(x; b, \delta, \xi) = \left[\frac{2b\delta^2 g(x;\xi)\overline{G}(x;\xi)}{[1-\overline{\delta}\overline{G}(x;\xi)]^3} \right] \left[1 - \frac{\delta^2 \overline{G}^2(x;\xi)}{[1-\overline{\delta}\overline{G}(x;\xi)]^2} \right]^{b-1} \tag{1.5}$$

and

$$F_{TLMOG}(x; b, \delta, \xi) = \left[1 - \frac{\delta^2 \overline{G}^2(x;\xi)}{[1-\overline{\delta}\overline{G}(x;\xi)]^2} \right]^{b} \tag{1.6}$$

respectively, with $b, \delta > 0$, $\overline{\delta} = 1 - \delta$, and ξ a vector of parameters from the baseline distribution.

Using the baseline distribution of the Weibull distribution, whose density function is,

$$g(x; \alpha) = \alpha x^{\alpha-1} e^{-x^{\alpha}}, \tag{1.7}$$

and whose distribution function is,

$$G(x; , \alpha) = 1 - e^{-x^{\alpha}}, \tag{1.8}$$

where $\alpha > 0$, Chipepa et al. (2020) derived the Topp-Leone Marshall-Olkin Weibull (TLMOW) distribution, whose distribution function is,

$$F_{TLMOW}(x; b, \delta, \alpha) = \left[1 - \frac{\delta^2 e^{-2x^{\alpha}}}{(1 - \bar{\delta} e^{-x^{\alpha}})^2}\right]^b, \tag{1.9}$$

and whose density function is,

$$f_{TLMOW}(x; b, \delta, \alpha) = \frac{2b\delta^2 \alpha x^{\alpha-1} e^{-2x^{\alpha}}}{(1 - \bar{\delta} e^{-x^{\alpha}})^3} \left[1 - \frac{\delta^2 e^{-2x^{\alpha}}}{(1 - \bar{\delta} e^{-x^{\alpha}})^2}\right]^{b-1}, \tag{1.10}$$

where $b, \delta, \alpha > 0$. Our aim in this article is to derive a flexible and versatile four-parameter Topp-Leone Marshall-Olkin Weibull Poisson (TLMOWP) distribution.

In Section 2 we derive the TLMOWP model, as well as its statistical properties. In Section 3 we derive measures of uncertainty such as the Shannon and Rényi Entropy. In Section 4, we derive the moments and moment generating function of the model, as well as the distribution of order statistics. In Section 5, we derive the maximum likelihood estimates. Section 6 provides results from a simulation study. In Section 7, we provide applications to three data sets. Finally, in Section 8 we provide concluding remarks.

2 The Model and Statistical Properties

In this section, we derive the TLMOWP model, sub-models, expansion of density function, and survival, hazard, and quantile functions.

2.1 The Model

Let $X = X_{(1)} = \min(X_1, X_2, ..., X_N)$ for $X_1, X_2, ..., X_n$ identically and independently distributed (iid) random variables following the TLMOW distribution in (1.9). We derive the TLMOWP distribution using the conditional distribution of X given $N = n$.

$$\begin{aligned} G_{X|N=n}(x) &= 1 - \prod_{i=1}^{n} (1 - G(x; \xi)) \\ &= 1 - S^n(x) \end{aligned} \tag{2.1}$$

where $S(x) = 1 - G(x;\xi)$ is the survival function. Thus, using the TLMO-G with the Weibull distribution as the baseline cdf, we obtain

$$G_{X|N=n}(x;b,\alpha,\delta) = 1 - \left\{1 - \left[1 - \frac{\delta^2 e^{-2x^\alpha}}{\left(1-\bar{\delta}e^{-x^\alpha}\right)^2}\right]^b\right\}^n, \tag{2.2}$$

where $b, \alpha, \delta > 0$. Hence, the distribution function of the life length of of the whole system, X, say F_θ, is given by

$$F_\theta(x) = 1 - \frac{C\left(\theta(S(x))\right)}{C(\theta)}, \tag{2.3}$$

and the corresponding pdf is given by

$$f_\theta(x) = \theta g(x) \frac{C'(\theta S(x))}{C(\theta)}, \tag{2.4}$$

where $g(x)$ is given in (1.10), and $C(\theta)$ and $C'(\theta)$ are from Johnson et al. (1994). Therefore, the distribution, density and hazard rate functions of the TLMOW-Poisson (TLMOWP) distribution are respectively given by

$$F(x;b,\alpha,\delta,\theta) = 1 - \frac{\exp\left\{\theta\left[1-\left(1-\frac{\delta^2 e^{-2x^\alpha}}{\left[1-\bar{\delta}e^{-x^\alpha}\right]^2}\right)^b\right]\right\} - 1}{\exp(\theta)-1}, \tag{2.5}$$

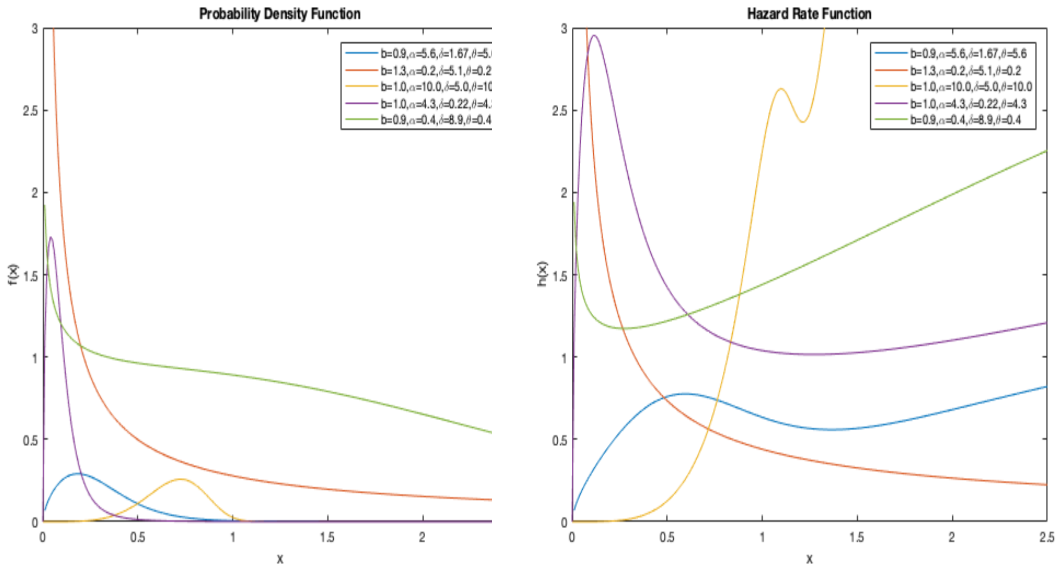
$$\begin{aligned} f(x;b,\alpha,\delta,\theta) = {} & \frac{2b\theta\delta^2\alpha x^{\alpha-1}e^{-2x^\alpha}}{\left[\exp(\theta)-1\right]\left[1-\bar{\delta}e^{-x^\alpha}\right]^3}\left(1-\frac{\delta^2 e^{-2x^\alpha}}{\left[1-\bar{\delta}e^{-x^\alpha}\right]^2}\right)^{b-1} \\ & \times \exp\left\{\theta\left[1-\left(1-\frac{\delta^2 e^{-2x^\alpha}}{\left[1-\bar{\delta}e^{-x^\alpha}\right]^2}\right)^b\right]\right\}, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} h(x) = {} & \frac{2b^\theta\delta^2\alpha x^{\alpha-1}e^{-2x^\alpha}}{\left[1-\bar{\delta}e^{-x^\alpha}\right]^3}\left(1-\frac{\delta^2 e^{-2x^\alpha}}{\left[1-\bar{\delta}e^{-x^\alpha}\right]^2}\right)^{b-1} \\ & \times \frac{\exp\left\{\theta\left[1-\left(1-\frac{\delta^2 e^{-2x^\alpha}}{\left[1-\bar{\delta}e^{-x^\alpha}\right]^2}\right)^b\right]\right\}}{\exp\left\{\theta\left[1-\left(1-\frac{\delta^2 e^{-2x^\alpha}}{\left[1-\bar{\delta}e^{-x^\alpha}\right]^2}\right)^b\right]\right\}-1}. \end{aligned} \tag{2.7}$$

for $b, \alpha, \delta, \theta > 0$.

Figure 2.1:
Plots of the pdf and hrf for the TL-MO-WP Distribution for Select Values



The plot of the pdf shows that the TLMOWP distribution can handle data that is left or right-skewed, while the hazard function exhibits bathtub, upside-down bathtub, and upside-down bathtub followed by bathtub shapes.

2.2 Sub-Models

In this section, we present some sub-models of the TLMOWP distribution.

- When $\alpha = 1$, the TLMOWP distribution becomes the Topp-Leone Marhsall-Olkin Exponential (TLMOE) distribution.
- When $\delta = 1$, the TLMOWP distribution becomes the Topp-Leone Weibull Poisson (TLWP) distribution.
- When $\delta = 1$ and $\alpha = 1$, the TLMOWP distribution becomes the Topp-Leone Exponential Poisson (TLEP) distribution.
- When $\alpha = 2$, the TLMOWP distribution becomes the Topp-Leone Marshall-Olkin Raleigh Poisson (TLMORP) distribution.
- When $\delta = 1$ and $\alpha = 2$, the TLMOWP distribution becomes the Topp-Leone Raleigh Poisson (TLRP) distribution.

2.3 Expansion of Density Function

In this section, we derive the series representation for the TLMOWP probability density function. The density function from equation (2.6) can be written in series form as follows:

$$f(x; b, \alpha, \delta, \theta) = \sum_{i,j,k,m,q=0}^{\infty} C(j, k, m, p, q; b, \theta, \delta) \times f(x; \alpha). \tag{2.8}$$

Therefore, the pdf of the TLMOWP distribution is a linear combination of the Weibull distribution with weights

$$C(i, j, k, m, q, b, \delta, \theta) = \frac{(-1)^{i+k+m}\binom{b-1}{i}\binom{j}{k}\binom{bk}{m}\binom{-(3+2i+2m)}{q}2b\theta^{j+1}\delta^{2+2i+2m}\overline{\delta}^{q}}{j!q!(e^{\theta}-1)(2+i+2m+1)}. \tag{2.9}$$

(See appendix for the derivation).

2.4 Quantile Function

In this section, we will derive the quantile function for the TLMOWP distribution. The quantile function for a probability distribution is useful for calculating quartiles. A quantile function is also sometimes referred to as the inverse distribution function, since the quantile function is the inverse of monotonically increasing distribution functions. Let $F(x)$ be the distribution function from equation (2.5). For any value $Q \in (0, \infty)$, the distribution equation $F(Q) = p \in (0, 1)$. We obtain quantile values for the TLMOWP distribution by solving the non-linear equation

$$Q(p) = \log\left\{\frac{\delta}{\left[1-\left(1-\frac{\log\left[(1-p)(e^{\theta}-1)+1\right]}{\theta}\right)^{\frac{1}{b}}\right]^{\frac{1}{2}}} + \overline{\delta}\right\}^{\frac{1}{\alpha}}. \tag{2.10}$$

2.5 Moment and Moment Generating Function

In this section, we derive the raw moments as well as the moment generating function for the TLMOWP distribution.

The r^{th} moment of the TLMOWP distribution is given by

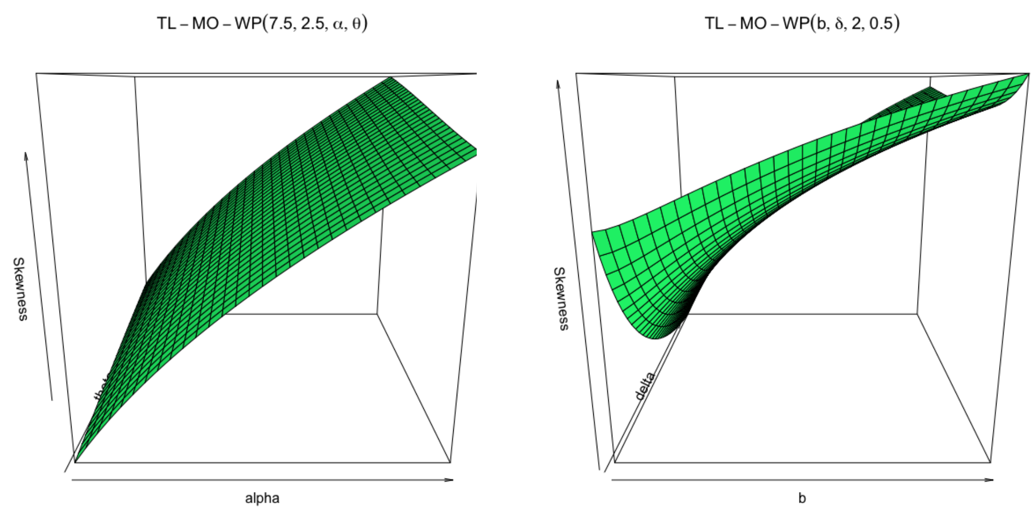
$$\mu_r' = E[X^r] = \int_{-\infty}^{\infty} X^r f_{\text{TLMOWP}}(X)\ \text{dx}. \tag{2.11}$$

Therefore, the r^{th} moment of the TLMOWP distribution is given by

$$\mu'_r = \sum_{i,j,k,m,q=0}^{\infty} (-1)^{i+k+m} \binom{b-1}{i} \binom{j}{k} \binom{bk}{m} \binom{-(3+2i+2m)}{q} \frac{2b\theta^{j+1}\delta^{2+2i+2m}\overline{\delta}^{q}}{j!(e^{\theta}-1)(2+i+2m+q)} \times \Gamma\left(\frac{r}{\alpha}+1\right). \tag{2.12}$$

We present plots of the skewness and kurtosis in Figures 2.2 and 2.3.

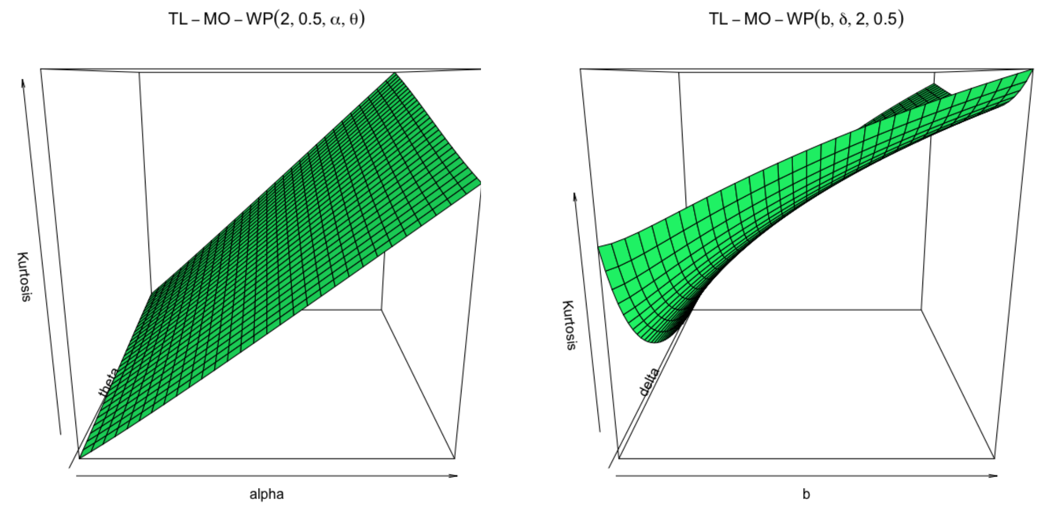
Figure 2.2:
Skewness



From the 3-D plots of skewness and kurtosis, we observe the following:

- When we fix the parameters b and δ, skewness of the TLMOWP distribution increases as α and θ increase.
- When we fix the parameters α and θ, skewness of the TLMOWP distribution increases as b and δ increase.
- When the parameters b and δ are fixed, the kurtosis of the TLMOWP distribution increases as α and θ increase.
- When we fix the parameters α and θ, kurtosis of the TLMOWP distribution increases as b and δ increase.

Figure 2.3:
Kurtosis



The moment generating function (mgf) of the TLMOWP distribution is given by

$$e^{tx} = \sum_{s=0}^{\infty} \frac{(tx)^s}{s!}. \tag{2.13}$$

Using the expansion, the moment generating function, $M_x(t)$, can be written

$$M_X(t) = E[e^{tX}] = \sum_{s=0}^{\infty} \frac{t^s}{s!} E[X^s]. \tag{2.14}$$

Substituting s in the r^{th} moment in (2.12)

$$M_X(t) = \sum_{i,j,k,m,q,s=0}^{\infty} (-1)^{i+k+m} \binom{b-1}{i} \binom{j}{k} \binom{bk}{m} \binom{-(3+2i+2m)}{q} \Gamma\left(\frac{s}{\alpha}+1\right) \times \frac{2b\theta^{j+1}\delta^{2+2i+2m}\overline{\delta}^q}{j!s!(e^{\theta}-1)(2+i+2m+q)^{\frac{r+\alpha}{\alpha}}}. \tag{2.15}$$

2.6 Distribution of Order Statistics

Order statistics play a vital role in statistics. In this section, we present the distribution of order statistics for the TLMOWP distribution. The probability density function of the i^{th} order statistic is given by

$$f_{i:n}(x) = \frac{n!f(x)}{(i-1)!(n-i)!}\left[F(x)\right]^{i-1}\left[1-F(x)\right]^{n-1}.$$

Therefore,

$$f_{i:n}(x) = \frac{2bn!}{(i-1)!(n-i)!}\sum_{p=0}^{n-i}\sum_{p,r,s,t,u,v,w=0}^{\infty}\binom{n-i}{p}\binom{p+i-1}{q}\binom{q}{r}\binom{b-1}{s}$$

$$\times\binom{t}{u}\binom{bu}{v}\binom{-(3+2s+2v)}{w}\cdot\frac{\overline{\delta}^{w}\theta^{t+1}\delta^{2+2s+2v}(r+1)^{t}(-1)^{p+q+r+s+t+u+v+1}}{(e^{\theta}-1)^{q+1}t!(2+2s+2v+w)}$$

$$\times g(x;\alpha), \tag{4.15}$$

where $g(x;\alpha)$ is the Weibull distribution with parameter $\alpha > 0$. Thus, the distribution of the i^{th} order statistic from the TLMOWP distribution is a linear combination of the Weibull distribution with parameter α, where

$$\frac{2bn!}{(i-1)!(n-i)!}\sum_{p=0}^{n-i}\sum_{p,r,s,t,u,v,w=0}^{\infty}\binom{n-i}{p}\binom{p+i-1}{q}\binom{q}{r}\binom{b-1}{s}$$
$$\times\binom{t}{u}\binom{bu}{v}\binom{-(3+2s+2v)}{w}\cdot\frac{\overline{\delta}^{w}\theta^{t+1}\delta^{2+2s+2v}(r+1)^{t}(-1)^{p+q+r+s+t+u+v+1}}{(e^{\theta}-1)^{q+1}t!(2+2s+2v+w)} \tag{4.16}$$

are the coefficients. (See appendix for derivation)

The r^{th} moment of the i^{th} order statistic is given by

$$E\left[X_{i:n}^{r}\right] = \frac{2bn!}{(i-1)!(n-i)!}\sum_{p=0}^{n-i}\sum_{p,q,r,s,t,u,v,w=0}^{\infty}\binom{n-i}{p}\binom{p+i-1}{q}\binom{q}{r}\binom{t}{u}\binom{bu}{v}$$
$$\times\binom{-(3+2s+2v)}{w}\frac{(-1)^{p+q+r+s+t+u+v+w+1}\theta^{t+1}\delta^{2+2s+2v}\overline{\delta}^{w}(r+1)^{t}}{(e^{\theta}-1)^{q+1}t!(2+2s+2v+w)}\cdot E_{g_w}(X^{r}), \tag{4.17}$$

where $E_{g_w}(X^r)$ is the r^{th} moment of the Weibull distribution with parameter α.

3 Uncertainty Measures

In this section, we present the Shannon entropy (Shannon, 1948) and the Rényi entropy (Rényi, 1961) for the TLMOWP distribution. Entropy plays an important role in information theory since it is a good measure of uncertainty. The entropy of a random variable is defined in terms of its probability distribution function.

3.1 Shannon Entropy

Shannon entropy for the TLMOWP distribution is defined to be

$$\mathrm{H}\left[f_{\mathrm{TLMOWP}}(X)\right]=E\left[-\log\left(f_{\mathrm{TLMOWP}}(X;b,\alpha,\delta,\theta)\right)\right]. \tag{3.1}$$

Therefore, Shannon entropy for TLMOWP distribution can be rewritten as

$$\begin{aligned}
H[f(X;b,\delta,\alpha,\theta)] &= \log\left[\frac{(e^{\theta}-1)}{2b\theta\delta^{2}\alpha}\right]+6\sum_{i,j,k,m,q=0}\frac{(-1)^{i+k+m+1}}{(e^{\theta}-1)j!}\binom{b-1}{i}\binom{j}{k}\binom{bk}{m}\\
&\times\binom{-(3+2i+2m)}{q}\frac{b\theta^{j+1}\delta^{2+2i+2m}\overline{\delta}^{a+q}}{2+i+2m+q+a}-(b-1)\\
&\times\sum_{d,i,j,k,m,q=0}^{\infty}\sum_{c=1}^{\infty}\frac{(-1)^{i+k+m+1}2\alpha b\theta^{j+1}\delta^{2(1+i+m+c)}}{j!(e^{\theta}-1)(2+i+2m+q+d+2c)c}\\
&\times\overline{\delta}^{q+d}\binom{b-1}{i}\binom{j}{k}\binom{bk}{m}\binom{-(3+2i+m)}{q}\binom{-2c}{d}\\
&-\theta\left[1-\sum_{d,f,g,i=0}^{\infty}\sum_{j,k,m,q=0}^{\infty}\sum_{c=1}^{\infty}\frac{(-1)^{2f+i+k+m}\binom{b-1}{i}\binom{j}{k}\binom{bk}{m}\binom{-(3+2i+m)}{q}\binom{b}{f}\binom{-2f}{g}}{j!(e^{\theta}-1)(2+i+2m+q+2f+g)}\right.\\
&\left.\times 2\alpha b\theta^{j+1}\delta^{2(1+i+m+c+f)}\overline{\delta}^{q+g}\right].
\end{aligned} \tag{3.2}$$

3.2 Rényi Entropy

Rényi entropy (1961) is an extension of Shannon entropy defined to be

$$I_R(\nu)=\frac{1}{1-\nu}\log\left(\int_0^{\infty}\left[f_{\mathrm{TLMOWP}(x;b,\alpha,\delta,\theta)}\right]^{\nu}\mathrm{dx}\right),\qquad \nu>0,\ \nu\neq 1. \tag{3.3}$$

Rényi entropy tends to Shannon entropy as $\nu\to 1$. Assume that $f(x)$ is the pdf of the TLMOWP distribution with parameters b, α, δ and θ. Consequently, Rényi entropy from (3.3) can be expressed as

$$\begin{aligned}
I_R(\nu) &= \frac{1}{1-\nu}\left[\frac{2b\theta\delta^{2}\alpha}{(e^{\theta}-1)}\right]^{\nu}\sum_{i,j,k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{i+k+m}(\theta\nu)^{j}\delta^{2i+2m}\overline{\delta}^{p}}{j!\alpha[2i+2\nu+2m+p]^{\nu-1+\frac{1}{\alpha}(1-\nu)}}\\
&\times\binom{b\nu-\nu}{i}\binom{j}{k}\binom{bk}{m}\binom{-(3\nu+2i+2m)}{p}\mathit{\Gamma}\left(\frac{1-\nu}{\alpha}+\nu-1\right)
\end{aligned} \tag{3.4}$$

for $\nu>0$ and $\nu\neq 1$.

4 Estimation

In this section, we use the maximum likelihood method to provide a basis for estimation of the parameters, b, α, δ, and θ. We denote the parameter vector for the TLMOWP distribution as $\Theta = [b, \alpha, \delta, \theta]^T$. The log-likelihood function, L, for a single observation from the TLMOWP distribution is given by

$$L = \log(2) + \log(b) + \log(\theta) + 2\log(\delta) + \log(\alpha) + (\alpha - 1)\log(x) - 2x^{\alpha} - \log(e^{\theta} - 1)$$
$$-3\log(1 - \overline{\delta}e^{-x^{\alpha}}) + (b-1)\log\left\{1 - \frac{\delta^2 e^{-2x^{\alpha}}}{[1 - \overline{\delta}e^{-x^{\alpha}}]^2}\right\} + \theta\left[1 - \left\{1 - \frac{\delta^2 e^{-2x^{\alpha}}}{[1 - \overline{\delta}e^{-x^{\alpha}}}]^2\right\}^b\right]. \tag{5.1}$$

The score function associated with the log-likelihood function is $U(\Theta) = \left[\frac{\partial L}{\partial b}, \frac{\partial L}{\partial \alpha}, \frac{\partial L}{\partial \delta}, \frac{\partial L}{\partial \theta}\right]^T$. The first-order partial derivatives of the log-likelihood function are given in the appendix section.

5 Simulation Study

In this section, we determine the reliability of the parameters of the TLMOWP distribution by performing multiple Monte-Carlo simulations of sample size $n = 25, 50, 100, 200, 400, 800$ using R. We consider 1000 samples for the parameter values given in Table 5.1 and Table 5.2. Table 5.1 and Table 5.2 list the mean maximum likelihood estimates of the TLMOWP parameters along with their associated bias and root mean square errors (RMSE). The bias and RMSE are given by

$$Bias(\hat{\theta}) = \frac{\sum\limits_{i=1}^{N}\hat{\theta}_i}{N} - \theta, \tag{6.1}$$

and

$$RMSE(\hat{\theta}) = \sqrt{\frac{\sum\limits_{i=1}^{N}(\hat{\theta}_i - \theta)^2}{N}}, \tag{6.2}$$

respectively.

From Tables 5.1 and 5.2, we can see that in all cases the RMSE and bias decreases within a sufficiently sized tolerance as the sample size increases. Furthermore, the mean value of the parameters is tending towards the true value. Hence, we can conclude that estimation of parameters is consistent under the TLMOWP model.

Table 5.1:
Simulation Results: Mean, RMSE and Average Bias (Set 1)

	$b = 1.00, \delta = 0.01, \alpha = 1.00, \theta = 0.01$			$b = 1.0, \delta = 0.1, \alpha = 1.0, \theta = 0.5$		
n	Mean	RMSE	Bias	Mean	RMSE	Bias
25	1.843489	3.097031	0.843489	3.870500	21.411908	2.870500
50	1.446717	1.597276	0.446717	1.521597	5.297191	0.521597
100	1.223600	0.659300	0.223600	1.128091	0.752150	0.128091
200	1.086978	0.265911	0.086978	1.049350	0.402809	0.049350
400	1.076036	0.201140	0.076036	1.025239	0.239318	0.025239
800	1.001685	0.014811	0.001685	1.014153	0.171294	0.014153
25	0.013984	0.011706	0.003984	0.166036	0.276563	0.066036
50	0.012677	0.009227	0.002677	0.145694	0.112544	0.045694
100	0.011518	0.004843	0.001518	0.142870	0.094887	0.042870
200	0.011209	0.003235	0.001209	0.133027	0.081933	0.033027
400	0.010985	0.002699	0.000985	0.126679	0.059867	0.026679
800	0.010247	0.001135	0.000247	0.120595	0.048488	0.020595
25	0.954881	0.200440	-0.045119	1.275077	0.900785	0.275077
50	0.960478	0.148973	-0.039522	1.155374	0.625781	0.155374
100	0.972630	0.108779	-0.027370	1.121939	0.488281	0.121939
200	0.980893	0.066908	-0.019107	1.085300	0.362137	0.085300
400	0.981837	0.061137	-0.018163	1.031612	0.162606	0.031612
800	0.999014	0.015110	-0.000986	1.017462	0.114379	0.017461
25	0.142985	0.397621	0.132985	1.589740	1.736631	1.089740
50	0.113219	0.337011	0.103219	1.497565	1.589730	0.997565
100	0.075056	0.162904	0.065056	1.453125	1.532443	0.953125
200	0.049199	0.127904	0.039199	1.226390	1.257832	0.726390
400	0.046310	0.088076	0.036310	1.073216	0.990926	0.573216
800	0.039803	0.059860	0.029803	0.936695	0.821942	0.436695

Table 5.2:
Simulation Results: Mean, RMSE and Average Bias (Set 2)

	$b = 1.0, \delta = 0.1, \alpha = 1.0, \theta = 1.5$			$b = 1.00, \delta = 0.01, \alpha = 1.00, \theta = 0.50$		
n	Mean	RMSE	Bias	Mean	RMSE	Bias
25	2.792206	12.694827	1.792206	1.712998	2.776500	0.712998
50	1.814854	5.855332	0.814854	1.540414	5.861355	0.540414
100	1.233464	1.085688	0.233464	1.237808	0.736014	0.237808
200	1.153543	0.516273	0.153542	1.127867	0.317205	0.127867
400	1.105255	0.305964	0.105255	1.099338	0.216721	0.099338
800	1.087791	0.234418	0.087791	1.018735	0.018735	0.018735
25	0.131138	0.132985	0.031138	0.012147	0.012617	0.002147
50	0.138438	0.115495	0.038438	0.011723	0.007965	0.001723
100	0.141833	0.110302	0.041833	0.011400	0.008701	0.001400
200	0.145513	0.118560	0.045513	0.010809	0.003882	0.000809
400	0.135397	0.104514	0.035397	0.010381	0.002594	0.000381
800	0.122123	0.077060	0.022123	0.009189	0.000811	-0.000811
25	1.207604	0.711097	0.207604	0.990924	0.195055	-0.009076
50	1.067594	0.512852	0.067594	0.974919	0.146616	-0.025081
100	1.034464	0.400540	0.034464	0.977473	0.123684	-0.022527
200	0.991656	0.251724	-0.008344	0.982021	0.083168	-0.017979
400	0.972777	0.163998	-0.027223	0.985595	0.062634	-0.014405
800	0.969457	0.122721	-0.030543	1.023479	0.023479	0.023479
25	2.152334	1.597134	0.652334	0.599887	0.411314	0.099887
50	2.158237	1.446099	0.658237	0.551596	0.287312	0.051596
100	2.135010	1.480205	0.635010	0.540552	0.282067	0.040552
200	2.040846	1.426922	0.540846	0.532534	0.213519	0.032534
400	1.852365	1.312707	0.352365	0.532896	0.078254	0.032896
800	1.686933	1.076211	0.186933	0.499949	0.000051	-0.000051

6 Application

In this section, we use the R package to fit three real data sets to the TL-MOWP model. In order to offer a meaningful comparison, we also fit the data to six other distributions: the Topp-Leone Marshall-Olkin Exponentiated Poisson (TLMOEP), Topp-Leone Marshall-Olkin Weibull (TLMOW), Beta Lindley Poisson (BLP), Burr XII Poisson (BXIIP), Exponentiated Power Lindley Poisson (EPLP), and Exponentiated Weibull Poisson (EWP) distributions.

The distributions have the following density functions:

$$f_{TLMOEP}(x;b,\delta,\theta)=\frac{2b^{\theta}\delta^{2}e^{-2x}}{[\exp(\theta)-1]\left[1-\overline{\delta}e^{-x}\right]^{3}}\left(1-\frac{\delta^{2}e^{-2x}}{\left[1-\overline{\delta}e^{-x}\right]^{2}}\right)^{b-1}\times\exp\left\{\theta\left[1-\left(1-\frac{\delta^{2}e^{-2x}}{\left[1-\overline{\delta}e^{-x}\right]^{2}}\right)\right]\right\}, \quad (7.1)$$

where $b,\delta,\theta>0$,

$$f_{BLP}(x,\beta,\theta,a,b)=\frac{\theta\beta^{2}(1+x)e^{-\beta x}e^{\omega}(1-e^{\omega})^{a-1}(e^{\omega}-e^{\theta})^{b-1}}{B(a,b)(\beta+1)(e^{\theta}-1)}(1-e^{\theta})^{2-a-b} \quad (7.2)$$

where

$$\omega=\theta\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right] \quad (7.3)$$

and $\beta,\theta,a,b>0$,

$$f_{BXIIP}(x;s,k,c,\lambda)=\frac{cks^{-c}\lambda}{1-e^{-\lambda}}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k-1}\exp\left\{-\lambda\left[1-\left(1+\left(\frac{x}{s}\right)^{c}\right)^{-k}\right]\right\} \quad (7.4)$$

where $s,k,c,\lambda>0$,

$$f_{EPLP}(x;\alpha,\beta,\omega,\theta)=\frac{\alpha\beta^{2}\omega\theta}{(\beta+1)(e^{\theta}-1)}(1+y^{\alpha})y^{\alpha-1}e^{-\beta y^{\alpha}}\left[1-\left(1+\frac{\beta y^{\alpha}}{\beta+1}\right)e^{-\beta y^{\alpha}}\right]^{\omega-1}\times\ \exp\left\{\theta\left[1-\left(1+\frac{\beta y^{\alpha}}{\beta+1}\right)e^{-\beta y^{\alpha}}\right]^{\omega}\right\}, \quad (7.5)$$

where $\alpha,\beta,\omega,\theta>0$, and

$$f_{EWP}(x;\alpha,\beta,\theta,\gamma)=\frac{\alpha\gamma\theta\beta^{\gamma}x^{\gamma-1}}{e^{\theta}-1}e^{-(\beta x)^{\gamma}}(1-e^{-(\beta x)^{\gamma}})^{a-1}e^{\theta(1-e^{-(\beta x)^{\gamma}})^{\alpha}}, \quad (7.6)$$

where $\alpha,\beta,\gamma,\theta>0$. See (1.10) for the density function of the TLMOW distribution.

We considered the values of the -2log-likelihood (-2log$L$), Akaike Information Criterion (AIC) given by $AIC=2p-2\log(L)$ (Akaike,1973), Consistent Akaike Information Criterion (CAIC) given by $AICC=AIC+\frac{2p(p+1)}{n-p-1}$ (Cavanaugh,1997), Bayesian Information Criterion (BIC) given by $BIC=p\log(n)-2\log(L)$ (Wit et al.,2012), two-sample Cramer von Mises criterion (W^{*}) (Anderson,2009), two-sample Anderson-Darling test (A^{*}) (Anderson,2009), sum of squares error (SS), and the p-value for the Kolmogorov-Smirnov (K-S) test. In the preceding sentence, L is the value of the maximum likelihood estimate for the distribution parameters, n is the size of the data set, and p is the number of parameters being estimated. These values were used as a measure of the goodness of fit between the data sets and the statistical models.

The AIC, CAIC, and BIC are used to compare the amount of information lost as a result of adding parameters between models, with the CAIC providing more accurate results, even in small samples. Hence, models with lower AIC, CAIC, and BIC are preferred. Similarly, models with lower SS, W^*, and A^* are preferred. For the K-S test, the best model is the one with the largest p-value.

Additionally, we performed a likelihood ratio test with the best fit parameters for nested models. That is, we will test the null hypothesis that the TLMOEP distribution is the true distribution against the alternate hypothesis that it is the TLMOWP distribution. The likelihood statistic, λ, was calculated using the difference of the -2log(L) values between the two distributions in question. The test statistic is the value of the χ^2 distribution with score λ and degrees of freedom given by the difference in number of parameters. Since the TLMOEP distribution has 3 parameters, and the TLMOWP distribution has 4 parameters, $df = 1$.

For each data set, we calculated the covariance matrix for the maximum log-likelihood parameter estimates. The entries in a covariance matrix represent the covariances of the row and column parameters. A corollary of this is that the main diagonal shows the variances of the parameters (Park,2018). By definition, the covariance matrices for the TLMOWP distribution are of size 4×4, based on the four parameters of the distribution. Let $I^{-1}(\Theta)$ denote the covariance matrix.

Glass Fiber Data

The first data set given in Table 6.1 is a collection of 63 measurements of the strength of 1.5 centimeter glass fibers from Smith and Naylor (1987).

0.55	0.93	1.25	1.36	1.49	1.52	1.58	1.61	1.64	1.68	1.73
1.81	2.00	0.74	1.04	1.27	1.39	1.49	1.53	1.59	1.61	1.66
1.68	1.76	1.82	2.01	0.77	1.11	1.28	1.42	1.50	1.54	1.60
1.62	1.66	1.69	1.76	1.84	2.24	0.81	1.13	1.29	1.48	1.50
1.55	1.61	1.62	1.66	1.70	1.77	1.84	0.84	1.24	1.30	1.48
1.51	1.55	1.61	1.63	1.67	1.70	1.78	1.89			

Table 6.1:
Glass Fiber Data Set

Table 6.1 shows the estimated parameters, along with the standard error in parentheses, for the glass fiber data set for each of the seven distributions being compared, as well as the goodness of fit statistics corresponding to them. From

Figure 6.1, we can see that the TLMOWP model is the best model under the Cramer von-Mises, Anderson-Darling, and sum of square error criteria since its W^*, A^*, and SS values are lower than any of the other distributions. However, if we consider the values of the AIC, CAIC, and BIC the TLMOW model seems to be outperforming the TLMOWP model. These are not the only criteria that can be used, hence the need for the K-S p-value, W^*, A^*, and SS. These values show that the TLMOWP model is better than the TLMOW model for this particular data set.

Figure 6.1:
Estimates of Models for Glass Fiber Data Set

	Estimates							Statistics							
Model	a	b	α	β	δ	θ	ω	$-2\log L$	AIC	$CAIC$	BIC	W^*	A^*	SS	p
TLMOWP	-	1.0410 (0.3005)	2.4245 (0.2931)	-	183.59 (177.45)	4.3727 (2.4092)	-	23.6	31.6	32.3	40.2	0.0872	0.4988	0.0766	0.4901
TLMOEP	-	3.5777 (0.7174)	-	-	28.0340 (24.1111)	165.07 (272.24)	-	28.8	34.8	35.2	41.2	0.1979	1.0941	0.1840	0.1453
TLMOW	-	0.9990 (0.2997)	2.4890 (0.1983)	-	46.5775 (30.4731)	-	-	24.6	30.6	31.0	37.0	0.0953	0.5508	0.0843	0.4599
BLP	0.6270 (0.4835)	212.73 (16.8359)	-	1.0992 (0.2785)	-	19.3516 (11.2307)	-	28.3	36.3	37.0	44.9	0.1716	0.9592	0.1664	0.1762
EPLP	-	-	4.5205 (1.0644)	0.2508 (0.1934)	-	2.5089 (1.6337)	0.5640 (0.3362)	25.6	33.6	34.3	42.2	0.1148	0.6525	0.0987	0.4226
EWP	-	-	5.5015 (1.3951)	0.6467 (0.05085)	-	2.7822 (1.5679)	0.5782 (0.3425)	26.0	34.0	34.7	42.5	0.1293	0.7283	0.1093	0.3711
	c	k	s	λ				$-2\log L$	AIC	$CAIC$	BIC	W^*	A^*	SS	p
BXIIP	5.8540 (0.6605)	11.9921 (197.38)	5.0439 (14.2096)	63.2451 (260.15)				30.5	38.5	39.1	47.0	0.2393	1.3140	0.2109	0.108

In order to compare the TLMOWP and the TLMOEP, we will test the hypothesis:

$$H_0 : TLMOEP(b, \delta, \theta)$$
$$H_a : TLMOWP(b, \alpha, \delta, \theta).$$

Since these are nested models, we will make use of the likelihood ratio test where the value of the likelihood ratio statistic is $\lambda = 28.8 - 23.6 = 5.2$. This follows a χ^2 distribution with 1 degree of freedom. The likelihood ratio test has p-value

$$\chi^2_1(5.2) = 0.022587 < 0.05.$$

Therefore, we reject the null hypothesis and can conclude that the TLMOWP distribution performs significantly better than the TLMOEP distribution for the glass fiber data set at the 5% significance level.

Figure 6.2 shows a plot of the fitted density functions superimposed over a histogram of the glass fiber data, as well as a plot of the observed probabilities of the different distributions against the expected probability. The plots show that the TLMOWP distribution provides the best fit for the data.

Figure 6.2:
Histogram with Fitted Densities and Sum of Square Errors for Glass Fiber Data

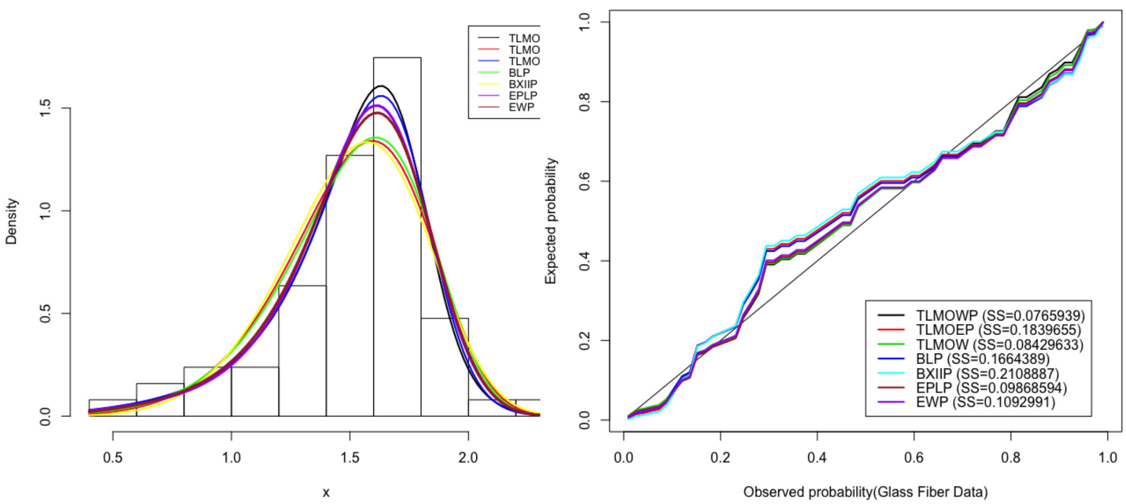


Figure 6.3:
Experimental CDF and Kaplan-Meier Survival Plot for Glass Fiber Data

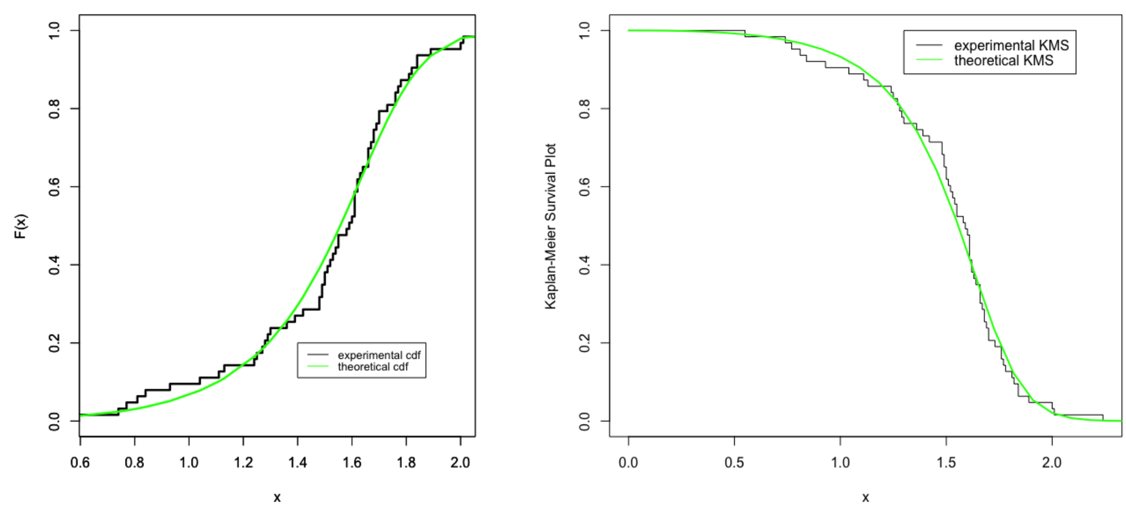
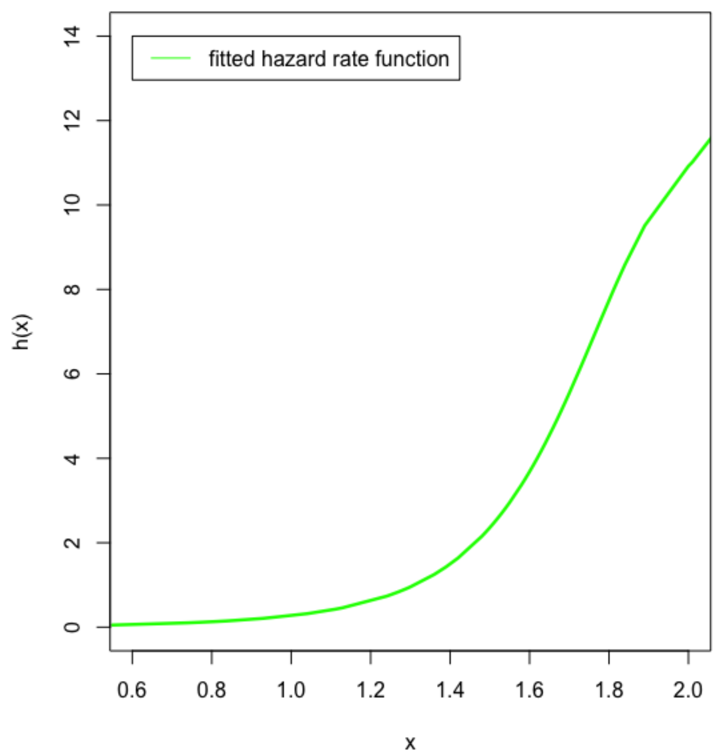


Figure 6.3 shows that the TLMOWP distribution provides a good fit for the glass fiber data set since the green lines representing the theoretical values match the black lines representing the data.

Figure 6.4:
Experimental Hazard Rate Function for Glass Fiber Data



The hazard rate function plot for the glass fiber data is increasing monotonically as shown in Figure 6.4.

The asymptotic covariance matrix for the glass fibers data set is given by

$$I^{-1}(\hat{\Theta}) = \begin{bmatrix} 0.09032 & -0.07355 & 45.4138 & 0.1583 \\ -0.07355 & 0.08590 & 35.9201 & -0.3352 \\ -45.4138 & 35.9201 & 31488 & 102.08 \\ 0.1583 & -0.3352 & 102.08 & 5.8040 \end{bmatrix}. \tag{7.8}$$

Thus, the two-sided 95% asymptotic confidence intervals for b, α, δ, and θ for the glass fiber data set are respectively given by 1.0410 ± 0.5890, 2.4245 ± 0.5744, 183.59 ± 347.7921, and 4.3727 ± 4.7218.

Kevlar Data

The second data set given in Table 6.2 contains 101 observations from Barlow, Toland, and Freeman (1984) representing the failure time in hours of kevlar

49/epoxy strands held under consistent pressure.

0.01	0.01	0.02	0.02	0.02	0.03	0.03	0.04	0.05	0.06	0.07	0.07	0.08	0.09	0.09
0.10	0.10	0.11	0.11	0.12	0.13	0.18	0.19	0.20	0.23	0.24	0.24	0.29	0.34	0.35
0.36	0.38	0.40	0.42	0.43	0.52	0.54	0.56	0.60	0.60	0.63	0.65	0.67	0.68	0.72
0.72	0.72	0.73	0.79	0.79	0.80	0.80	0.83	0.85	0.90	0.92	0.95	0.99	1.00	1.01
1.02	1.03	1.05	1.10	1.10	1.11	1.15	1.18	1.20	1.29	1.31	1.33	1.34	1.40	1.43
1.45	1.50	1.51	1.52	1.53	1.54	1.54	1.55	1.58	1.60	1.63	1.64	1.80	1.80	1.81
2.02	2.05	2.14	2.17	2.33	3.03	3.03	3.34	4.20	4.69	7.89				

Table 6.2:
Kevlar Data Set

Figure 6.5 contains the estimated parameters, along with the standard error in parentheses, for the kevlar data set for each of the seven distributions, as well as their corresponding goodness-of-fit statistics. Figure 6.5 shows that the TLMOWP model is superior to all of the other distributions under the -2log$L$ criterion. If we consider the values of the AIC, CAIC, and BIC statistics, it seems that the TLMOW model is outperforming the TLMOWP model. However, the values for W^*, A^*, SS, and the K-S p-value show that the TLMOWP model is superior

Figure 6.5:
Estimates of Models for Kevlar Data Set

Model	Estimates							Statistics							
	a	b	α	β	δ	θ	ω	$-2\log L$	AIC	$CAIC$	BIC	W^*	A^*	SS	p
TLMOWP	-	1.0705 (0.2678)	0.7063 (0.1420)	-	13.9417 (8.0463)	5.1813 (2.3586)	-	202.9	210.9	211.4	221.4	0.1152	0.7096	0.1165	0.7215
TLMOEP	-	0.7110 (0.08009)	-	-	24.5313 (17.0023)	3.8805 (1.6147)	-	209.8	215.8	216.0	223.6	0.0937	0.7414	0.0703	0.8229
TLMOW	-	1.0503 (0.2726)	0.7539 (0.09013)	-	2.6198 (0.8346)	-	-	204.5	210.5	210.7	218.3	0.1353	0.8156	0.1321	0.6207
BLP	0.7090 (0.1693)	0.5159 (0.5235)	-	2.2512 (1.8595)	-	0.7317 (1.2986)	-	204.8	212.8	213.2	223.3	0.1082	0.7091	0.1030	0.6819
EPLP	-	-	0.7894 (0.2025)	1.7951 (0.6112)	-	1.1683 (1.2586)	0.9385 (0.3828)	204.4	212.4	212.9	222.9	0.1349	0.8141	0.1292	0.6921
EWP	-	-	0.8717 (0.2410)	1.3032 (0.7400)	-	1.2662 (1.2008)	0.8589 (0.3681)	204.6	212.6	213.0	223.1	0.1408	0.8416	0.1347	0.6639
	c	k	s	λ				$-2\log L$	AIC	$CAIC$	BIC	W^*	A^*	SS	p
BXIIP	0.9352 (0.08536)	3.1903 (43.2769)	414.91 (4968.77)	89.7902 (526.60)				206.0	214.0	214.4	224.4	0.2029	1.1302	0.1996	0.3714

In order to compare the TLMOWP model to the TLMOEP model, we will test the hypothesis

$$H_0 : TLMOEP(b, \delta, \theta)$$
$$H_a : TLMOWP(b, \alpha, \delta, \theta).$$

Since these are nested models, we will make use of the likelihood ratio test where the value of the likelihood ratio statistic is $\lambda = 209.8 - 202.9 = 6.9$. The likelihood ratio test has p-value

$$\chi^2_1(6.9) = 0.00862 < 0.01.$$

Therefore, we reject the null hypothesis and conclude that the TLMOWP distribution performs significantly better than the TLMOEP distribution for the kevlar data set at the 1% significance level.

Figure 6.6 shows a plot of the fitted density functions over a histogram of the kevlar data set, as well as a plot of the observed probabilities of the different distributions against their expected probability. The plots show that the TLMOWP model provides the best fit for the kevlar data set.

Figure 6.6:
Histogram with Fitted Densities and Sum of Square Errors for Kevlar Data

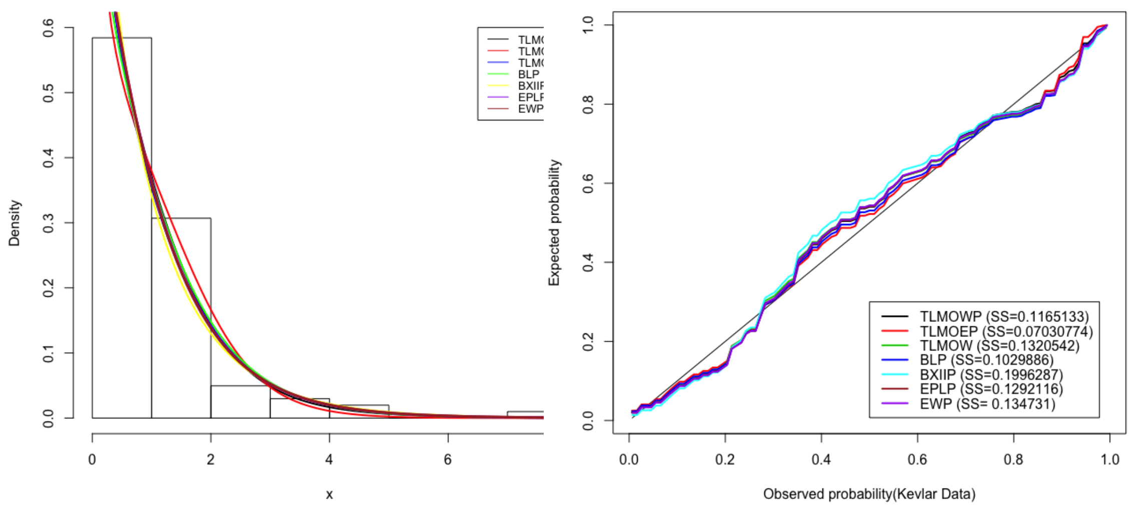


Figure 6.7:
Experimental CDF and Kaplan-Meier Survival Plot for Kevlar Data

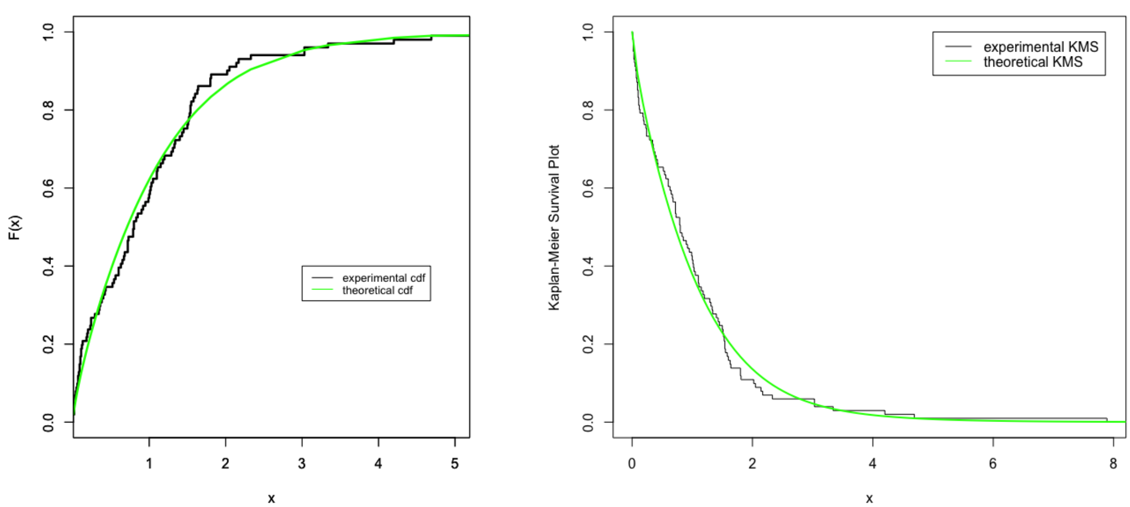


Figure 6.7 shows that the TLMOWP distribution provides a good fit for the kevlar data set since the green lines representing the theoretical values match the black lines representing the data.

Figure 6.8:
Experimental Hazard Rate Function for Kevlar Data

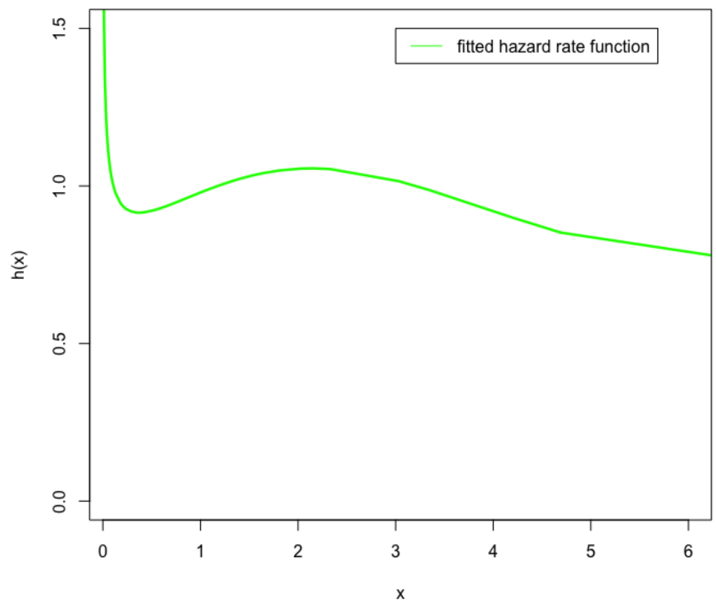


Figure 6.8 shows the plot of the hazard rate function for the kevlar data set. The hazard rate function exhibits a bathtub shape followed by an upside-down bathtub shape for the kevlar data set.

The covariance matrix for the kevlar data set is given by

$$I^{-1}(\hat{\Theta}) = \begin{bmatrix} 0.02018 & -0.03311 & 0.3062 & -0.1898 \\ -0.03311 & 0.07173 & -1.2323 & 0.2078 \\ 0.3062 & -1.2323 & 64.7431 & 10.6272 \\ -0.1898 & 0.2078 & 10.6272 & 5.5629 \end{bmatrix}. \tag{7.9}$$

Thus, the two-sided asymptotic confidence intervals for b, α, δ, and θ for the kevlar data set are respectively given by 1.0705 ± 0.2784, 0.7063 ± 0.5249, 13.9417 ± 15.7704, and 5.1813 ± 4.6227.

Carbon Fiber Data

The final data set under consideration is from Nicholas and Padgett (2006). It features 100 observations of the breaking stress of 50mm carbon fibers.

0.98	5.56	2.83	3.68	2.00	3.51	0.85	1.61	3.28	2.95	5.08	0.39	1.57	3.19	4.90
2.74	2.73	2.50	3.60	3.11	2.93	2.85	2.77	2.76	1.73	2.48	3.22	3.70	3.27	2.87
1.47	3.11	4.42	2.81	3.15	1.92	1.84	1.22	2.17	1.61	2.12	3.09	2.97	4.20	2.35
1.41	1.59	1.12	1.69	2.79	1.89	1.87	3.39	3.33	2.55	3.68	3.19	1.71	1.25	4.70
2.88	3.68	1.08	3.22	3.75	2.96	2.55	2.59	2.97	1.57	2.17	4.38	2.03	2.82	2.53
3.31	2.38	1.36	0.81	1.17	1.84	12.40	3.15	2.67	3.31	2.81	2.56	2.17	4.91	1.59
1.18	2.48	2.03	1.69	2.43	3.39	3.56	0.80	2.05	3.65					

Table 6.3:
Carbon Fiber Data Set

The entries in Figure 6.9 include estimates of the estimated parameters, along with the standard error in parentheses, and their corresponding goodness-of-fit statistics. The TLMOWP model is the best model under the W^* and A^* criteria. If we consider the values of the -2log$L$, AIC, $CAIC$, and BIC criteria, it appears that the BXIIP model outperforms the TLMOWP model. However, since the K-S p-value is higher and the SS is lower for the TLMOWP distribution, it is a better model for these criteria for the carbon fiber data set.

Figure 6.9:
Estimates of Models for Carbon Fiber Data Set

	Estimates							Statistics							
Model	a	b	α	β	δ	θ	ω	$-2\log L$	AIC	$CAIC$	BIC	W^*	A^*	SS	p
TLMOWP	-	3.6217 (1.6086)	0.6460 (0.1084)	-	10.9712 (7.3996)	6.4091 (2.1705)	-	304.4	312.4	312.8	322.8	0.0999	0.5528	0.0987	0.55
TLMOEP	-	1.4828 (0.3015)	-	-	45.7478 (35.2760)	2.5588 (1.9474)	-	318.9	324.9	325.2	332.8	0.1014	0.7668	0.0610	0.8688
TLMOW	-	3.9169 (1.7232)	0.8131 (0.06507)	-	4.9673 (2.4409)	-	-	310.4	316.4	316.7	324.2	0.1443	0.8366	0.1329	0.4631
BLP	1.5154 (2.1919)	0.9155 (0.3740)	-	1.3124 (0.3012)	-	4.6274 (7.9490)	-	311.3	319.3	319.7	329.7	0.1468	0.8539	0.1352	0.4384
EPLP	-	-	0.7687 (0.1789)	2.1012 (0.7763)	-	3.4312 (2.0343)	6.3559 (5.8534)	309.5	317.5	317.9	327.9	0.1541	0.8603	0.1403	0.4504
EWP	-	-	0.8250 (0.2062)	1.7712 (1.2125)	-	3.5049 (2.0109)	6.3803 (5.8957)	309.6	317.6	318.0	328.0	0.1574	0.8778	0.1432	0.4368
	c	k	s	λ				$-2\log L$	AIC	$CAIC$	BIC	W^*	A^*	SS	p
BXIIP	3.1850 (0.3386)	0.9863 (0.9393)	4.9099 (1.4830)	6.4271 (2.7424)				303.4	311.4	311.8	321.8	0.1084	0.5689	0.1057	0.5033

In order to compare the TLMOWP model to the nested TLMOEP model, we will test the hypothesis

$$H_0 : TLMOEP(b, \delta, \theta)$$
$$H_a : TLMOWP(b, \alpha, \delta, \theta).$$

Since these are nested models, we will use the likelihood ratio test, where the value of the likelihood ratio statistic is $\lambda = 318.9 - 304.4 = 14.5$. This follows a χ^2 distribution with 1 degree of freedom. The likelihood ratio test has p-value:

$$\chi^2_1(14.5) = 0.00014 < 0.01.$$

Therefore, we reject the null hypothesis and conclude that the TLMOWP distribution performs significantly better than the TLMOEP distribution for the carbon fiber data set at the 1% significance level.

Figure 6.10 shows a plot of the distributions superimposed over a histogram of the data set, as well as a plot of the models against a line representing the expected probability. From these plots, we observe the goodness-of-fit of the TLMOWP model compared to the others.

Figure 6.10:
Histogram with Fitted Densities and Sum of Square Errors for Carbon Fiber Data

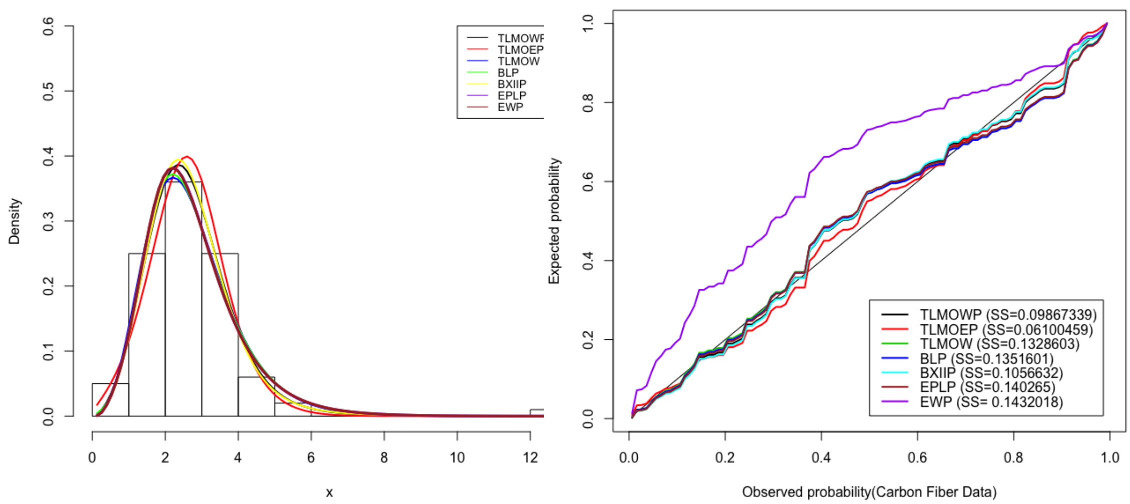


Figure 6.11:
Experimental CDF and Kaplan-Meier Survival Plot for Carbon Fiber Data

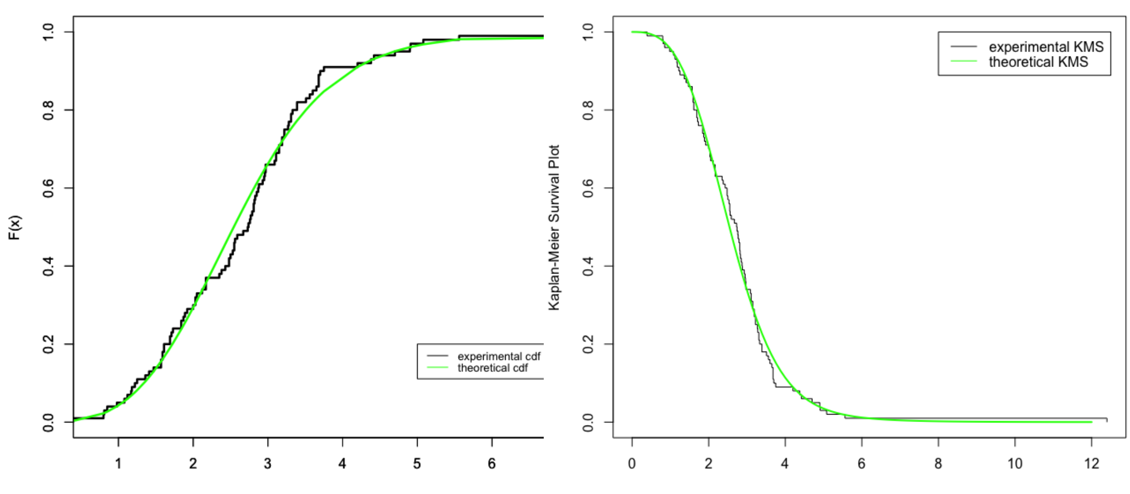


Figure 6.11 shows that the TLMOWP model provides a good fit to the car-

bon fiber data, since the green curves representing the theoretical probabilities align with the black lines representing the data.

Figure 6.12:
Experimental Hazard Rate Function for Carbon Fiber Data

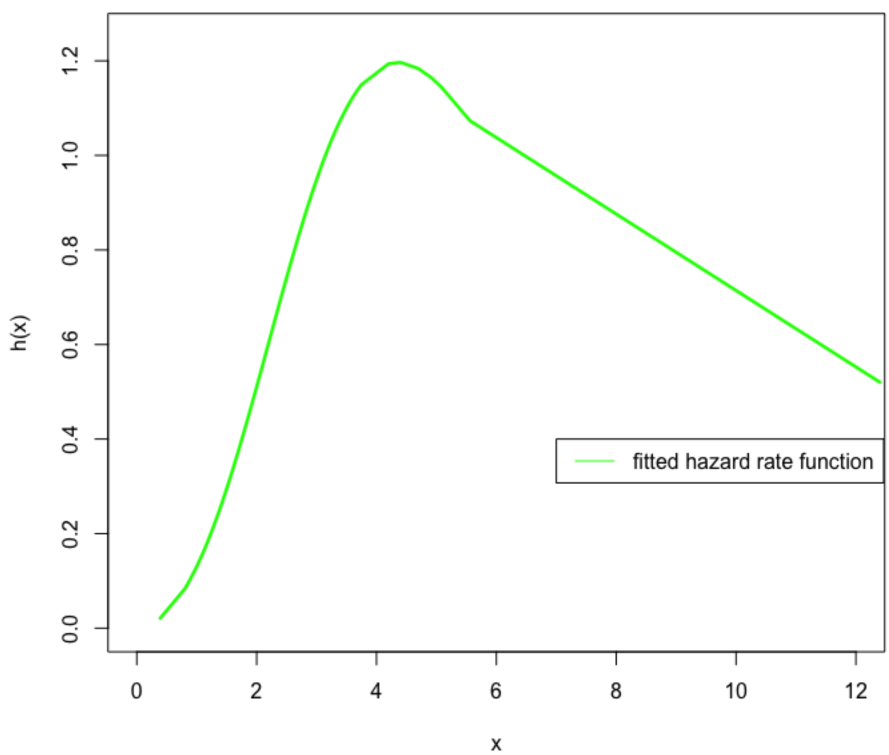


Figure 6.12 shows that the carbon fiber data exhibits an upside-down bathtub hazard rate.

The covariance matrix for the carbon fiber data set is given by

$$I^{-1}(\hat{\Theta}) = \begin{bmatrix} 2.5875 & -0.1518 & -11.6664 & 0.8842 \\ -0.1518 & 0.01176 & 0.6813 & -0.1324 \\ -11.6664 & 0.6813 & 54.7542 & -1.9870 \\ 0.8842 & -0.1324 & -1.9870 & 4.7109 \end{bmatrix}. \tag{7.10}$$

Thus, the two-sided asymptotic confidence intervals for b, α, δ, and θ for the carbon fiber data set are respectively given by 3.6217 ± 3.1527, 0.6460 ± 0.2125, 10.9712 ± 14.5029, and 6.4091 ± 4.2540.

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APPENDIX

The following url contains all the derivations referred to in the document `https://drive.google.com/file/d/15oSSjpvrX-tcyygcyxHfwup2QY3A0B7p/view?usp=sharing`