On an extension of Chebyshev-Padé approximants

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Abstract

The Padé approximation considered as interpolation problem by rational fractions is widely used to accelerate power series because to their accuracy. Its generalization in the orthogonal Chebyshev basis, a family of polynomials that presents a behaviour uniform, have been applied successfully in the resolution to various dependent problems of a variable. In this article, our approach aims to extend this generalization to functions of two variables. Numerical implementations are also presented.

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1 Introduction

Approximation methods have always been the subject of intense investigation because they have been for most of the times inescapable in the resolution to some partial differential equations. Among the ones, these consisting to approach some functions written under forms to series by rational fractions have proved their efficiency.
Padé approximants, since the pioneer paper [16] of 1892, are up to date thanks to applications in physical sciences, mathematics, and other applied sciences with the advent of computers in the 1950s as tool of convergence acceleration [1, 2]. These approximants are the locally best rational approximants to a power series. The posed problem by Padé is the following: let f a given function through its Taylor series expansion at the origin

$$f(z) = c_0 + c_1 z + c_2 z^2 + ... + c_n z^n + ... \tag{1}$$

where $c_k = f^{(k)}(0)/k!$, $k = 1, 2, ...$ It concerns to find a set of rational fractions $p(z)/q(z)$ which validly approach $f(z)$ [2].

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There are enough works on the Padé approximants [2, 3, 4, 5, 17]. Also, to generalize certain concepts of Padé approximants to functions of several variables, we can consult for example [6, 8, 10, 12, 13].
In its article [14], using the Padé approximants, H. J. Maehly has given the starting point of a method to convert Chebyshev series into rational expressions involving Chebyshev polynomials. To this we must add an other variant proposed in [7]. The term *Chebyshev-Padé approximants* will refer Padé approximants in the orthogonal Chebyshev polynomials basis.

The present paper proposes to extend the contained ideas in [7] and [14] to functions of two variables. Pseudo-spectral methods are a motivation for the research of these approximants in the aim to accelerate their convergence.
This paper is organized as follows. In the section 2, are briefly presented Padé approximants. Results on the extension of Padé approximants in the Chebyshev basis are given in section 3: the subsection 3.1 presents the uniform approximation, subsection 3.2 is devoted to Chebyshev-Padé approximants of univariate functions, and the subsection 3.3 is dedicate to the approach of Chebyshev-Padé approximants for functions of two variables. In section 4 some examples are chosen to show the high accuracy of the approach. Conclusions of the study are summarized in section 5.

2 Padé approximants

Considering the power series (1), it is possible to construct, under some conditions, a double sequences of rational fractions $p(z)/q(z)$ whose the numerator p is a polynomial of degree m and denominator q of degree n [3]:

$$\frac{p(z)}{q(z)} = \frac{a_0 + a_1 z + a_2 z^2 + ... + a_m z^m}{b_0 + b_1 z + b_2 z^2 + ... + b_n z^n}. \tag{2}$$

The coefficients a_i et b_j $(i = 0, ..., m; j = 0, ..., n)$ such as $a_m \neq 0$, $b_n \neq 0$ of (2) can be compute so that its increasing power expansion to z coincides with the one to $f(z)$ as far as possible, ie generally until z^{m+n} including term. In other words, the difference between the rational fraction (2) and the power series (1) will begin with a term of degree $m + n + 1$.

Definition 2.1. *The rational fraction (2) is said Padé approximant of the function f of order m, n if*

$$f(z) - \frac{p(z)}{q(z)} = O\left(z^{m+n+1}\right), \quad z \longrightarrow 0 \tag{3}$$

and one denote by $[m/n]_f$ (or $[p/q](z)$) this approximant.

Theorem 2.1. *[2] If the Padé approximant $[m/n]_f(z)$ exists, then it is unique.*

Concerning the convergence of these approximants, we choose the following result[2]:

Theorem 2.2 (Montessus de Balloré)**.** *Let f be a holomorphic function from the disc $\{z : |z| \leq R\}$ with the poles z_1, z_2,..., z_k. Let n the total order of multiplicity of poles. Then, the Padé approximant $[m/n]_f$ converge on f, uniformly on any compact subset $\{z : |z| < R, z \neq z_j, j = 1, 2, ..., k\}$, as $m \longrightarrow \infty$.*

In [9], this theorem genelirazed to the multivariate case.

3 Extension of Padé approximants in the Chebyshev polynomial basis

3.1 Uniform approximation

Let f a continuous function on the interval $[-1, 1]$ and expanded in the form of the Chebyshev series

$$f(x) = \sum_{k=0}^{\infty}{}' c_k T_k(x), \quad k = 0, 1, ..., \tag{4}$$

where $\sum'$ means here and in the rest of our work that the first term in the summation is halved, and a_k is defined by the relation

$$c_k = \frac{2}{\pi} \int_{-1}^{1} \left(1 - x^2\right)^{-\frac{1}{2}} f(x) T_k(x) \, dx. \tag{5}$$

The c_k are called the Chebyshev coefficients, $T_k(x) = \cos(k \arccos(x))$ is the Chebyshev polynomial of degree k of the first kind and $\omega(x) = (1 - x^2)^{-\frac{1}{2}}$ the weight function [15]. These polynomials verify the property

$$T_i(x) T_j(x) = \frac{1}{2} \left[T_{i+j}(x) + T_{|i-j|}(x)\right]. \tag{6}$$

The zeros of Chebyshev polynomials $T_k(x)$ in the interval $[-1, 1]$ are

$$x_n = \cos\left(\frac{(2n+1)\pi}{2k}\right), \; n = 0, 1, ..., k - 1. \tag{7}$$

It is indeed known that the truncated expansion of a function in the form of Chebyshev series is the near-best polynomial approximant, in the sense of the uniform norm, on the interval $[-1, 1]$. To obtain uniformly the accuracy to rational fractions approximations, we use the Chebyshev polynomials, a family of orthogonal polynomials that present an uniform behaviour.

3.2 Chebyshev-Padé approximants of functions of single variable

As in section 2, we can approach f under its form (4) by a rational fraction $[m/n]$ ([7], [14]). But, we formulate the following definition:

Definition 3.1. *One calls a Chebyshev-Padé approximant of the power series (4) every rational fraction*

$$\frac{p(x)}{q(x)} = \frac{\sum_{k=0}^{m}{}' a_k T_k(x)}{\sum_{k=0}^{n}{}' b_k T_k(x)}, \tag{8}$$

satisfying

$$\Big(fq - p\Big) = O\Big(T_{m+n+1}(x)\Big), \tag{9}$$

where the notation $O\Big(T_{m+n+1}(x)\Big)$ means that the first term nonzero in the orthogonal expansion of the function has an index greater than or to $m+n+1$.

Theorem 3.1. *[2] If it exists, the Chebyshev-Padé approximant (8) satisfying the condition (9) is unique.*

To extend the definition (8) to functions of two variables, we recall the Padé-Chebyshev method in the case univariate functions showed in [7] for $n \leq m$:

$$\frac{1}{2}\sum_{j=0}^{n}{}' b_j \left(c_{i+j} + c_{|i-j|}\right) = 0, \quad i = m+1, m+2, \ldots, m+n, \tag{10}$$

$$\frac{1}{2}\sum_{j=0}^{n}{}' b_j \left(c_{i+j} + c_{|i-j|}\right) = a_i, \quad i = 0, 1, 2, \ldots, m. \tag{11}$$

The equations (10) determine the coefficients b_j, and the equations (11) product the a_i. Experience has shown that the system of equations (10) is fairly well conditioned.
In the following discussion, we propose an analogous approach to (8) for functions of two variables i.e the calculus of coefficients a_{ij} and b_{rs} in expressions

$$p(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{m} a_{ij} T_i(x) T_j(y) \tag{12}$$

$$q(x,y) = \sum_{r=0}^{n}\sum_{s=0}^{n} b_{rs} T_r(x) T_s(y) \tag{13}$$

3.3 Chebyshev-Padé approximants of fonctions of two variables

We start by expressing the following theorem:

Theorem 3.2. *[15] Let $f : [-1,1] \times [-1,1] \longrightarrow \mathbb{C}$ a continuous function and of bounded variation in the interval $I = [-1,1] \times [-1,1]$ (see also [15] for a definition of bounded variation for bivariate functions). If one of its partial derivatives exists and is bounded in I, the function f has a bivariate Chebyshev expansion,*

$$f(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij} T_i(x) T_j(y), \tag{14}$$

converges uniformly on I.

Proof. We can refer to [15]. $\square$

This means that the truncated bivariate Chebyshev (14) in x and y of respective degrees n and m can being defined for functions satisfying hypothesis of theorem 3.2 by

$$f(x,y) \approx \sum_{i=0}^{n}\sum_{j=0}^{m} c_{ij} T_i(x) T_j(y). \tag{15}$$

where the coefficients c_{ij} are calculed in [11]:

$$c_{ij} = \frac{\epsilon_{ij}}{(n+1)(m+1)} \sum_{k=0}^{n}\sum_{l=0}^{m} f(x_k, y_l) \cos\left(\frac{i(2k+1)\pi}{2(n+1)}\right) \cos\left(\frac{j(2l+1)\pi}{2(m+1)}\right) \tag{16}$$

with

$$\begin{array}{lllllllll} \epsilon & = & 4 & \text{for} & i \neq 0 & \text{and} & j \neq 0, & & \\ \epsilon & = & 2 & \text{for} & i = 0 & \text{and} & j \neq 0 & \text{or} & i \neq 0 \quad \text{and} \quad j = 0, \\ \epsilon & = & 1 & \text{for} & i = 0 & \text{and} & j = 0. & & \end{array}$$

By the relation (9) and applying the property (6), we have:

$$\begin{aligned} f(x,y)\,Q(x,y) &= \left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij} T_i(x) T_j(y)\right)\left(\sum_{r=0}^{n}\sum_{s=0}^{n} b_{rs} T_r(x) T_s(y)\right) \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n} c_{ij} b_{rs} T_i(x) T_r(x) T_j(y) T_s(y) \\ &= \frac{1}{4}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n} c_{ij} b_{rs} \Big(T_{i+r}(x) T_{j+s}(y) + T_{i+r}(x) T_{|j-s|}(y) \\ &+ T_{|i-r|}(x) T_{j+s}(y) + T_{|i-r|}(x) T_{|j-s|}(y)\Big). \end{aligned} \tag{17}$$

In order to simplify different expressions, we formulate the following propositions:

Proposition 3.1.
Let $(\alpha_i)_{i\geq 0}$ and $(\beta_i)_{i\geq 0}$ two series of real numbers, and $(\gamma_i(x))_{i\geq 0}$ a series of functions, we have:

$$\sum_{i=0}^{\infty}\sum_{r=0}^{n} \alpha_i \beta_r \gamma_{i+r}(x) = \sum_{i=0}^{\infty}\sum_{r=0}^{n} \tilde{\alpha}_{i-r} \beta_r \gamma_i(x) \tag{18}$$

$$\sum_{i=0}^{\infty}\sum_{r=0}^{n} \alpha_i \beta_r \gamma_{i-r}(x) = \sum_{i=0}^{\infty}\sum_{r=0}^{n} \tilde{\alpha}_{i+r} \beta_r \gamma_i(x) \tag{19}$$

$$\begin{aligned} \sum_{i=0}^{\infty}\sum_{r=0}^{n} \alpha_i \beta_r \gamma_{|i-r|}(x) &= \sum_{i=0}^{\infty}\sum_{r=0}^{n} \tilde{\alpha}_{r-i} \beta_r \gamma_i(x) + \\ & \sum_{i=1}^{\infty}\sum_{r=0}^{n} \alpha_{i+r} \beta_r \gamma_i(x) \end{aligned} \tag{20}$$

where the coefficients $\widetilde{\alpha}_i$ *verify*

$$\widetilde{\alpha}_i = \begin{cases} \alpha_i, & \text{if } i \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

Proof. To prove (18), we made a change of index $\widetilde{i} = i + r$, it follows:

$$\begin{aligned} \sum_{i=0}^{\infty} \sum_{r=0}^{n} \alpha_i \beta_r \gamma_{i+r}(x) &= \sum_{r=0}^{n} \sum_{\widetilde{i}=r}^{\infty} \alpha_{\widetilde{i}-r} \beta_r \gamma_{\widetilde{i}}(x) \\ &= \sum_{r=0}^{n} \sum_{i=r}^{\infty} \alpha_{i-r} \beta_r \gamma_i(x) \\ &= \sum_{i=0}^{\infty} \sum_{r=0}^{n} \widetilde{\alpha}_{i-r} \beta_r \gamma_i(x) \end{aligned}$$

The proof (19) is similar to (18), it suffices to set $\widetilde{i} = i - r$.
Indeed to show (20), first a decomposition gives:

$$\sum_{i=0}^{\infty} \sum_{r=0}^{n} \alpha_i \beta_r \gamma_{|i-r|}(x) = \sum_{r=0}^{n} \sum_{i=0}^{r} \alpha_i \beta_r \gamma_{r-i}(x) + \sum_{r=0}^{n} \sum_{i=r+1}^{\infty} \alpha_i \beta_r \gamma_{i-r}(x). \tag{22}$$

Then, the change of respective index $\widetilde{i} = r - i$ and $\widetilde{i} = i - r$ for two terms of the right-hand side of (22), it comes

$$\begin{aligned} \sum_{i=0}^{\infty} \sum_{r=0}^{n} \alpha_i \beta_r \gamma_{|i-r|}(x) &= \sum_{r=0}^{n} \sum_{\widetilde{i}=r}^{0} \alpha_{r-\widetilde{i}} \, \beta_r \gamma_{\widetilde{i}}(x) + \sum_{r=0}^{n} \sum_{\widetilde{i}=1}^{\infty} \alpha_{\widetilde{i}+r} \beta_r \gamma_{\widetilde{i}}(x) \\ &= \sum_{r=0}^{n} \sum_{i=0}^{r} \alpha_{r-i} \, \beta_r \gamma_i(x) + \sum_{r=0}^{n} \sum_{i=1}^{\infty} \alpha_{i+r} \beta_r \gamma_i(x) \\ &= \sum_{i=0}^{\infty} \sum_{r=0}^{n} \widetilde{\alpha}_{r-i} \, \beta_r \gamma_i(x) + \sum_{i=1}^{\infty} \sum_{r=0}^{n} \alpha_{i+r} \beta_r \gamma_i(x) \end{aligned}$$

with the $\widetilde{\alpha}_i$ are defined as in (21). $\square$

In the following we consider the coefficients $(\widetilde{c}_{ij})$ such that

$$\widetilde{c}_{ij} = \begin{cases} c_{ij}, & \text{if } i \geq 0 \text{ et } j \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Using the proposition 3.1 we obtain:

$$
\begin{aligned}
4f\left(x,y\right)Q\left(x,y\right) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n}\widetilde{c}_{i-r,j-s}b_{rs}T_i(x)T_j(y) \\
&+ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n}\widetilde{c}_{i-r,s-j}b_{rs}T_i(x)T_j(y) \\
&+ \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n}\widetilde{c}_{i-r,s+j}b_{rs}T_i(x)T_j(y) \\
&+ 2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n}\widetilde{c}_{r-i,s-j}b_{rs}T_i(x)T_j(y) \\
&+ 2\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n}\widetilde{c}_{i+r,s-j}b_{rs}T_i(x)T_j(y) \\
&+ \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n}\widetilde{c}_{r-i,s+j}b_{rs}T_i(x)T_j(y) \\
&+ \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n}\widetilde{c}_{i+r,s+j}b_{rs}T_i(x)T_j(y) \qquad (23)
\end{aligned}
$$

Since (12), (13) and (14) must verify (9), this helps by identification of coefficients for $i,j=m+1,...,m+n$, to obtain the following homogeneous system of unknowns b_{rs}:

$$
\sum_{r=0}^{n}\sum_{s=0}^{n}\left(\widetilde{c}_{i-r,j-s}+\widetilde{c}_{i-r,s+j}+\widetilde{c}_{i+r,s+j}\right)b_{rs}=0. \qquad (24)
$$

The system (24) is homogeneous and includes n^2 equations and $(n+1)^2$ unknowns. It is thus a system over determined. Generally, it does not admit solutions. A way possible to solve is to normalize certain terms. In our case, $(2n+1)$ conditions of normalizations are needed (recall that other normalizations are possible):

$$
\begin{aligned}
b_{00} &= 1, && (25)\\
b_{r0} &= 1 \quad \text{with} \quad r=1,2,...,n, && (26)\\
b_{0s} &= 1 \quad \text{with} \quad s=1,2,...,n. && (27)
\end{aligned}
$$

In these specific cases, the system (24) contains n^2 equations and n^2 unknowns:

$$
\sum_{r=1}^{n}\sum_{s=1}^{n}\Theta_{ijrs}b_{rs}=-3c_{ij}-\sum_{r=1}^{n}\Theta_{ijr0}-\sum_{s=1}^{n}\Theta_{ij0s}, \qquad (28)
$$

with

$$
\Theta_{ijrs}=\widetilde{c}_{i-r,j-s}+\widetilde{c}_{i-r,s+j}+\widetilde{c}_{i+r,s+j}.
$$

In the same way, the coefficients a_{ij} are determined by solving the following system:

$$\begin{cases} \frac{1}{2}\sum_{r=0}^{n}\sum_{s=0}^{n}\widetilde{c}_{rs}b_{rs} = a_{00}, & i,j=0, \\ \frac{1}{4}\sum_{r=0}^{n}\sum_{s=0}^{n}\left(\widetilde{c}_{r,s+j}+2\widetilde{c}_{r,s-j}\right)b_{rs} = a_{0j}, & 1\leq j\leq m, \\ \frac{1}{4}\sum_{r=0}^{n}\sum_{s=0}^{n}\left(2\widetilde{c}_{r+i,s}+\widetilde{c}_{i-r,s}+2\widetilde{c}_{r-i,s}\right)b_{rs} = a_{i0}, & 1\leq i\leq m, \\ \frac{1}{4}\sum_{r=0}^{n}\sum_{s=0}^{n}\left(\widetilde{c}_{i-r,s+j}+2\widetilde{c}_{i+r,s-j}+\widetilde{c}_{r-i,s+j}+\widetilde{c}_{i-r,j-s}+\widetilde{c}_{i-r,s-j}\right. & \\ \left.+\,2\widetilde{c}_{r-i,s-j}\right)b_{rs} = a_{ij}, & 1\leq i,j\leq m. \end{cases} \tag{29}$$

Remark 3.1. *Different normalisations enable to prove the existence of these approximants but no the uniqueness.*

4 Numerical experimentation

To illustrate the efficiency of the previous method, we give three estimated examples at the $N = 30$ Chebyshev points. One the one hand, we have established a table of values showing absolute errors between exact values of each function and its approximate by truncate Chebyshev series f_{ap}, and one the other hand with its Chebyshev-Padé approximant $f(x,y)_{ChebPade}$ on the same grid points $N \times N$. Thus, we could exhibit a positive integer n (degree of denominator in x and y) for which $f(x,y)_{ChebPade} \longrightarrow f(x,y)$ varying m (degree numerator in x and y).

[m/n]	$\|\|f - f_{ap}\|\|_\infty$	$\|\|f - f_{ChebPade}\|\|_\infty$
[15/7]	0.17844680	0.00989083
[16/7]	0.17219250	0.00956546
[17/7]	0.16663555	0.00768792
[18/7]	0.16210627	0.00761395
[19/7]	0.15787886	0.00666526
[20/7]	0.15341211	0.00605857

Table 1: $f(x,y) = \sin(\pi x)\sin(\pi y)$

[m/n]	$\|\|f - f_{ap}\|\|_\infty$	$\|\|f - f_{ChebPade}\|\|_\infty$
[20/9]	0.08544526	0.00993883
[21/9]	0.08331408	0.00964092
[22/9]	0.08153698	0.00945850
[23/9]	0.07971592	0.00868023
[24/9]	0.07783491	0.00813102
[25/9]	0.07594786	0.00775546

Table 2: $f(x,y) = \exp(-xy)$

[m/n]	$\|\|f - f_{ap}\|\|_\infty$	$\|\|f - f_{ChebPade}\|\|_\infty$
[25/10]	0.12392750	0.00955737
[26/10]	0.12102090	0.00917107
[27/10]	0.11822960	0.00813607
[28/10]	0.11557987	0.00802139
[29/10]	0.11303214	0.00714170
[30/10]	0.11060684	0.00335623

Table 3: $f(x, y) = \exp(x)\,(\sin(y) + xy^2)$

Three tables of absolute errors show clearly that it is more accurately to approach functions by rational functions than by polynomials with the same number of degrees of freedom. Therefore, this method confirms the convergence acceleration of pseudo-spectral methods.

5 Concluding remarks

In this article, an approach of Chebyshev-Padé approximants for functions of two variables have been proposed. We obtained algebraic systems to determine the numerator and denominator coefficients of these approximants. Numerical examples made proved the efficiency of this method in the convergence acceleration of pseudo-spectral methods.

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