Nonnegative solutions of a nonlinear system and applications to elliptic BVPs *

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Abstract

In this communication, we study the existence of nonnegative solutions of a nonlinear system in Banach spaces. These maps involved in the system defined on cone do not necessarily take values in the cone. Using fixed point theorems just established for this type of mappings, nonnegative solutions of the system are obtained and used to investigate elliptic boundary value problems(BVPs).

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1 Introduction and preparatories

It has been well-known that nonlinear systems arise in many fields of scientific researches and engineering practices, the study on existence of nonnegative solutions for that is very interesting and of great importance, see, for example, [1, 2] and the references there. However, the nonlinear terms involved in that take negative values in many cases, to the best of our knowledge, there is little study on it.

Recently, Yang, one of authors of [3], and Lan established a new fixed point index theory for nowhere normal-outward compact maps [4] and proved a few fixed point theorems, which were used to population models with sign-changing nonlinearities [5]. In this communication, we expand some results [3] to systems and apply them to investigate elliptic boundary value problems(BVPs).

We recall some knowledge on r-nowhere normal-outward maps and a fixed point theorem for these maps obtained in [3].

Let K be a closed convex set in a Banach space X with norm $\|\cdot\|$, and let $r : X \to K$ be a retraction, that is, r is continuous and satisfies $r(y) = y$ for $y \in K$. Recall that a map $A : D \subset K \to X$ is called to be a r-nowhere normal-outward map on D relative to K if

$$Ax \in \left(X \setminus r^{-1}(x)\right) \cup \{x\} \quad \text{for } x \in D.$$

The following criterion was obtained in [3, Proposition 2.1].

Lemma 1.1. *Let $A : D \subset K \to X$ be a map and let $r : X \to K$ be a retraction. If*

$$x = A(r(x)) \quad \text{for some } x \in r^{-1}(D) \tag{1.1}$$

implies $x \in D$, then A is a r-nowhere normal-outward map on D relative to K.

A map $A : D \subset X \to X$ is said to be compact if A is continuous and $A(S)$ is relatively compact for each bounded subset S of D.

Let $D_K = D \cap K$, $\overline{D}_K = \overline{D} \cap K$ and $\partial D_K = \partial D \cap K$. In [3], we established the following known result (see [3, Theorem 3.2]).

Theorem 1.1. *Let K be a a closed convex set in X and let $r : X \to K$ be a retraction. Let D^1, D be bounded open sets in X such that $\overline{D_K^1} \subset D_K$ and $D_K^1 \neq \emptyset$. Assume that $A : \overline{D}_K \setminus D_K^1 \to X$ is compact such that the following conditions hold.*

(h_1) There exists $x_0 \in D_K$ such that $tA + (1-t)\widehat{x_0}$ is r-nowhere normal-outward on $\overline{D}_K$ relative to K for $t \in (0, 1]$.

(LS) $x \neq tAx + (1-t)x_0$ for $x \in \partial D_K$ and $t \in (0, 1)$.

(h_2) There exists $e \in K \setminus \{0\}$ such that $A + \lambda e$ is a r-nowhere normal-outward on $\overline{D_K^1}$ relative to K for $\lambda > 0$.

(E) $x \neq Ax + \lambda e$ for $x \in \partial D_K^1$ and $\lambda \geq 0$.

Then A has a fixed point in $\overline{D}_K \setminus \overline{D}_K^1$.

Let K be a cone in X. Then K defines a partial order $\leq$ in X by $x \leq y$ if and only if $y - x \geq 0$. A cone K is said to be *reproducing* if $X = K - K$, to be *total* if $X = \overline{K - K}$ and to be *normal* if there exists $\sigma > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq \sigma \|y\|$ [1].

Recall that a real number λ is called an eigenvalue of a linear operator $L : X \to X$ if there exists $\varphi \in X \setminus \{0\}$ such that $\lambda \varphi = L\varphi$. The radius of the spectrum of L in X, denoted by $r(L)$, is given by $r(L) = \lim_{m \to \infty} \sqrt[m]{\|L\|^m}$. We write

$$\mu_1(L) = \frac{1}{r(L)}. \tag{1.2}$$

We denote by $\mathcal{L}(K)$ the set of compact linear operators $L : X \to X$ satisfying $L(K) \subset K$ and $r(L) > 0$. By Krein-Rutman theorem (see [6, Theorem 3.1] or [7]), if K is a total cone and $L \in \mathcal{L}(K)$, then there exists an eigenvector $\varphi \in K \setminus \{0\}$ such that

$$\varphi = \mu_1(L)L\varphi. \tag{1.3}$$

2 Nonnegative solutions of a nonlinear system

Let $n \geq 2$, $K^n = K \times K \times .... \times K$ and $A_i : K^n \to X$. We shall establish the existence results of nonzero solutions in K^n for the following system of the form

$$x_i = A_i(x) \text{ for } i \in I_n := \{1, 2, ..., n\}, \tag{2.1}$$

where $x = (x_1, x_2, ..., x_n) \in X_n$.

Let $X_n = X \times X \times .... \times X$ with the maximum norm $\|x\| = \max\{\|x_1\|, \|x_2\|, ..., \|x_n\|\}$ and $r_n x = (rx_1, rx_2, ..., rx_n)$. Then $r_n : X_n \to K^n$ is a retraction. A map $A : K^n \to X$ is called to be a r-nowhere normal-outward map with respect to component $x_i (i \in I_n)$ on K^n relative to K, if $x \in X_n$, $x_i = A_i r_n x$ implies $x_i \in K$.

Using Theorem 1.1, we prove

Theorem 2.1. *Let K be a total and normal cone in X, and let $r : X \to K$ be a retraction. Assume that $A_i : K^n \to X$ is compact and satisfies the following conditions:*

(h_n) *For any $i \in I_n$, there exists $x_i \in K$ such that $tA_i + (1 - t)\widehat{x_i}$ is r-nowhere normal-outward map with respect to component x_i on K^n relative to K for $t \in (0, 1]$.*

$(LS)_n$ *For any $i \in I_n$, there exist $v_i \in K \setminus \{0\}$, $L_i \in \mathcal{L}(K)$ and $\varepsilon_i \in (0, \mu_1(L_i))$ such that*

$$A_i x \leq \big(\mu_1(L_i) - \varepsilon_i\big) L_i(x_i) + v_i \quad for\ x \in K.$$

$(E)_n$ *There exist $i_0 \in I_n$, $\rho_0 > 0$ and $L_0 \in \mathcal{L}(K)$ such that*

$$A_{i_0} x \geq \mu_1(L_0) L_0(x_{i_0}) \ \ for\ x \in \partial K_{\rho_0}.$$

Then (2.1) has a solution in $K^n \setminus \{0\}$.

Proof. Let $A : K^n \to X_n$ be defined by $Ax = (A_1 x, A_2 x, ..., A_n x)$. Then A is compact since A_i is compact. Let $x \in X_n$ such that $x = tArx + (1-t)x_0$, where $x_0 = (x_1, x_2, ..., x_n)$. Then $x_i = tA_i rx + (1-t)x_i$. By (h_n), we see $x_i \in K$. Hence $tA + (1-t)\widehat{x_0}$ is a r_n-nowhere normal-outward map on K^n relative to K^n and (h_1') of Theorem 1.1 holds.

Let $\rho > 0$, $K_\rho^n = \{x : x \in K^n, \|x\| < \rho\}$ and $\partial K_\rho^n = \{x : x \in K^n, \|x\| = \rho\}$.

Since

$$r((\mu_i(L_i) - \varepsilon_i)L_i) = (\mu_1(L_i) - \varepsilon_i)r(L_i) < 1,$$

$(I - (\mu_1(L_i) - \varepsilon_i)L_i)^{-1}$ exists and is a bounded linear operator such that

$$(I - (\mu_1(L_i) - \varepsilon_i)L_i)^{-1}(K) \subset K,$$

where $I : X \to X$ is the identical mapping $I(z) = z$.

Let σ be the normality constant of K and

$$\rho^* = \max\{\rho_0, \sigma\|(I - (\mu_i(L_i) - \varepsilon_i)L_i)^{-1}(v_i + x_i)\|, i \in I_n\}.$$

Let $\rho > \rho^*$ and $x_0 = (x_1, x_2, ..., x_n)$. Then $x_0 \in K^n$. We prove that

$$x \neq tAx + (1-t)x_0 \quad for\ x \in \partial K_\rho^n\ and\ t \in (0,1]. \tag{2.2}$$

In fact, if not, there exist $x \in \partial K_\rho^n$ and $t \in (0,1]$ such that $x = tAx + (1-t)x_0$. This, together with $(LS)_n$, implies

$$\begin{aligned} x_i = tA_i x + (1-t)x_i &\leq t[(\mu_1(L_i) - \varepsilon_i)L_i(x) + v_i] + (1-t)x_i \\ &\leq (\mu_1(L_i) - \varepsilon_i)L_i(x) + v_i + x_i \end{aligned}$$

and

$$(I - (\mu_i(L_i) - \varepsilon)L_i)x_i \leq v_i + x_i\ for\ i \in I_n.$$

This, together with $(I - (\mu_i(L_i) - \varepsilon_i)L_i)^{-1}(K) \subset K$, implies

$$x_i \leq (I - (\mu_i(L_i) - \varepsilon_i)L_i)^{-1}(v_i + x_i).$$

4

Since K is a normal cone with normality constant σ, it follows that

$$\|x_i\| \leq \sigma \|(I - (\mu_i(L_i) - \varepsilon_i)L_i)^{-1}(v_i + x_i)\| \leq \rho^* \text{ for } i \in I_n.$$

Hence, we have $\rho = \|x\| \leq \rho^* < \rho$, a contradiction. (LS) of Theorem 1.1 holds.

Without loss of generality, we may assume A has not fixed point in ∂P_{ρ_0} (otherwise, the result has been proved). Since K is total, it follows from the Krein-Rutman theorem that there exists $\varphi_0 \in K \setminus \{0\}$ such that $\varphi_0 = \mu_1(L_0)L_0(\varphi_0)$. Let $e = (\underbrace{0, 0, ..., 0}_{i_0 - 1}, \varphi_0, \underbrace{0, ..., 0}_{n - i_0})$. Let $x \in \partial K_{\rho_0}^n$ such that $x = Arx + \lambda e$. When $i \neq i_0$, we have $x_i = A_i rx$ and $x_i \in K$ by (h_n) with $t = 1$. When $i = i_0$, by $(E)_n$, $x_{i_0} = A_{i_0} rx + \lambda \varphi_0 \geq \lambda \varphi_0$. Hence $x \in K^n$ and $A + \lambda \widehat{e}$ is r-nowhere normal-outward map on $\overline{K}_{\rho_0}^n$ relative to K^n.

We prove that

$$x \neq Ax + \lambda \varphi_0 \quad \text{for } x \in \partial K_{\rho_0}^n \text{ and } \lambda \geq 0. \tag{2.3}$$

In fact, if not, there exist $x \in \partial K_{\rho_0}^n$ and $\lambda \geq 0$ such that

$$x = Ax + \lambda \varphi. \tag{2.4}$$

This implies $x_{i_0} = A_{i_0} x + \lambda \varphi_0$ and $\lambda > 0$. By $(E)_n$ and (2.4), we have $A_{i_0} x \geq \mu_1(L_0)L_0(x_{i_0}) \geq 0$ and $x_{i_0} \geq \lambda \varphi_0$. Let

$$\lambda_1 = \sup\{\lambda > 0 : x_{i_0} \geq \lambda \varphi_0\}. \tag{2.5}$$

Then $0 < \lambda \leq \lambda_1 < \infty$, $x_{i_0} \geq \lambda_1 \varphi_0$ and $L_0(x_{i_0}) \geq \lambda_1 L_0(\varphi_0) = \dfrac{\lambda_1}{\mu_1(L_0)} \varphi_0$. By (2.4) and $(E)_n$,

$$x_{i_0} = A_{i_0}(x) + \lambda \varphi_0 \geq \mu_1(L_0)L_0(x_{i_0}) + \lambda_1 \varphi_0 \geq (\lambda_1 + \lambda)\varphi_0.$$

Hence, by (2.5), we have $\lambda_1 \geq \lambda_1 + \lambda > \lambda_1$, a contradiction. (E) of Theorem 1.1 holds.

By Theorem 1.1, A has a fixed point in $\overline{K}_\rho^n \setminus \overline{K}_{\rho_0}$, that is, (2.1) has a solution in $K^n \setminus \{0\}$. $\qquad \square$

The following result shows the nome-type compression and expansion theorem of (2.1).

Theorem 2.2. *Let K be a total, normal cone in X, $r : X \to K$ be a retraction, $A_i : K^n \to X$ be compact and satisfy:*

$(h_n)'$ *tA_i is r-nowhere normal-outward map with respect to component x_i on K^n relative to K for any $i \in I_n$ and $t \in (0, 1]$.*

Assume that the following conditions hold: there exist $\rho_0, \rho_1 \in (0, \infty)$ with $\rho_0 < \rho_1$ such that for any $i \in I_n$

5

$(E)_n$ $A_i x \in K$ *for* $x \in K^n_{\rho_0}$ *and* $\|A_i x\| > \|x_i\|$ *for* $x \in K^n_{\rho_0}$ *with* $\|x_i\| = \rho_0$.

(H_n) $\|A_i x\| \le \|x_i\|$ *and* $x \in K^n_{\rho_1}$ *with* $\|x_i\| = \rho_1$.

Then (2.1) has a solution in $K^n \setminus \{0\}$.

Proof. Let A be defined by Theorem 2.1 and $x_i = 0 (i \in I_n)$. Then $(h_n)'$ implies that tA is r_n-nowhere normal-outward map on K^n relative to K_n for $t \in (0, 1]$.

Let $x \in \partial K^n_{\rho_0}$. Then there is $x_i \in P$ such that $\|x_i\| = \rho_0$ and $\|Ax\| \ge \|A_i x\| > \|x_i\| = \rho_0 = \|x\|$ by (E_n). Since $A : K^n_{\rho_0} \to K^n$, the standard argument shows $i_{r,K}(A, K^n_{\rho_0}) = i_K(A, K^n_{\rho_0}) = 0$, for example, see [6].

Without loss of generality, we may assume that A has no a fixed point in $\partial K^n_{\rho_1}$ (otherwise, the result has been proved). We prove $x \ne \lambda A x$ for $x \in K^n_{\rho_1}$ and $0 < \lambda \le 1$.

In fact, if there exist $x \in \partial K^n_{\rho_1}$ and $0 < \lambda \le 1$ such that $x = \lambda A x$, then $\lambda < 1$ since A has no a fixed point in $K^n_{\rho_1}$. From $x_i = \lambda A_i x$ for any $i \in I_n$, we have $\|x_i\| = \lambda\|A_i x\|$. Let $i_0 \in I_n$ such that $\|x_{i_0}\| = \rho_1$. By (H_n), $\|x_{i_0}\| = \lambda\|A_{i_0}x\| < \|A_{i_0}x\| \le \|x_{i_0}\|$, it is a contradiction. By Theorem 3.1 [3], $i_{r,K}(A, K^n_{\rho_1}) = 1$.

Hence, A has a fixed point in $K^n_{\rho_1} \setminus \overline{K^n_\rho}$ and (2.1) has a solution in $K^n \setminus \{0\}$. $\square$

Remark 2.1. One may refer to [8] for the nome-type compression and expansion theorem, where maps defined on cone and take values in the cone.

3 Nonegative solutions of elliptic boundary value problems

In this section, we investigate the existence of nonzero nonnegative (classical) solutions of the following elliptic systems:

$$\begin{cases} \mathcal{L}_i u_i(z) = f_i(z, \mathbf{u}(z)) \text{ in } \Omega \text{ and for each } i \in I_n, \\ u_i(z) = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^m$ $(m \in \mathbb{N}, m \ge 2\})$ is a bounded domain,

$$\mathcal{L}_i u_i(z) = -\sum_{k,j=1}^m \frac{\partial}{\partial z_k}\left(a^{(i)}_{kj}(z)\frac{\partial u_i(z)}{\partial z_j}\right) + \sum_{j=1}^m b^{(i)}_j(z)\frac{\partial u_i(z)}{\partial z_j} + c_i(z)u_i(z), \tag{3.2}$$

where $\mathbf{u}(z) = (u_1(z), u_2(z), ..., u_n(z))$, $z = (z_1, z_2, ..., z_m)$, $f_i : \overline{\Omega} \times \mathbb{R}^n_+ \to \mathbb{R}$, $f_i \in C^\mu(\Omega \times \mathbb{R}^n)$ and $\mu \in (0, 1)$ is a given constant. When $f_i \in C^\mu(\Omega \times \mathbb{R}^n) \to \mathbb{R}_+$, system (3.1)was studied in [10]. A single Elliptic BVP with Dirichlet boundary condition was studied in [5]. The definitions of $\mathcal{L}$ and Ω are same as in [2, 10].

u is called to a (classical) solution of (3.1), we mean a function $u_i \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying (3.1) pointwise. A solution **u** of (3.1) is said to be nonnegative if $u_i \in P$, where

$$P = \{u \in C(\overline{\Omega}) : u(z) \geq 0 \quad \text{for } z \in \overline{\Omega}\} \tag{3.3}$$

is the positive cone in $C(\overline{\Omega})$, which is total and normal.

We always assume that the following conditions hold for each $i \in I_n$:

(C_1) $a_{kj}^{(i)}, b_j^{(i)}, c^{(i)} \in C^{\hat{\mu}}(\overline{\Omega})$ for $k, j \in I_m$, and $c^{(i)}(z) \geq 0$ for $z \in \overline{\Omega}$.

(C_2) $a_{kj}^{(i)}(z) = a_{jk}^{(i)}(z)$ for $z \in \overline{\Omega}$ and $k, j \in I_m$, and there exists $\mu_i > 0$ such that

$$\sum_{k,j=1}^{m} a_{kj}^{(i)}(z)\xi_k\xi_j \geq \mu_i|\xi|^2 \quad \text{for } x \in \overline{\Omega} \text{ and } \xi = (\xi_1, ..., \xi_m) \in \mathbb{R}^m.$$

(C_3) $\frac{\partial a_{kj}^{(i)}}{\partial z_k} \in C^{\hat{\mu}}(\overline{\Omega})$ for $k, j \in I_m$.

(C_4) $\frac{\partial b_j^{(i)}}{\partial z_j} \in C(\overline{\Omega})$ and $\frac{\partial b_j^{(i)}(z)}{\partial x_j} \leq 2c^{(i)}(z)$ for $x \in \Omega$ and $j \in I_m$.

(h_1) $f_i \in C^{\hat{\mu}}(\overline{\Omega} \times \mathbb{R}_+^n)$.

(h_2) (Positivity condition) $f_i(z, \hat{y}_0^i) \geq 0$ for $z \in \Omega$, where $\hat{y}_0^i = (y_1, y_2, ..., y_{i-1}, 0, y_{i+1}, ..., y_n)$ and $y_i \in \mathbb{R}_+$.

Let $\mathscr{L}$ be defined by [5] and all assumptions on $\mathscr{L}$ hold. Following the known results [5], there is a bounded linear operator L satisfying

(1) L maps $C(\overline{\Omega})$ to $C^{2+\hat{\mu}}(\overline{\Omega})$.

(2) If $u \in C^{2+\hat{\mu}}(\overline{\Omega})$ and $v \in C^{\hat{\mu}}(\overline{\Omega})$ satisfy $u = Lv$, then u and v satisfy

$$\begin{cases} \mathscr{L}u(z) = v(z) & \text{in } \overline{\Omega}, \\ u(z) = 0 & \text{on } \partial\Omega \end{cases} \tag{3.4}$$

see [1, Theorem 4.2]

Let $\psi \in C^{\hat{\mu}}(\overline{\Omega})$ and let

$$L_\psi(u) = L(\psi u) \quad \text{for } u \in C(\overline{\Omega}) \tag{3.5}$$

and

$$\underline{\psi} = \min\{\psi(z) : z \in \overline{\Omega}\}.$$

7

It is well known that if $\psi \in C^{\hat{\mu}}(\overline{\Omega})$ with $\psi > 0$, then $L_\psi : C(\overline{\Omega}) \to C^\sigma(\overline{\Omega}) \subset C(\overline{\Omega})$ is a compact linear operator such that $\overline{L_\psi}(P) \subset P$ for each $\sigma \in (0,2)$ and there exists $\varphi \in P \cap C_0^{2+\hat{\mu}}(\overline{\Omega}) \setminus \{0\}$ such that

$$\varphi = \mu_1(L_\psi)L_\psi\varphi, \tag{3.6}$$

where $\mu_1(L_\psi) = 1/r(L_\psi)$ and $r(L_\psi)$ is the spectral radius of L_ψ, see [1, Theorem 4.2]

We define an operator $A_i : P^n \to C(\overline{\Omega})$ by

$$(A_i\mathbf{u})(z) = (L_iF_i\mathbf{u})(z) \tag{3.7}$$

where L_i is an operator corresponding to L when $\mathscr{L} = \mathscr{L}_i$ in (3.4) and $F_i : P^n \to C(\overline{\Omega})$ is a Nemytskii operator defined by

$$(F_i\mathbf{u})(z) = f_i(z, \mathbf{u}(z)). \tag{3.8}$$

Let $A : P^n \to C(\overline{\Omega})_n$ by $A\mathbf{u} = (A_1\mathbf{u}, A_2\mathbf{u}, .., A_n\mathbf{u})$. By the known results, it is easy to verify under the conditions (h_1) that A is is compact, and $\mathbf{u} \in P^n$ is a solution of the following fixed point equation

$$\mathbf{u}(z) = A\mathbf{u}(z) \quad \text{for } z \in \overline{\Omega}, \tag{3.9}$$

if and only if $u_i \in C_0^{2+\hat{\mu}}(\overline{\Omega})$ and $\mathbf{u}$ is a nonnegative solution of (3.1), see [5, Lemma 2.2].

Let $r : C(\overline{\Omega}) \to P$ be defined by

$$r(u)(z) = u^+(z) := \max\{u(z), 0\}. \tag{3.10}$$

Then it is easy to know that r is a retraction from $C(\overline{\Omega})$ to P, $r^{-1}(P) = C(\overline{\Omega})$ and $r : C(\overline{\Omega}) \to P$ is a Lipschitz continuous map with Lipschitz constant 1.

Theorem 3.1. *Assume that (C_4), (h_1) and (h_2) hold. Then tA is a r_n-nowhere normal-outward map on P^n relative to P^n for $t \in (0,1]$, where A_i and r are same as in (3.7) and (3.10), respectively.*

Proof. Let $\mathbf{u} = (u_1, u_2, ..., u_i, ...u_n) \in C(\overline{\Omega})_n$ such that $\mathbf{u}(z) = tA(r_n\mathbf{u})(z)$. Then $u_i(z) = tA_i(ru_i)(z)$ for any $i \in I_n$. Similarity to the proof of Theorem 3.1 [5], we obtain $u_i \in P$ and $\mathbf{u} \in P^n$. $\square$

By using Theorems 2.1 and 3.1, we prove our main result on the existence of nonzero nonnegative solutions of (3.1).

Theorem 3.2. *Assume that (C_4), (h_1), (h_2) and the following conditions hold.*
   (H_1) For any $i \in I_n$, there exist $\phi_i \in C^{\hat{\mu}}(\overline{\Omega})$ with $\underline{\phi_i} > 0$, $\varepsilon_i \in (0, \mu_1(L_{\phi_i}))$ and $u_0^i \in P$ such that

$$f_i(z, y) \leq u_0^i(z) + \left(\mu_1(L_{\phi_i}) - \varepsilon_i\right)\phi_i(y)y_i \quad \text{for } z \in \overline{\Omega} \text{ and } y \in \mathbb{R}_+^n.$$

   (H_2) There exists $i_0 \in I_n$ and $\psi_\rho \in C^{\hat{\mu}}(\overline{\Omega})$ with $\underline{\psi_\rho} > 0$ such that

$$f_{i_0}(z, y) \geq \mu_1(L_{\psi_\rho})\psi_\rho(y)y_{i_0} \quad \text{for } z \in \overline{\Omega}, \ |y| \in [0, \rho] \text{ and } y \in \mathbb{R}_+^n.$$

   Then (3.1) has a solution $\boldsymbol{u} \in P^n \setminus \{0\}$ with $u_i \in C_0^{2+\hat{\mu}}(\overline{\Omega})$.

Proof. By Theorem 3.1, we have that tA is a r_n-nowhere normal-outward map on P^n relative to P^n for $t \in (0, 1]$.
   Since $L_i(P) \subset P$, by (H_1) we have

$$A_i\boldsymbol{u}(z) \leq u_1^i(z) + \left(\mu_1(L_{\phi_i}) - \varepsilon\right)L_{\phi_i}(u_i) \quad \text{for } \boldsymbol{u} \in P^n \text{ and } z \in \overline{\Omega}.$$

where $u_1^i(z) = L(u_0^i)(z) \geq 0$ for $z \in \overline{\Omega}$. Hence, A satisfies Theorem 2.1 $(LS)_n$.
   By (H_2) and $L_{i_0}(P) \subset P$, we have

$$A_{i_0}\boldsymbol{u}(z) \geq \mu_1(L_{\psi_\rho})L_{\psi_\rho}(u_{i_0}) \quad \text{for } \boldsymbol{u} \in \partial P_\rho^n \text{ and } z \in \overline{\Omega}.$$

and A satisfies Theorem 2.1 $(E)_n$.
   By Theorem 2.1, there exists $\boldsymbol{u} \in P^n \setminus P_\rho^n$ such that $\boldsymbol{u} = A\boldsymbol{u}$. It follows that $\boldsymbol{u}$ is a solution of (3.1) and $u_i \in C_0^{2+\hat{\mu}}(\overline{\Omega})$. $\qquad\square$

   Notation: Let $\psi_i : \overline{\Omega} \to (0, \infty)$ with $\underline{\psi_i} > 0$. For $y_i \in (R^+ \setminus \{0\})$, let

$$\underline{f_{\psi_i}}(y_i) = \inf_{(z,y)\in\overline{\Omega}\times R_+^n} \frac{f_i(z,y)}{\psi_i(z)}, \quad \overline{f_{\psi_i}}(y_i) = \sup_{(z,y)\in\overline{\Omega}\times R_+^n} \frac{f(z,y)}{\psi_i(z)},$$

$$(f_{\psi_i})_0 = \liminf_{y_i \to 0+} \underline{f_{\psi_i}}(y_i)/y_i, \quad (f_{\psi_i})^\infty = \limsup_{y_i \to \infty} \overline{f_{\psi_i}}(y_i)/y_i.$$

As a special case of Theorem 3.2, we obtain the following result.

Corollary 3.1. *Assume that (C_4), (h_1), (h_2) and there exists $\psi_i \in C^{\hat{\mu}}(\overline{\Omega})$ with $\underline{\psi_i} > 0$ such that the following condition holds.*

$(H_1)'$ $(f_{\psi_i})^\infty < \mu_1(L_{\psi_i})$ for any $i \in I_n$

$(H_2)'$ $\mu_1(L_{\psi_{i_0}}) < (f_{\psi_{i_0}})_0$ for some $1 \leq i_0 \leq n$.

(H_3) *For any $r \in (0, \infty)$ and i, there is $M_r^i > 0$ such that $f_i(z, y) \leq M_r^i$ for $z \in R_+^n$ and $y_i \in [0, r]$*

Then (3.1) *has a solution* $\boldsymbol{u} \in P^n \setminus \{0\}$ *with* $u_i \in C_0^{2+\hat{\mu}}(\overline{\Omega})$.

Proof. By $(H_1)'$ and (H_3), there exists $\varepsilon_i > 0$ such that

$$f_i(z, y) \leq \left(\mu_1(L_{\psi_i}) - \varepsilon_i\right)\psi_i(z)y_i \quad \text{for } (z, y) \in \overline{\Omega} \times R_+^n \text{ and } y_i \in [r_i, \infty)$$

and

$$f_i(z, y) \leq M_{r_i}^i + \left(\mu_1(L_{\psi_i}) - \varepsilon_i\right)\psi_i(z)y_i \quad \text{for } (z, y) \in \overline{\Omega} \times R_+^n.$$

By $(H_2)'$, it is easy to see that there exists $\rho > 0$ such that $f_i(z, y) \geq 0$ for $z \in \overline{\Omega}$, $\|y\| \leq \rho$ for every $i \in I_n$ and

$$f_{i_0}(z, y) \geq \mu_1(L_{\psi_{i_0}})\psi_{i_0}(z)y_{i_0} \quad \text{for } (z, y) \in \overline{\Omega} \times R_+^n \text{ and } |y| \in [0, \rho].$$

The result follows from Theorem 3.2. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.1. Let $f_i(z, y) = \psi_i(z)y_i e^{\alpha_i - \Sigma_{j=1}^n \beta_{ij}y_j - \frac{m_i}{1 + \Sigma_{j=1}^n k_{ij}y_j}} - d_i y_i^{\sigma_i}$ or $f_i(z, y) = \frac{s_i y_i^{\gamma_{ii}}}{1 + y_i^{\gamma_{ii}} + \Sigma_{j=1, j \neq i}^n a_j y_j^{\gamma_{ij}}} - d_i y_i^{\sigma_i}$, all the parameters are nonnegative, $\sigma_i \geq 1$. When $\sigma_i = 1$, population models of Ricker types (3.1) and Beverton-Holt types (4.1) in [10] were studied. However, the results can be not utilized to discuss a case of $\sigma_i > 1$, but we can treat the case of $\sigma_i > 1$ since $f_i(z, \hat{y}_0^i) \geq 0$.

4 Conclusions and Remarks

We study the existence of nonnegative solutions of a nonlinear system consisting of nowhere normal-outward maps in Banach spaces, this type of maps defined on a cone does not necessarily take values in the cone and contains weakly inward maps and generalized inward maps under the appropriate conditions [3]. In history, someone established the fixed point index of weakly inward maps or generalized inward maps, but it is difficult to use the index since verifying the weakly inward maps or generalized inward maps is very difficult [9]. From the discussion in this paper, we know that verifying the nowhere normal-outward map is easy in applications. The existence of nonnegative solutions [5] of a single elliptic BVP is extended to system (3.1) via using Theorem 2.1, system (3.1) has many uses such as population models [10] or Volterra-Lotka competition models [11] or steady-state periodic solutions for parabolic systems [12]. We hope that the results of this paper will be applied in many fields.

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