Order, Chaos and Symmetry in the Deterministic Chains of Elements

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Abstract

A new concept of order and chaos has been suggested, whereby any system's level of order can be determined provided existence of the order establishment procedure. This work develops a mathematical tool that realizes the above concept while analyzing a level of order in the deterministic chains of elements. Compositions of open chains (words) are being analyzed. Determined hereby are compositions of words enjoying ideal symmetry, as well as the procedure of step-by-step transformation of open sequence of elements into the ideal-symmetry composition. The level of order in a word is determined by the number of steps in the defined procedure that transforms this word into the symmetric-state composition. To describe a word or a word composition the A matrix is being used, components of which are the closest neighbors' numbers of pairs. It has been shown that the level of order is calculated by expansion of the A matrix to matrices that correspond to the ideal-symmetry compositions. The theorem has been formulated and proved as to the type of the pair matrix expansion to matrices that correspond to the ideal-order compositions. Maximum and minimum symmetry word structures have been found. It has been shown that, in general, the minimum symmetry state is a far cry from the maximum entropy condition found in the Shannon's classic information theory.

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1 Introduction

Order and chaos issue is one of the most vague problems in the modern natural science. In near-homogenous physical systems degree of order is being described by a continuous quantity - entropy. In case a system consists of deterministically interconnected discrete elements, the notion of entropy, generally speaking, is no longer applicable. However, the notion of entropy, as well as that of complexity, remain today the key research tool [1]- [8]. Let us consider a chain of interconnected physical elements or a deterministic sequence of symbols. The following questions are worth detailed discussion:

- What is the amount of information in this chain?
- How complex is this chain?
- How much meaning does the chain's information carry?
- How much is this chain ordered? (What is the amount of order?)

The Shannon's classic information theory answers the first question [2]- [4]. In this case we shall simulate a deterministic sequence of elements as a Markov random chain, having determined a statistical conditional probability of this element's occurrence at the known m of the preceding ones. The theory of algorithmic complexity of different types answers the second question [5, 6]. Recently, both questions are being actively discussed in the context of the biological systems' research [7, 8]. Diverse methods of the complexity's statistical estimation are being offered. As to the meaning contained in information, both the Shannon's classic information theory and the Kolmogorov complexity operate exclusively with the syntactic, and not the substantial content of information. It should be noted, however, there are attempts today to experimentally extract substantial content in the DNA complexity assessment [7]. Quite naturally, the modern ways of information and complexity calculation are based upon probabilistic and statistic approach.

Let us now address the level of order issue. Level of order is generally deemed to be determined by entropy. Therefore the problem could seem irrelevant. The classic theory accepts that the more information (entropy) a system has, the less the order. Chaos turns out to be more informative. A number of works published recently assert that relation between entropy and order in physical systems could be reverse [9]- [11]. In any case, until now all disputes on order, chaos and level of symmetry have been carried out based exclusively upon the notion of entropy or algorithmic complexity. For the deterministic sequence of elements, however, it is very much expedient to determine the level of order without resorting to the probability theory's methods. This is why we suggest introducing a purely combinatorial measure that would account, among other things, for the certain symmetry properties. The philosophical concept of such measure's building had been formulated by us earlier [12]. *A system's degree of order can be determined provided order establishment rules have been formulated - either those set up a priori, or those corresponding to the real physical laws.* At this very stage we shall only discuss the order establishment rules, determined based upon some common-sense expectations. When considering a sequence of elements, one of those possible order establishment methods is to denote certain chain states as ideal and to set a procedure that will transform the chain from the condition with the fixed location of elements to the ideal state. It is advisable to select high-symmetry conditions as the ideal ones. Above in [12], the degree of order has been determined for a composition of closed words, composed of equal number of each of the fixed k of different elements. Research has been based upon expansion of the number of neighboring pairs' matrix into matrices describing symmetric states. In corresponding matrices the sums of elements in each line and each column are equal. In this work essential generalizations have been made. The ideal order states have been defined for a composition built of open words. Expansion of a number of neighboring pairs' matrix into the ideal state matrices in case of arbitrary number of each of the fixed k of different elements has been received. In this paper the general condition has been found, in which a word reaches the maximum asymmetry state. This state has been compared with the state of maximum entropy in the classical information theory. In the Shannon's theory maximum entropy describes chaotic, e.c. equiprobable, memoryless distribution of elements in the word. It may seem that the state of chaos is the state of minimal symmetry. But this is not always true. It has been illustrated, that the maximum asymmetry state, in general, is a far cry from the state of chaos by Shannon.

2 Composition of Open Words and the Ideal Symmetry State

Let there be k different elements of any nature. Let us suppose that each element have two connections - left and right. These connections allow the elements to create one-dimensional chains. In this work, in contrast to the previous one, we shall consider open chains, in which the first chain element has a free left connection and the last chain element - a free right connection. We shall call the chains under review words, by analogy with the theory of information. Let there be a certain set of such words, where each of the k elements occurs n times exactly ($n \geq 1$). Let us call this set a composition. Each word, included in such composition, we shall call a composition fragment. Let us give an example of composition for $k = 4$, $n = 5$, that has three fragments:

$$abda \qquad acccb \qquad bddddaaccbb.$$

Any open fragment that creates a periodic chain in the ring closing we shall call "periodic". For example, we call periodic the fragment *abcabc*. Open periodic fragments with the same period may differ in length and first element. Let us agree to hereinafter bind all fragments having identical period and identical first element into a single fragment. For example:

$$abcabc \quad abc \quad abc \rightarrow abcabcabcabc$$

Equal number of all different elements creating a composition, allows to define the most symmetric compositions we shall hereby call "ideal". This definition will be different from the corresponding one in the previous paper [12] because we consider a composition built of open words.

Let us call a composition the "ideal" one, provided:

1. All composition fragments are periodic. Number of elements, denoted by l, in the minimal period, complies with inequation $1 \leq l \leq k$. All elements in the minimal period are different.

2. Each of the k elements belongs to only one of the periods, but it may also belong to different fragments.

3. An open fragment may begin with any element that belongs to a relevant period.

Let us build an ideal composition consisting of nk elements. First, let us build a set of short fragments out of k initial elements in such a way, that each element would belong to one fragment only ($n = 1$). Let us arrange these fragments ascending in length, equal to the number of elements. Let m_1 fragments have length of l_1, m_2 fragments have length of l_2, m_i fragments have length of l_i, at that $l_i > l_{i-1}$ ($1 \leq i \leq t$, where t is the number of the fragments' different lengths). For example, for 8 initial elements designated by digits 1-8 one can build the following set of 5 fragments ($m_1 = 2$, $l_1 = 1$, $m_2 = 3$, $l_2 = 2$):

$$1 \quad 2 \quad 34 \quad 56 \quad 78.$$

Apparently, for the total number of short fragments we have:

$$1 \leq \sum_{i=1}^{t} m_i \leq k. \tag{1}$$

And, besides:

$$\sum_{i=1}^{t} m_i l_i = k. \tag{2}$$

Elements that entered each of the short fragments in some order, we shall consider a minimal period of the fragment of the ideal composition being created. Let us now expand each of the created fragments by repeating its period n times. The set thus created presents an ideal-order composition with the fixed set of periods and minimum number of fragments. Let us denote number of fragments by s and ideal-order composition with minimum number of fragments by $Id(k, n, s = s_{min})$. Let us give an example of an ideal composition, where $Id(k = 6, n = 5, s_{min} = 3)$. (The elements are denoted by digits 1-6.)

$$11111 \quad 2323232323 \quad 546546546546546. \tag{3}$$

All fragments of the ideal composition with minimum number of fragments have different periods. An ideal composition with arbitrary number of fragments $Id(k, n, s)$ may include fragments with identical periods, different in length and first element. Two ideal compositions $Id(k, n, s_1)$ and $Id(k, n, s_2)$ we shall consider equivalent, provided they consist of the same set of different periods, notwithstanding the number of fragments and which element of the fragment occurs first. Let us give an example of an ideal composition $Id(k=6, n=5, s=5)$, equivalent to (3):

$$11111 \quad 232323 \quad 3232 \quad 546546 \quad 465465 \quad 654. \tag{4}$$

Apparently, any ideal composition is equivalent to one of the ideal compositions with minimum number of fragments. The maximum possible number of an ideal composition's fragments is limited by a number of elements the fragment may begin with, that is $s_{max} = k$, unlike an ideal composition with closed fragments [12], where $s_{max} = nk$.

Let us calculate number $Q(m_1, ..., m_t, l_1, ..., l_t)$ of different nonequivalent ideal compositions with minimum number of fragments for the fixed set of numbers m_i, l_i. Clearly, for calculation it will suffice to consider compositions with short fragments, i.e. to assume $n = 1$. Let us arrange all k elements in a row. In total there are $k!$ such arrangements. Let us now divide a row of such arrangements into consecutive fragments - as above, ascending in length: (m_i fragments l_i long, where $1 \leq i \leq t$). Following the aforementioned division, a part of $k!$ arrangements leads, for the given set of m_i, l_i to equivalent compositions. Specifically: rearrangement of equal-length fragments ($m_i!$ combinations) and cyclic interchange inside every fragment, whereby the last element falls into the place of the first one (l_i combinations), give birth to equivalent ideal compositions. Consequently we have:

$$Q(m_1, ..., m_t, l_1, ..., l_t) = \frac{k!}{l_1^{m_1} l_2^{m_2} ... l_t^{m_t} m_1! m_2! ... m_t!}. \tag{5}$$

where m_i, l_i satisfy formulas (1),(2). The total number of different nonequivalent ideal compositions with minimum number of fragments:

$$\sum_{m_i, l_i} Q(m_1, ..., m_t, l_1, ..., l_t) = k!, \tag{6}$$

where the sum goes on all possible $m_1, ..., m_t$, $l_1, ..., l_t$ values.

3 Description of the Words Symmetry by Means of the Closest Neighbors Pair Matrix

Sequence of any two following each other left to right symbols in the word we shall call a pair, for brevity sake. Let us designate elements of the alphabet, the composition had been built from, by indices i, j:

$$1 \leq i \leq k, \qquad 1 \leq j \leq k.$$

Let us denote the total number of pairs in the composition words, which have i element to the left and j element to the right, as a_{ij} . Now, in order to describe the composition, let us introduce the pair matrix A of order k, components of which a_{ij} are nonnegative integers:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1k} \\ a_{21} & a_{22} & ... & a_{2k} \\ ... & ... & ... & ... \\ a_{k1} & a_{k2} & ... & a_{kk} \end{pmatrix}. \tag{7}$$

In the matrix (7), unlike [12], the sum of elements in each line and each column is not necessarily a fixed number. Thus, expansion of the matrix (7) we are about to obtain, covers both open-word compositions and compositions with unequal number of elements contained therein. We are about to demonstrate that the pair matrix (7) in the present review may be expanded into the ideal-order matrices - the same way we did in [12]. Let us examine an I_1 unit matrix of order k, as well as all possible I_m matrices, received from the former by line rearrangement. I_m matrices shall be called the ideal-order matrices. Apparently, in total there are $k!$ of different I_m matrices. We shell denote $I = \{I_1, I_2, ...I_{k!}\}$ the set of all I_m matrices. Let us denote by $U_m(s)$ matrices of equivalent ideal compositions with different number of fragments s $(s_{min} \leq s \leq s_{max})$. The matrix $U_m(s)$ is similar to matrix I_m. To each ideal composition $Id(k, n, s_{max})$ having maximum possible number of fragments $s_{max} = k$ corresponds a $U_m(s_{max})$ pair matrix that may be expressed by means of the I_m matrix in the following way:

$$U_m(s_{max}) = (n-1)I_m. \tag{8}$$

But if $s < s_{max}$, certain nonzero elements of the $U_m(s)$ matrix will equal n, and not $n-1$, in contrast to (8), whereas zero elements will occupy the same, as in (8) places. Should composition consist of a single open periodic fragment, then $k-1$ of the $U_m(1)$ nonzero elements will equal n, and one will equal $n-1$.

4 Expansion of a Square Matrix With Nonnegative Integer Components Into the Ideal-order Matrices

Theorem 1 *Square matrix A, ,components of which a_{ij} are nonnegative integers, may be expanded as follows:*

$$A = \sum_{l=1}^{r} x_l I_{m_l} + A_r, \tag{9}$$

where x_l - natural positive expansion coefficients, $I_{m_l} \in I$, $r \leq (k-1)^2 + 1$, and A_r matrix does not allow any further expansion, that is to say there are no such x_{r+1} and $I_{m_{r+1}}$ to which all $A_r - x_{r+1} I_{m_{r+1}}$ matrix components might be nonnegative. The set of I_{m_l} matrices in (9) for $1 \leq l \leq r$, as well as the A_r matrix, have not been uniquely defined. However, for the set of I_{m_l} matrices, expansion (9) of which has been performed, there is a single set of expansion coefficients x_l.

Proof. First of all, let us note that for $k = 2$ expansion (9) is unique and it is obvious that $r \leq 2$. Therefore, the theorem proof offered below falls into the $k \geq 3$ case. To begin with, let us describe the procedure that leads to expansion (9). Let us assume there will be an ideal-order matrix I_{m_1}, nonzero elements of which $(I_{m_1})_{ij} \neq 0$ are inconsistent with the corresponding zero elements $a_{ij} = 0$ of the A matrix, if any of such zero elements exist. Let us superimpose I_{m_1} matrix on A matrix. Let the smallest element in the A matrix transverse with the I_{m_1} matrix nonzero elements equal x_1. Then

$$A = x_1 I_{m_1} + A_1. \tag{10}$$

In locations, where in the A matrix elements' transverse with the I_{m_1} matrix nonzero elements there were elements equal to x_1, the residual matrix A_1 will have zeroes. Now, we can try to expand the A_1 matrix according to the scheme (10). Provided expansion is possible, we shall have $A_1 = x_2 I_{m_2} + A_2$, where A_2 is the next residual matrix. Proceeding with the similar expansion, we reach the finit recurrent procedure:

$$A_{l-1} = x_l I_{m_l} + A_l. \tag{11}$$

In(11) we have: $1 \leq l \leq r$,$A = A_0$. Following l steps of the recurrent procedure (11), we get:

$$A = Y_l + A_l, \tag{12}$$

$$Y_l = \sum_{s=1}^{l} x_s I_{m_s}. \tag{13}$$

Further recurrent procedure (11-13) becomes impossible as soon as during superimposition of the A_l residual matrix on any of the I_m matrices a one of the latter becomes consistent with a zero of the former. We need to prove that the number of steps of the recurrent procedure (11), required to reach indecomposability of A_l, does not exceed $(k-1)^2+1$. Y_l matrix elements, hereinafter denoted as $y_{ij}(l)$, have properties as follows:

1. The sums of elements in any line or column are equal:

$$\sum_{j=1}^{k} y_{ij}(l) = \sum_{j=1}^{k} y_{ji}(l) = n(l), 1 \leq i \leq k, \tag{14}$$

   where

$$n(l) = \sum_{s=1}^{l} x_s. \tag{15}$$

2. At least l elements of the Y_l matrix are equal to the corresponding elements of the A matrix. These elements are fixed, i.e. during the next step of the recurrent procedure they remain unchanged.

Properties (14) may be considered as a system of linear homogeneous equations in relation to k^2+1 variables (all $y_{ij}(l)$ elements and $n(l)$). In (14) there are $2k-1$ independent linear equations and, therefore, $(k-1)^2+1$ free variables. (Variables, located in i line and j column of the Y matrix, may be considered basic, whereas the rest of the $Y(l)$ matrix elements and $n(l)$ may be considered free.) Following each step of the recurrent procedure, one or several elements, not equal to zero in A_{l-1} matrix, become zero in

A_l matrix. Let one of these elements occur in i_l line and j_l column. The corresponding elements of A and $Y(l)$ matrices then become equal:

$$y_{i_l,j_l}(l-1) \neq a_{i_l,j_l}, \tag{16}$$

$$y_{i_l,j_l}(l) = a_{i_l,j_l}. \tag{17}$$

(In case during a single step of the recurrent procedure (11) several A_l matrix elements become zeroes, this is due to possible additional linear relations between its elements). Following p iterations in (11), the Y_l matrix elements, for which, (17) has been performed, become fixed for all $l \geq p$, i.e. they remain unchanged in further iterations. After each iteration system of equations (14) for $y_{ij}(l)$ and $n(l)$ can be expanded by attaching equation (17) for a single pair (i_l, j_l) to the system (14). The new system is not homogeneous and contains $2k-1+p$ linearly independent equations. Indeed, equation (17) with $l = p$ cannot be a derivative of equation (14) and $p-1$ of any preceding equations (17). If this were the case, then from equations (14) and $p-1$ equations (17) it would follow

$$y_{i_p j_p} = a_{i_p j_p}. \tag{18}$$

Equation (18) would mean that element $y_{i_p j_p}$ remains unchanged under transition from iteration $l = p-1$ to iteration $l = p$, which contradicts conditions of (17,16).

The iteration procedure completes as soon as the number of variables of the system (14,17) equals the number of independent equations of the same system, i.e. p reaches its highest possible value when:

$$k^2 + 1 = 2k - 1 + p. \tag{19}$$

Thus, the maximum possible number of iterations is:

$$r \leq p_{max} = k^2 - 2k + 2. \tag{20}$$

In order to prove the uniqueness of expansion (9) for determinated set of I_{m_l} matrixes, it will suffice to ascertain linear independence of $I_{m_1}, I_{m_2}, ..., I_{m_r}, A_r$ matrices, provided $A_r \neq 0$, or linear independence of $I_{m_1}, I_{m_2}, ..., I_{m_r}$, provided $A_r = 0$. Let us equate linear combination of these matrices to zero:

$$\sum_{l=1}^{r} \alpha_l I_{m_l} + \alpha_{r+1} A_r = 0. \tag{21}$$

Let us assume that during the first step of the recurrent procedure a certain element, not equal to zero in A matrix, $((A)_{i_1 j_1} \neq 0)$ becomes zero in A_1 matrix:

$$(A_1)_{i_1 j_1} = 0. \tag{22}$$

Then, as it follows from the stated recurrent procedure:

$$\forall l \geq 2 : (I_{m_l})_{i_1 j_1} = 0, \qquad (A_r)_{i_1 j_1} = 0. \tag{23}$$

From (21,23) it follows that $\alpha_1 = 0$. In the similar way, for each subsequent step s for $2 \leq s \leq r-1$ we have $(A_{s-1})_{i_s j_s} \neq 0$,

$$(A_s)_{i_s j_s} = 0, \quad \forall l \geq s+1 \quad (I_{m_l})_{i_s j_s} = 0, \quad (A_r)_{i_s j_s} = 0 \Rightarrow \alpha_s = 0. \tag{24}$$

The last step of the recurrent procedure will give us

$$(A_{r-1})_{i_{r-1} j_{r-1}} \neq 0, \qquad (A_r)_{i_{r-1} j_{r-1}} = 0,$$

from which $\alpha_r = 0$. Now it arises from (21) that either $\alpha_{r+1} = 0$ or $A_r = 0$. In this way the required linear independence has been proven.

The foregoing makes it clear that the iteration procedure may be completed at $p < p_{max}$. Let us single out here two special cases. Inasmuch as the sum of elements in any line or column of the Y matrix equals n, then, as soon as all elements in any such line (or column) become fixed, no further iterations become possible. Provided the initial A matrix had no zeroes, the minimum number of iterations in this case equals k. Another interesting case is when at step l, M^2 elements of the Y matrix, filling the square, for the elements with indices $1 \leq i, j \leq M$, were $2M > k > 2$, become fixed. In this case, maximum after $l = M^2$ iterations, a one in any of the I_m matrices shall necessarily superimpose on zero in A_l.

Note to the Theorem 1. Let A matrix have following properties:

- Elements a_{ij} of the A matrix are tied by s linear, homogenous and linearly independent conditions, which, together with $2k-2$ linearly independent conditions, found in equations

$$\sum_{j=1}^{k} a_{ij} = \sum_{j=1}^{k} a_{ji}, \quad 1 \leq i \leq k, \tag{25}$$

  form a system of $2k-2+s$ homogenous, linearly independent equations.

- There exists a recurrent procedure (11), in which the same s conditions are valid for A_l (and, consequently, for Y_l) at any l.

Then, in addition to (14) we have more s linear conditions at each iteration step and, consequently, in expansion (9):

$$r \leq p_{max} = k^2 - 2k + 2 - s. \tag{26}$$

The described requirements can be met by, for example, conditions of the $a_{ij} = 0$ type. (Because of (25) the number of zeroes in A in this case may exceed s.) The similar situation may exist provided existence of the symmetry conditions:

$$a_{i_1,j_1} = a_{i_2,j_2},\ where\ i_1 \neq i_2, j_1 \neq j_2.$$

5 Transformation of the Open-end Word into the Ideal-order composition

Let there be given a single word, built out k elements, each occurring exactly n times. Let us build an A pair matrix for this word and then expand it according to the Theorem 1. Let us then describe the step-by-step transformation procedure of this word into the ideal-order composition that corresponds to the first matrix I_{m_1} in the expansion (9). Let us perform first x_2 steps. In the word, let us cut k_1 pairs, being described by the I_{m_2} matrix, but not occurring in I_{m_1} ($k_1 \leq k$). We get $k_1 + 1$ fragments. Let us patch these fragments so that the new composition adds up the maximum possible number of pairs, corresponding to the I_{m_1} matrix. Provided possibility of adding k_1 such pairs, then value x_1 in expansion (9) for the new composition's $A^{(1)}$ matrix will increase by 1 in comparison to A . Provided it is impossible to add all k_1 pairs in the above manner, a certain part of fragments will remain unpatched and we will get a composition of fragments, not a single word. In any case, expanding an $A^{(1)}$ matrix into the same ideal matrices as above, instead of x_2 we shall get x_2-1 . This completes the first step. If $x_2 > 1$, then in the composition received we shall execute further cuts, corresponding to I_{m_2}, but not occurring in I_{m_1} and perform further maximum possible number of fragment patchings to increase coefficient x_1 in (9). In the same manner we execute x_2 steps, upon which the expansion coefficient ahead of I_{m_2} in (9) becomes zero. Then, in the similar way, let us perform further $x_3, x_4, ...x_r$

steps of cuts - patchings that correspond to matrices $I_{m_3}, ..., I_{m_r}$. The last step we shall consider the cuts to be performed in order to zero A_r in the final composition. The described procedure has the total of $n - x_1$ steps. Examples of arrangement can be found in the appendix.

6 Degree of Order and the Minimum Symmetry State

Let us examine a certain open word and all possible expansions (9) of its pair matrix. Let us select one of the I_m matrices that occurs in at least one of these expansions. Of all these expansions, let us select the one, in which coefficient x_m in front of I_m has the maximum possible value. Let us denote:

$$g_m = max\{x_m\}, \tag{27}$$

where maximum is taken by all possible expansions (9), in which I_m matrix occurs. The g_m value equals the smallest of A matrix elements, located in places, corresponding to the non-zero elements of the I_m matrix. The g_m value shows, how close this word is to the state of symmetry, described by the I_m matrix. Specifically, $g_m = n - 1$ means the word is very close to the ideal symmetry state and it takes no more than one step to transform it into the ideal composition, corresponding to I_m. Let us arrange each of the possible expansions (9) so that the expansion coefficients will not increase:

$$x_1 \geq x_2 \geq x_3 \geq ... \geq x_r.$$

Of all expansions arranged in this manner, let us select the one having maximum x_1 and determine:

$$G = max\{x_1\} = max\{g_m\}. \tag{28}$$

Then value

$$N = n - G. \tag{29}$$

describes the minimum number of the arrangement procedure steps required to transfer the word in question to the closest ideal-order composition. In this sense G is a measure to describe the given word's symmetry level. The closer the G value is to n , the more symmetric is the word.

Further it would be interesting to determine the word structures, under which G reaches its maximum and minimum values at the given n. Maximum degree of order $G_{max} = n-1$ we have for composition with pare matrix $U_m(s)$. Maximum degree of order for an open single word is being reached in two cases:

- A word represents an ideal composition, consisting of one fragment, for example:

  123123123123123.

- In a single arrangement procedure step a word may be transformed into an ideal-order composition, for example:

  11111222233334444.

In both cases we have:

$$G_{max} = n - 1. \tag{30}$$

In our model G value is the function of the pair matrix $G = G(A)$. Let us look for such pair matrix structure at the given n, at which G reaches its minimum value. Value G_{min} will correspond to such words, for which the number of steps N, required for transition to the closest ideal-symmetry composition, is maximum. It may naturally be expected, that in this case the numbers of steps required for transitions to any of the ideal-order states, described by one of the matrices in expansion (9) are roughly equal (that is g_i differ by not more than 1). This is substantiated by the following theorem.

Theorem 2 *Let A , matrix be given, in which the sum of elements in any line and any column equals n . Some of the A matrix elements may equal zero. Let in this matrix $i \neq j$, be found, such that*

$$g_i - g_j \geq 2. \tag{31}$$

Then there exists A' matrix with the same value of n, in which:

- *In both A and A' zeroes are located in the same very places;*

- *for all $i \neq j$ of the A' matrix we have:*

$$|g_i - g_j| \leq 1. \tag{32}$$

- $G(A') \leq G(A)$.

Conclusion, formulated above, follows from the g_i balancing possibility by multiple application of the following procedure:

$$A_1 = A + \beta_{ij}(I_j - I_i), \tag{33}$$

where $\beta_{ij} = \lfloor 0.5(g_i - g_j) \rfloor$, I_i, I_j are the ideal symmetry matrices, corresponding to g_i, g_j respectively. At that, ones of the I_i, I_j are inconsistent with zeroes of the A matrix. In reality, we shall execute each step of the procedure(33) for such g_i, g_j pair, in which g_i has its maximum value, and g_j is at its minimum. Then, at every step the minimum g_j will increase at least by one and, as the final result, we shall get the A' matrix.

From the Theorem 2 it follows that for a matrix, possessing property (32) at any $i \neq j$, G value reaches its local structural minimum. Let us initiate our search for a structure, whereby the absolute value of G_{min} , is being reached, from a special case, when for all possible I_i in expansion (9) we have $g_i = L$. In this case the A pair matrix expands according to the iteration procedure (11) into r matrices having equal expansion coefficients $x_i = L$; at that, $A_r = 0$, and the sum of elements of any line and any column in the A matrix equals $n = rL$ (in this way n is aliquot to r). The structure that corresponds to G_{min} will be reached at the maximum possible value of $r = r_{max}$. From symmetry considerations it is apparently enough to look for the A' matrix, in which r elements, equal to L, fill up a certain rectangle. Let us denote the number of this rectangle's horizontal elements as x, and the number of its vertical elements as y ($x \geq 1, y \geq 1$). Let us divide the A' matrix into 4 rectangular blocks, as shown below:

$$\mathbf{A}' = \begin{pmatrix} L & \dots & L & a_{1,x+1} & \dots & a_{1,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ L & \dots & L & a_{y,x+1} & \dots & a_{y,k} \\ a_{y+1,1} & \dots & a_{y+1,x} & a_{y+1,x+1} & \dots & a_{y+1,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k,1} & \dots & a_{k,x} & a_{k,x+1} & \dots & a_{k,k} \end{pmatrix}, \tag{34}$$

The first necessary condition of expanding A' into r matrices with equal coefficients is obvious:

$$xy = r. \tag{35}$$

The second necessary condition follows from the prerequisite that at every expansion step only one element of the $1 \leq i \leq y, 1 \leq j \leq x$ rectangle must be set to zero. In the latter case, $k-1$ elements of the I_{m_l} matrix in (9) occur respectively in $x-1$ columns under the rectangle with elements L and in $y-1$ lines to the right of the said rectangle. In this way we get:

$$x + y = k + 1. \tag{36}$$

Inasmuch as the sum of elements in any line and any column equals rL, we have:

$$\forall\, 1 \leq i \leq y : \sum_{j=x+1}^{k} a_{ij} = L(r-x), \tag{37}$$

$$\forall\, 1 \leq j \leq x : \sum_{i=y+1}^{k} a_{ij} = L(r-y), \tag{38}$$

$$\sum_{i=1}^{y} \sum_{j=x+1}^{k} a_{ij} = yL(r-x). \tag{39}$$

Hence, using (35, 36) we have:

$$\sum_{i=y+1}^{k} \sum_{j=x+1}^{k} a_{ij} = Lr(k-x) - yL(r-x) = 0. \tag{40}$$

And inasmuch as $a_{ij} \geq 0$ from (40) we get:

$$\forall\, y+1 \leq i \leq k\,, x+1 \leq j \leq k\ : a_{ij} = 0. \tag{41}$$

So, all elements of the rectangle, located in the lower right corner of the A' matrix, must equal zero. From (41) it follows that conditions (35), (36) are sufficient for only one element of the $1 \leq i \leq y, 1 \leq j \leq x$ rectangle being set to zero at every iteration procedure step. The maximum possible r we find from (35), (36). If k is odd, the maximum possible value of r is being reached at $x_{max} = y_{max} = 0.5(k+1)$, and equals:

$$r_{max} = \frac{(k+1)^2}{4}. \tag{42}$$

If k is even, the maximum possible value of r is being reached at $x_{max} = 0.5k$, $y_{max} = 0.5k + 1$ or at $x_{max} = 0.5k + 1, y_{max} = 0.5k$, and equals:

$$r_{max} = \frac{k}{2}(\frac{k}{2} + 1). \tag{43}$$

Level of order of the word, described by the pair matrix (34) is a function of the variable $r : G = G(r)$. At $r = r_{max}$ we get the minimum value for G:

$$G_{min} = L = \frac{n}{r_{max}}. \tag{44}$$

Let us denote the pair matrix that at the given n has the minimum G value as $A_{asym} = A'(r_{max})$. There is no difficulty in understanding that minimum G, described by formula (44) and corresponding to the A' matrix (34), is also the smallest possible value among all matrices, for which n is aliquot to r_{max}. Now, let n be not aliquant to r_{max}, that is $n = r_{max}L + M$, were $1 \leq M < r_{max}$. Then it is possible to build an A_{asym} matrix, using expansion into the same very I_m matrices that occur in the matrix expansion (34):

$$A_{asym} = \sum_{m=1}^{r}(LI_m + x'_m I_m). \tag{45}$$

In (45) at least one of the coefficients x'_m equals zero, whereas all those different from zero equal one. In this case, matrix A_{asym} is similar to (34).The only difference is that in the rectangle $1 \leq i \leq y$, $1 \leq j \leq x$ we get M elements that equal $L + 1$ and $r_{max} - M$ elements that equal L. Level of symmetry for the word, being described by matrix A_{asym}, in this case will also be minimal and equal to $L + 1$, that is:

$$G_{min} = \left\lceil \frac{n}{r_{max}} \right\rceil. \tag{46}$$

Now let us examine an open word, in which every of the k elements occurs n times. In the corresponding pair matrix the sum of elements in one of the lines and one of the columns equals $n - 1$. Let $n > r_{max}$. Then, the A_{asym} matrix that corresponds to the minimal symmetry sequence, is being built similar to the aforementioned method. The only difference is that in the A_{asym} matrix' expansion into the ideal-order matrices, there appears the residual matrix A_r, being one of the I_m matrices with a single 1 deleted. Results for the degree of order (44-46) however, hold. In this way we come to the conclusion as follows.

Theorem 3 *Minimal value of the symmetry level for the words built out of nk elements is:*

$$G_{min} = \left\lceil \frac{n}{r_{max}} \right\rceil, \qquad (n > r_{max}), \tag{47}$$

$$r_{max} = \begin{cases} \frac{(k+1)^2}{4}, & k \text{ iz odd}, \\ \\ \frac{k}{2}(\frac{k}{2}+1), & k \text{ iz even}. \end{cases} \tag{48}$$

The result (47) has been found earlier in [12] for $k = 3$ and $k = 4$.

7 Comparison with Information Theory

Let us examine the above described deterministic open word as stochastic one. In accordance with our model, let us assume that absolute frequency of every element in the word is n and a_{ij} is the frequency of appropriate pare. Thus, statistical probability of occurrence of any element is:

$$p(i) = \frac{1}{k}.$$

There are $nk - 1$ pares in the sequence, therefore probability of occurrence of pare ij is:

$$p(ij) = \frac{a_{ij}}{nk-1}. \tag{49}$$

For conditional probability of element i given j we have:

$$p(i/j) = \frac{ka_{ij}}{nk-1}. \tag{50}$$

Making use of the first order Markov chain's model [4], from equations (49) and (50) we obtain next expression for an entropy H of the word:

$$H = -\frac{1}{k}\log_2\frac{1}{k} - \sum_{i,j} a_{ij}\log_2\frac{ka_{ij}}{nk-1}. \tag{51}$$

For equiprobable distribution of elements the state of maximum entropy (chaos) is being reached when information source is memoryless:

$$p(ij) = p(i)p(j).$$

Components a_{ij} are integers, therefore the maximum value of entropy in (51) is being reached when A matrix elements a_{ij} differ by not more than one. In this case in (9) $r = k$. Given the chaos word structure we have the local minimum of symmetry level G , which equals:

$$G_{chaos} = \left\lceil \frac{n}{k} \right\rceil. \tag{52}$$

At $k = 2$ local minimum of symmetry level (52) is absolute and equals to symmetry level calculated from (47) and 48. Consequently at $k = 2$ chaos by Shannon and minimum of symmetry are being reached at one and the same word structure. However, provided $k \geq 3$, the symmetry level value (52) does not equal to absolute minimum of G, determined in (47). At $k \geq 3$, degree of asymmetry in (47) is substantially higher than in (52). Chaotic distribution appears to be not the most asymmetric.

8 Conclusion

Now we have a simple algorithm for calculating the level of symmetry of elements' chain. To get this level we have to find maximum value of g_m for every m whereby

$$\forall i, j \quad (A - g_m I_m)_{ij} \geq 0.$$

Then the value of symmetry level will be:

$$G = max\{g_1, g_2, ...g_{k!}\}.$$

9 Appendix

Let us give an example of the open word's arrangement into the ideal-symmetry state.

$$112431234224413133441322 \tag{53}$$

The word's pair matrix (53) and its expansion:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 1 & 2 \\ 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 \end{pmatrix} = 2 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} +$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{54}$$

The closest ideal-symmetry state corresponds to the first matrix in the expansion (54). We have $n = 6, G = 2, N = 4$. Below please find the step-by-step procedure of arrangement into the closest ideal-symmetry state. Stars represent all necessary cuts. Following cuts, patching, possible at the given step, are performed. Cross linking points are shown by the break sign. Spaces separate different fragments. Step 1 (corresponds to removal of the second matrix in the expansion)

$$11 * 24312 * 34224413133 * 44 * 1322$$

$$11\_34224413133 \quad 44 \quad 1322\_24312$$

Step 2 (corresponds to removal of the third matrix in the expansion)

$$113422 * 44 * 13133 \quad 44 \quad 13 * 2224312$$

$$13133\_13 \quad 44\_44 \quad 113422\_2224312$$

Step 3 (corresponds to removal of the forth matrix in the expansion)

$$1313 * 313 \quad 4444 \quad 1 * 134 * 22222 * 4312$$

$$1313 \quad 313 \quad 1 \quad 134\_4444\_4312\_22222$$

Step 4 (corresponds to removal of the residual matrix)

$$1313 \quad 313 \quad 1 \quad 13 * 444444 * 31 * 222222$$

$$1313\_13 \quad 313\_1\_31 \quad 444444 \quad 222222$$

Final ideal state: $Id(k = 4, n = 6, s = 4)$:

$$131313 \quad 313131 \quad 444444 \quad 222222$$

Further, let us give an example of the word in the state farthest from the symmetry state and its arrangement procedure.

$$231223113213231223113213122313$$

$$\mathbf{A_{asym}} = \begin{pmatrix} 2 & 3 & 5 \\ 2 & 3 & 5 \\ 6 & 3 & 0 \end{pmatrix} = 3 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + 3 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \\ + 2 \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{55}$$

Arrangement procedure into the 123 periodic state:
Step 1 (corresponds to reduction of the coefficient at the second matrix)

$$23122311 * 32132312 * 23113213122313$$

$$23122311\_23113213122313 \quad 32132312$$

Step 2 (corresponds to reduction of the coefficient at the second matrix)

$$231223112311 * 321312 * 2313 \quad 32132312$$

$$231223112311\_2313 \quad 321312\_32132312$$

Step 3 (corresponds to removal of the second matrix)

$$2312 * 231123112313 \quad 321 * 31232132312$$

$$321\_231123112313 \quad 2312 \quad 31232132312$$

Step 4 (corresponds to reduction of the coefficient at the third matrix)

$$32 * 1231123112313 \quad 2312 \quad 3123 * 21 * 32312$$

$$21\_2312\_32312\_3123\_1231123112313 \quad 32$$

Step 5 (corresponds to removal of the third matrix)

$$2 * 123123 * 23123123123112311231 * 3 \quad 32$$

$$23123123123112311231\_2\_3123123 \quad 32$$

Step 6 (corresponds to removal of the forth matrix)

$$231231231231 * 1231123123123123 \quad 3 * 2$$

231231231231   2_3_1231123123123123

Step 7 (corresponds to removal of the residual matrix)

$$231231231231 \quad 231231 * 123123123123$$

231231231231_231231   123123123123

Final ideal state: $Id(k = 3, n = 10, s = 2)$:

231231231231231231   123123123123

$N = 10 - 3 = 7$ steps in total.

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