**THE TRANSMUTED GOMPERTZ DISTRIBUTION**

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In this article, we generalize the Gompertz distribution using the quadratic rank transmutation map studied by Shaw et al. [20] to develop a transmuted Gompertz distribution. Some structural properties of the new distribution are derived such as mean, variance, moment generating function, hazard functions, and order statistic. The parameters of the proposed model are estimated using the maximum likelihood estimation and obtain the observed information matrix. The usefulness of the transmuted Gompertz distribution for modeling reliability data is illustrated using real data.

*Key Words*: Gompertz distribution, Moment generating function, Reliability Function, maximum likelihood estimation.

1. **INTRODUCTION**

The Gompertz distribution is a ﬂexible distribution that can be skewed to the right and to the left. This distribution is a generalization of the exponential distribution and is commonly used in many applied problems, particularly in lifetime data analysis Johnson et al. [10]. The Gompertz distribution is considered for the analysis of survival, in some sciences such as gerontology Brown and Forbes [5], computer Ohishi et al. [19], biology Economos [6], and marketing science Bemmaor and Glady [4]. Recently, a generalization based on the idea of Gupta and Kundu [9] was proposed by El-Gohary and Al-Otaibi [8]. This new distribution is known as generalized Gompertz distribution which includes the exponential, generalized exponential, and Gompertz distributions. In this article we present a new generalization of Gompertz distribution called the transmuted Gompertz distribution.

A random variable X is said to have transmuted distribution, according to the quadratic rank transmutation map (QRTM), if its cumulative distribution function is given by

 (1)

where is the cdf of the base distribution. Note that if we have the distribution of the base random variable. Further details can be found in Shaw and Buckley [21]. Recently several distributions have been proposed from the transmuted family of lifetime distributions. Aryal and Tsokos [1,2] proposed the transmuted extreme value and transmuted Weibull distributions and examined various structural properties of these models with applications. Khan and King [11,12] studied the transmuted modified Weibull distribution and the transmuted generalized Inverse Weibull distribution and discussed some theoretical properties of these distributions. Recently Khan and King [13,14] proposed the transmuted Inverse Weibull and transmuted modified Inverse Rayleigh distributions for modelling reliability data. More recently Khan et al. [15] proposed the transmuted Chen distribution and investigated various structural properties with an application to reliability data. Transmuted Lomax distribution is presented by Ashour and Eltehiwy [3]. Recently Merovci [17] studied the transmuted Rayleigh distribution with application to balder cancer data. Merovci [18] studied the transmuted generalized Rayleigh distribution. Yuzhu et al. [22] proposed and studied the transmuted linear exponential distribution and discussed some of its structural properties. Elbatal et al. [7] introduced the transmuted generalized linear exponential distribution and formulated some structural properties of this distribution. In the present study we will provide mathematical formulation of the transmuted Gompertz distribution and some of its properties.

1. **TRANSMUTED GOMPERTZ DISTRIBUTION**

A random variable X is said to have a Gompertz distribution with parameters if its probability density function (pdf) is given by

 (2)

The corresponding cumulative distribution function (c.d.f.) is:

 (3)

Now using (1) and (3), we have the cdf of a transmuted Gompertz distribution

 (4)

Hence, the pdf of transmuted Lindley distribution with parameter is

 (5)

Note that the transmuted Gompertz distribution is an extended model to analyze more complex data and it generalizes some of the widely used distributions. The Gompertz distribution is clearly a special case for . Figure 1 illustrates some of the possible shapes of the pdf of a transmuted Gompertz distribution for selected values of the parameters and . Figure 2 illustrates some of the possible shapes of the CDF of a transmuted Gompertz distribution for selected values of the parameters and .



Figure 1: Pdf of Transmuted Gompertz distribution



Figure 2: CDF of Transmuted Gompertz distribution

1. **MOMENTS AND QUANTILES**

Now let us consider the diﬀerent moments of the transmuted Gompertz distribution. Suppose X denote the transmuted Gompertz distribution random variable with parameters , then

In particular, the mean and variance of the transmuted Gompertz distribution can be worked out as

 (6)

 (7)

Where is the -function defined by

 (8)

and is the gamma function.

The quantile of the transmuted Gompertz distribution can be obtained from (4) as

 (9)

1. **MOMENT GENERATING FUNCTION**

Let have a transmuted Gompertz distribution. Then the moment generating function of , say , is

After simplification, the moment generating function of a transmuted Gompertz distribution is

 (10)

Where is the exponential integral function defined by

1. **RELIABILLITY ANALYSIS**

The transmuted Gompertz distribution can be a useful characterization of failure time of a given system because of the analytical structure. The reliability function , which is the probability of an item not failing prior to some time , is deﬁned by . The reliability function of a transmuted Gompertz distribution is given by

 (11)

Figure 3 illustrates the reliability function of a transmuted Gompertz distribution for different values of the parameter keeping .



Figure 3: Reliability function of Transmuted Gompertz distribution

The other characteristic of interest of a random variable is the hazard rate function deﬁned by

which is an important quantity characterizing life phenomenon. It can be loosely interpreted as the conditional probability of failure, given it has survived to the time . The hazard rate function for a transmuted Gompertz random variable is given by

 (12)

Figure 4 illustrates the hazard rate function of a transmuted Gompertz distribution for different values of the parameter keeping .



Figure 4: Hazard rate function of Transmuted Gompertz distribution

1. **ORDER STATISTICS**

The order statistic of a statistical sample is equal to its smallest value. Together with rank statistics, order statistics are among the most fundamental tools in non-parametric statistics and inference. For a sample of size n, the nth order statistic (or largest order statistic) is the maximum, that is,

The sample range is the difference between the maximum and minimum. It is clearly a function of the order statistics:

We know that if denotes the order statistics of a random sample from a continuous population with cdf and pdf then the pdf of is given by

The pdf of the order statistic for transmuted Gompertz distribution is given by

Therefore, the pdf of the largest order tatistic is given by

 (13)

and the pdf of the smallest order statistic is given by

 (14)

1. **MAXIMUM LIKELIHOOD ESTIMATION**

In this section, we determine the maximum-likelihood estimates (MLEs) of the parameters of the transmuted Gompertz distribution. Consider is a random sample from transmuted Gompertz distribution, then the likelihood function for the vector of parameters can be expressed as

 (15)

Then the log-likelihood function for the vector of parameters can be expressed as

 (16)

The first order partial derivative with respect to then equating it to zero, we obtain the component of the score vector is given by

 (17)

 (18)

 (19)

The maximum likelihood estimates for the parameter vector can be obtained by solving the simultaneous equation of the first order partial derivatives. An iterative process such as Newton-Raphson method has to be adopted to solve this system of equations for . For testing of hypothesis and confidence interval for the parameter vector of , we considered the multivariate normal distribution with the variance covariance matrix and its inverse of the expected information matrix are given by

 (20)

and

 (21)

The unit observed information matrix, whose elements are given by

 (22)

 (23)

 (24)

 (25)

 (26)

The multivariate normal distribution can be used to obtain as an approximate confidence intervals for the parameters can be determined as

 , , ,

Where is the upper percentiles of the standard normal distribution.

**8. SIMULATION STUDY**

In this section we evaluate the performance of the MLEs for the three parameter transmuted Gompertz distribution. Let denote the uniform random variable over the interval . We generate the random variable of the subject distribution by using the equation

 (27)

which is the formula used for generating random numbers. we generate the random samples of size from the three parameter transmuted Gompertz distribution. We consider the fixed choice of parameter values . The simulation process is repeated for 1000 times and the results were displayed in Table 1. Simulation study has been performed in order to evaluate the mean estimates, bias, standard error (S.E) and mean square error (MSE). These results of simulation suggest that as the sample size increases the method of MLEs provide the better estimates.

Table 1: Mean, Bias, standard Error, and MSE of the TG distribution

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| N | Parameter Estimates | Mean | Bias | S.E | MSE |
| 20 |  | 1.27250283.89328802.0686510 | - 0.4016622.2740426- 0.075619 | 0.00984850.56551802.1147490 | 0.22748762.56596800.5752596 |
| 50 |  | 1.20206003.59162201.5816597 | - 0.4409881.8670116- 0.246196 | 0.00302390.23324781.4074296 | 0.27502282.40398900.5317689 |
| 80 |  | 1.14631593.2846840 1.2294745 | - 0.4690061.6898342- 0.348686 | 0.00204350.15844820.7095365 | 0.30333411.59314870.4460581 |
| 100 |  | 1.18120143.03032001.0896427 | - 0.4538541.5311851- 0.389988 | 0.00152330.11942920.4723524 | 0.29276381.38300130.4136119 |
| 200 |  | 1.20699592.13255560.8486543 | - 0.4389981.3593263- 0.452703 | 0.00080410.05807830.2186997 | 0.27399081.16644190.3356811 |
| 300 |  | 1.161298801.92277830 0.7664156 | - 0.4597211.3610612- 0.476566 | 0.00058950.04094450.1333565 | 0.29046081.05180710.3154879 |
| 400 |  | 1.15135841.7393392 0.7097958 | - 0.4654171.3557573- 0.489956 | 0.00044210.03148790.0984372 | 0.29761050.98321860.2936824 |
| 500 |  | 0.45067721.29737230.2589096 | - 0.1804370.5169367- 0.195562 | 0.00013840.00978350.0308347 | 0.11448760.08554960.1082714 |

1. **APPLICATION**

In this section, we use a real data set to show that the transmuted Gompertz distribution can be a better model than one based on the Gompertz distribution. The data set given in Table 2 represents an uncensored data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang [16].

Table (2).

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 0.08 | 2.09 | 3.48 | 4.87 | 6.94 | 8.66 | 13.11 | 23.63 | 0.20 | 2.23 |
| 3.52 | 4.98 | 6.97 | 9.02 | 13.29 | 0.40 | 2.26 | 3.57 | 5.06 | 7.09 |
| 9.22 | 13.80 | 25.74 | 0.50 | 2.46 | 3.64 | 5.09 | 7.26 | 9.47 | 14.24 |
| 25.82 | 0.51 | 2.54 | 3.70 | 5.17 | 7.28 | 9.74 | 14.76 | 26.31 | 0.81 |
| 2.62 | 3.82 | 5.32 | 7.32 | 10.06 | 14.77 | 32.15 | 2.64 | 3.88 | 5.32 |
| 7.39 | 10.34 | 14.83 | 34.26 | 0.90 | 2.69 | 4.18 | 5.34 | 7.59 | 10.66 |
| 15.96 | 36.66 | 1.05 | 2.69 | 4.23 | 5.41 | 7.62 | 10.75 | 16.62 | 43.01 |
| 1.19 | 2.75 | 4.26 | 5.41 | 7.63 | 17.12 | 46.12 | 1.26 | 2.83 | 4.33 |
| 7.66 | 11.25 | 17.14 | 79.05 | 1.35 | 2.87 | 5.62 | 7.87 | 11.64 | 17.36 |
| 1.40 | 3.02 | 4.34 | 5.71 | 7.93 | 11.79 | 18.10 | 1.46 | 4.40 | 5.85 |
| 8.26 | 11.98 | 19.13 | 1.76 | 3.25 | 4.50 | 6.25 | 8.37 | 12.02 | 2.02 |
| 3.31 | 4.51 | 6.54 | 8.53 | 12.03 | 20.28 | 2.02 | 3.36 | 6.76 | 12.07 |
| 21.73 | 2.07 | 3.36 | 6.93 | 8.65 | 12.63 | 22.69 | 5.49 |  |

In order to compare the three parameter transmuted Gompertz distribution with the two parameter Gompertz distribution, we consider criteria like −2ℓ̂ (maximized likelihood), AIC (Akaike information criterion), CAIC (corrected Akaike information criterion), and BIC (Bayesian information criterion) for the data set. The better distribution corresponds to smaller −2ℓ, AIC, CAIC, and BIC values:

and

Where is the number of parameters in the statistical model, the sample size and ℓ is the maximized value of the log-likelihood function under the considered model.

Table 3: Criteria for comparison.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Model** | **-2*l*** | **AIC** | **CAIC** | **BIC** |
| Transmuted Gompertz distribution | 646.35968 | 652.35968 | 652.55319 | 660.91577 |
| Gompertz distribution | 668.0626 | 672.0626 | 672.1586 | 677.76666 |

Table 3 shows the values of −2ℓ, AIC, CAIC, and BIC values. The values in Table 3, indicate that the transmuted Gompertz distribution leads to a better ﬁt than the Gompertz distribution.

**10. CONCLUDING**

We introduced a new generalization of the Gompertz distribution called the transmuted Gompertz distribution. The subject distribution is generated by using the quadratic rank transmutation map and taking the Gompertz distribution as the base distribution. We derive expansions for the mean, variance and the moment generating function. The estimation of parameters is approached by the method of maximum likelihood, also the information matrix is derived. The hazard rate function and reliability behavior of the transmuted Gompertz distribution shows that the subject distribution can be used to model reliability data. The importance and flexibility of the transmuting parameter in the transmuted Gompertz distribution is accessed by simulation. An application of transmuted Gompertz distribution to real data shows that the new distribution can be used quite effectively to provide better fits than Gompertz distribution.

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**REFERENCES**

[1] G R. Aryal, and C. P. Tsokos, "On the transmuted extreme value distribution with applications. Nonlinear Analysis: Theory, Methods and applications," Nonlinear Analysis: Theory, Methods and applications, Vol. 71, 2009, pp1401-1407.

[2] G R. Aryal, and C. P. Tsokos, "Transmuted Weibull distribution: A Generalization of the Weibull Probability Distribution,"European Journal of Pure and Applied Mathematics, Vol. 4, no. 2, 2011, pp89-102.

[3] S. K. Ashour, M. A. Eltehiwy, "Transmuted Lomax Distribution, " American Journal of Applied Mathematics and Statistics, vol. 1, no. 6, 2013, pp. 121−127.

[4] A. C. Bemmaor, and N. Glady, "Modeling purchasing behavior with sudden “death”: A ﬂexible customer lifetime model," Management Science vol. 58 no 5, 2012, pp.1012–1021.

[5] K. Brown, and W. Forbes, "A mathematical model of aging processes, " Journal of Gerontology vol. 29, no. 1, 1974, pp. 46–51.

[6] A. C. Economos, "Rate of aging, rate of dying and the mechanism of mortality, " Archives of Gerontology and Geriatrics, vol. 1, no. 1, 1982, pp. 46–51.

[7] I. Elbatal, L.S. Diab and N. A. Abdul Alim, "Transmuted Generalized Linear Exponential Distribution, " International Journal of Computer Applications, vol. 83, no. 17, Dec. 2013, pp. 29-37.

[8] A. El-Gohary, and A. N. Al-Otaibi, "The generalized Gompertz distribution, " Applied Mathematical Modelling, vol. 37, no. 1-2, 2013, pp.13–24.

[9] R. D. Gupta. and D. Kundu, "A new class of weighted exponential distributions, " Statistics: A Journal of Theoretical and Applied Statistics, vol. 43, no. 6, 2009, pp. 621-634.

[10] N.L Johnson, S. Kotz and N. Balakrishnan, "Continuous Univariate Distribution,". John Wiley & Sons, vol. 1, 2nd, 1995, New York.

[11] M. Khan Shuaib, King Robert, "Transmuted Modified Weibull Distribution: A Generalization of the Modified Weibull Probability Distribution, " European Journal of Pure And Applied Mathematics, vol. 6, no. 1, 2013, pp. 66-88.

[12] M. Khan Shuaib, King Robert, "Transmuted Generalized Inverse Weibull Distribution, " Journal of Applied Statistical Sciences, vol. 20, no. 3, 2013, pp. 15-32.

[13] M. Khan Shuaib, King Robert, "A New Class of Transmuted Inverse Weibull Distribution for Reliability Analysis, " American Journal of Mathematical and Management Sciences, vol. 33, no. 4, 2014, pp. 261-286.

[14] M. Khan Shuaib, King Robert, "Transmuted Modified Inverse Rayleigh Distribution, " Austrian Journal of Statistics, vol. 44, no. 3, 2015, pp. 17-29.

[15] M. Khan Shuaib, King Robert and Hudson Irene, "A new three parameter transmuted Chen lifetime distribution with application, "Journal of Applied Statistical Sciences, vol. 21, no. 3, 2015, pp. 239-259.

[16] E.T. Lee and J.W. Wang "Statistical Methods for Survival Data Analysis, " Wiley, NewYork, 3rd ed., 2003.

[17] F. Merovci, "Transmuted Rayleigh distribution," Austrian Journal of Statistics, vol. 42, no. 1, 2013, pp. 21-31.

[18] F. Merovci, "Transmuted generalized Rayleigh distribution, " J. Stat. Appl. Pro. vol. 3, no. 1, 2014, pp. 9-20.

[19] K. Ohishi, H. Okamura and T. Dohi, "Gompertz software reliability model: Estimation algorithm and empirical validation, " Journal of Systems and Software, vol. 82, no. 3,2009, pp. 535–543.

[20] W. T. Shaw, and I. R. C. Buckley, "The alchemy of probability distributions: beyond Gram–Charlier expansions, and a skew-kurtotic normal distribution from a rank transmutation map, " Technical report, <http://arxiv.org/abs/0901.0434>, 2009.

[21] W. T. Shaw, and I. R. C. Buckley, "The alchemy of probability distributions: beyond Gram-Charlier expansions and a skewkurtotic-normal distribution from a rank transmutation map, ". Research report, King’s College, London, U.K.

[22] Yuzhu Tian, Maozai Tian and Qianqian Zhu, (2014). "Transmuted Linear Exponential Distribution: A New Generalization of the Linear Exponential Distribution, " Communications in Statistics - Simulation and Computation, vol. 43, no.10, 2014, pp. 2661-2677.

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