Stability Analysis of a Damped Wave Coupled With Heat Conduction

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Abstract

We investigate stability properties of a damped wave coupled with a thermal effects modelled by Cattaneo's law. The well-posedness and uniform exponential stability of the whole system are obtained using semi-group theory. The asymptotic behaviour of the solution of the system through polynomial decay is also discussed by employing multiplier technique.

Keywords: Cattaneo's law, Thermal effect, Energy decay estimate. C_0-semigroup. Exponential stability. Asymptotic stability.

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1 Introduction and Mathematical Formulation

Vibrations stability and controllability of strings or beams arising from different engineering backgrounds has attracted attention of many researchers. There are many coupled systems describing both the elastic behavior of a system and the heat conduction within the system. Such thermoelastic systems have been treated by many authors, for a survey on classical thermoelastic system we cite Jiang and Racke [15], Messaoudi and Said-Houari [12], Racke [14], Grasselli, Rivera and Pata [6] and the references therein. The question of energy decay estimates in the context of stabilization problems has earlier been studied by several author (cf. Chen [3], Komornik and Zuazua [10], Lagnese [7] and the references therein). The linear differential equation describing the vibrations of flexible structures has of the form

$$u_{tt} - a^2\Delta u - a^2\beta\Delta u_t = 0 \quad \text{on} \quad \Omega \times \mathbb{R}^+, \tag{1.1}$$

where Ω is a bounded connected domain in $\mathbb{R}^n$ with a piecewise smooth boundary $\partial\Omega$ and $u = u(x,t)$ denote the deflection of the flexible structure at any point (x,t), $\beta > 0$ is a constant and $a > 0$ is the constant wave velocity. The stabilization of an equation like (1.1) subject to mixed boundary conditions was studied by Bose and Gorain [2]. Recently, Alves *et al* [1] consider a coupled system of realistic linear model, which models the behaviour of a viscoelastic material coupled to a heat conduction equation governed by Fourier's law of heat conduction. Keeping in view with the ideas of (1.1) and Alves *et al* [1], we are concerned mathematically the following system of equations

$$u_{tt} - a^2\Delta u - a^2\beta\Delta u_t + \eta\Delta\theta = 0 \quad \text{on} \quad \Omega \times \mathbb{R}^+, \tag{1.2}$$

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$$\theta_t - \Delta\theta - \eta\Delta u_t = 0 \quad \text{on} \quad \Omega\times\mathbb{R}^+. \tag{1.3}$$

On the other hand, the classical model for the propagation of heat turns into the well-known equations for the temperature θ (difference to a fixed constant reference temperature) and the heat flux vector q as

$$\theta_t + \zeta \operatorname{div} q = 0, \tag{1.4}$$

and

$$q + \nu \nabla\theta = 0, \tag{1.5}$$

with positive constants γ and κ. Relation (1.5) represents the assumed Fourier's law of heat conduction and, plugged into (1.4), yields the parabolic heat equation

$$\theta_t - \zeta\,\nu\Delta\theta = 0. \tag{1.6}$$

It is well known that Fourier's law of heat conduction predicts the physical paradox of infinite speed of heat propagation. Hence any thermal disturbance at one point has an instantaneous effect elsewhere in the body. The use of Cattaneo's law removes this paradox and still keeps the essentials of a heat conduction process. The simplest Cattaneo law replacing Fourier's law (1.5) is

$$\tau q_t + q + \kappa\nabla\theta = 0 \tag{1.7}$$

where $q := q(x,t)$ is the heat flux vector and parameter $\tau > 0$ is the relaxation time describing the time lag in the response of the heat flux to a gradient in the temperature. Keeping in view with the ideas of (1.2) – (1.7), herein we are concerned mathematically the following system of equations

$$u_{tt} - a^2\Delta u - a^2\beta\Delta u_t + \eta\Delta\theta = 0 \quad \text{on} \quad \Omega\times\mathbb{R}^+, \tag{1.8}$$

$$\theta_t - \eta\Delta u_t + \kappa \operatorname{div} q = 0 \quad \text{on} \quad \Omega\times\mathbb{R}^+, \tag{1.9}$$

$$\tau\, q_t + q + \kappa\nabla\theta = 0 \quad \text{on} \quad \Omega\times\mathbb{R}^+. \tag{1.10}$$

Additionally, we have initial conditions

$$u(x,0) = u_0, \quad u_t(x,0) = u_1, \quad \theta(x,0) = \theta_0, \quad q(x,0) = q_0 \quad \text{for} \quad x\in\Omega. \tag{1.11}$$

But, boundary conditions have several choices, depends on the physical situation one wants to deal with. In these present paper we investigate the system with two set of boundary conditions. The first one is corresponding to a rigidly clamped structure with temperature held constant on the boundary

$$u = 0 \quad \text{and} \quad \theta = 0 \quad \text{on} \quad \partial\Omega, \tag{1.12}$$

and the other one is corresponding to a rigidly clamped structure with zero heat flux on the boundary

$$u = 0 \quad \text{and} \quad q = 0 \quad \text{on} \quad \partial\Omega. \tag{1.13}$$

Our purpose in this work is to investigate analytically the stability and regularity of the system (1.8) – (1.10), subject to the boundary conditions (1.12) or (1.13) and initial conditions (1.11). To achieve the results, we adopt two different approaches, one is direct method by constructing suitable Lyapunov like functional associated with the energy functional of the system and other is semigroup theory by constructing suitable infinitesimal generator associated with the system.

Energy estimate of the system:

Lemma 1.1. *Let $\beta > 0$. For every solutions (u, θ, q) of the system* (1.8)-(1.10), *the total energy $\mathcal{E}_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is given at time t by*

$$\mathcal{E}_1(t) = \frac{1}{2}\left[a^2 \int_\Omega |\nabla u|^2 + \int_\Omega {u_t}^2 + \int_\Omega \theta^2 + \tau \int_\Omega q^2\right], \tag{1.14}$$

satisfies

$$\frac{d}{dt}\mathcal{E}_1(t) = -a^2\beta \int_\Omega |\nabla u_t|^2 - \int_\Omega q^2. \tag{1.15}$$

Proof. We multiply the equation (1.8), (1.9) and (1.10) by u_t, θ and q respectively and then integrate with respect to x over Ω, using Green's formula together with (1.12), we obtain the result (1.15).

Remark 1.1. *We have seen that $\dfrac{d\mathcal{E}_1(t)}{dt} \neq 0$, it follows from* (1.15) *that it is not energy conserving. Also, the negativity of the right hand side of* (1.15) *shows that some amount of energy of the system is dissipating due to consideration of damping of the structure.*

Integrating (1.15) with respect to t over $[0,\, t]$, we have

$$\mathcal{E}_1(t) - \mathcal{E}_1(0) = -a^2\beta \int_0^t \int_\Omega |\nabla u_s|^2 ds - \int_0^t \int_\Omega q^2 ds \quad \text{for} \quad t \geq 0, \tag{1.16}$$

where

$$\mathcal{E}_1(0) = \frac{1}{2}\left[a^2 \int_\Omega |\nabla u_0|^2 + \int_\Omega {u_1}^2 + \int_\Omega {\theta_0}^2 + \tau \int_\Omega {q_0}^2\right]. \tag{1.17}$$

In view of (1.16) and (1.17) we may conclude that if $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $\theta_0 \in L^2(\Omega)$ and $q_0 \in L^2(\Omega)$, where

$$H_0^1(\Omega) = \left\{\phi : \, \phi \in H^1(\Omega), \quad \phi = 0 \quad \text{on} \quad \partial\Omega\right\},$$

the subspace of the classical Sobolev space

$$H^1(\Omega) := \left\{\phi : \, \phi \in L^2(\Omega), \quad \nabla\phi \in L^2(\Omega)\right\}$$

of real valued functions of order one, then

$$\mathcal{E}_1(t) \leq \mathcal{E}_1(0) < \infty, \quad t \geq 0. \tag{1.18}$$

Now, taking time derivative of the system (1.8)-(1.10), another energy like functional $\mathcal{E}_2$: $\mathbb{R}^+ \to \mathbb{R}^+$ given by

$$\mathcal{E}_2(t) = \frac{1}{2}\left[a^2 \int_\Omega |\nabla u_t|^2 + \int_\Omega {u_{tt}}^2 + \int_\Omega {\theta_t}^2 + \tau \int_\Omega {q_t}^2\right], \tag{1.19}$$

satisfies

$$\frac{d}{dt}\mathcal{E}_2(t) = -a^2\beta \int_\Omega |\nabla u_{tt}|^2 - \int_\Omega {q_t}^2. \tag{1.20}$$

Remark 1.2. *The first energy estimate i.e.* $\mathcal{E}_1(t)$ *will allow us to investigate well-posedness with the point of view of semigroups (cf. Pazy [13]). While the other energy estimate i.e.* $\mathcal{E}_2(t)$ *will be necessary to study the asymptotic behaviour of the system* $(1.8)-(1.10)$ *with boundary conditions* (1.12).

2 Well-possedness of the problem

In this section, we obtain the existence and uniqueness of solutions for the coupled system $(1.8)-(1.10)$ with initial and boundary conditions. We will use the following standard $L^2(\Omega)$ space, the scaler product and norm are denoted by

$$\langle u, v\rangle_{L^2(\Omega)} = \int_\Omega u\,\overline{v}\,dx, \qquad ||u||^2_{L^2(\Omega)} = \int_\Omega |u|^2 dx.$$

We have the Poincaré inequality

$$||u||^2_{L^2(\Omega)} \leq C_p\,||\nabla u||^2_{L^2(\Omega)}, \quad \text{for all} \quad u \in H^1_0(\Omega),$$

where C_p is the Poincaré constant.

Taking $u_t = v$, the initial boundary value problem $(1.8)-(1.10)$ can be reduced to the following abstract initial value problem

$$\frac{d}{dt}U(t) = \mathcal{A}U(t), \qquad U(0) = U_0, \qquad \text{for all} \quad t > 0, \tag{2.1}$$

with $U(t) = (u, v, \theta, q)^T$ and $U_0 = (u_0, u_1, \theta_0, q_0)^T$, where the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is given by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \theta \\ q \end{pmatrix} = \begin{pmatrix} v \\ a^2\Delta u + a^2\beta\Delta v - \eta\Delta\theta \\ \eta\Delta v - \kappa\,\mathrm{div}\,q \\ -\dfrac{1}{\tau}\,(q + \kappa\nabla\theta) \end{pmatrix}. \tag{2.2}$$

We introduce the phase space $\mathcal{H} = H^1_0(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ endowed with the Hilbertian product given by

$$\langle U, U_1\rangle_{\mathcal{H}} = a^2 \int_\Omega \nabla u\,\nabla\overline{u_1}\,dx + \int_\Omega v\,\overline{v_1}\,dx + \int_\Omega \theta\,\overline{\theta_1}\,dx + \int_\Omega \tau\,q\,\overline{q_1}\,dx, \tag{2.3}$$

where $U=(u,v,\theta,q)$, $U_1=(u_1,v_1,\theta_1,q_1)$ and the norm given by

$$||(u,v,\theta,q)||^2_{\mathcal{H}} = ||a\nabla u||^2_{L^2(\Omega)} + ||v||^2_{L^2(\Omega)} + ||\theta||^2_{L^2(\Omega)} + ||\sqrt{\tau}\, q||^2_{L^2(\Omega)}. \tag{2.4}$$

We can easily show that the norm $\|\cdot\|_{\mathcal{H}}$ is equivalent to usual norm in $\mathcal{H}$.

The domain of the operator $\mathcal{A}$ denoted by $\mathcal{D}(\mathcal{A})$, depends upon the boundary conditions under consideration. For the boundary conditions (1.12), we define

$$\begin{aligned}\mathcal{D}(\mathcal{A})=\mathcal{D}_1 \;=\; &\big\{(u,v,\theta,q)\in\mathcal{H} : u\in H^1_0(\Omega), a^2u+a^2\beta v-\eta\theta\in H^2(\Omega)\cap H^1_0(\Omega),\\ &\theta\in H^1_0(\Omega),\ q,\operatorname{div} q\in H^1(\Omega)\big\}.\end{aligned} \tag{2.5}$$

For the boundary conditions (1.13), we have

$$\begin{aligned}\mathcal{D}(\mathcal{A})=\mathcal{D}_2 \;=\; &\big\{(u,v,\theta,q)\in\mathcal{H} : u\in H^1_0(\Omega),\ a^2u+a^2\beta v-\eta\theta\in H^2(\Omega)\cap H^1_0(\Omega),\\ &\theta\in H^1(\Omega),\ q,\operatorname{div} q\in H^1_0(\Omega)\big\}.\end{aligned} \tag{2.6}$$

Now, existence and uniqueness result of the system (1.8)-(1.10) with initial and boundary conditions describe in the next theorem as follows

Theorem 2.1. *For any $U_0\in\mathcal{D}(\mathcal{A})$ (either $\mathcal{D}_1$ or $\mathcal{D}_2$), there exist a unique global solution*

$$\begin{aligned}&u\in C^1(\mathbb{R}^+;H^1_0(\Omega))\cap C^2(\mathbb{R}^+;L^2(\Omega))\\ &\theta\in C(\mathbb{R}^+;H^1_0(\Omega))\cap C^1(\mathbb{R}^+;L^2(\Omega))\\ &q\in C(\mathbb{R}^+;L^2(\Omega))\\ &a^2u+a^2\beta v-\eta\theta\in(\mathbb{R}^+;H^2(\Omega)\cap H^1_0(\Omega))\\ &div\, q\in(\mathbb{R}^+;L^2(\Omega))\end{aligned}$$

of the system $(1.8)-(1.10)$.

Proof. To prove the above theorem, we first need some proposition as follows

Proposition 2.1. *Let $\beta>0$. The operator $\mathcal{A}$ generates a C_0-semigroup $\mathcal{S}_{\mathcal{A}}(t)$ of contractions on the space $\mathcal{H}$.*

Proof. We will show that $\mathcal{A}$ is a dissipative operator and 0 belongs to resolvent set of $\mathcal{A}$, denoted by $\varrho(\mathcal{A})$. Then our conclusion will follow using the well known Lumer–Phillips theorem (cf. [13]). We observe that if $U=(u,v,\theta,q)\in\mathcal{D}(\mathcal{A})$ (either $\mathcal{D}_1$ or $\mathcal{D}_2$) then by using (2.2) and (2.3), we have

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= a^2 \int_\Omega \nabla v\, \nabla \overline{u}\, dx + \int_\Omega v_t\, \overline{v}\, dx + \int_\Omega \theta_t\, \overline{\theta}\, dx + \tau \int_\Omega q_t\, \overline{q}\, dx. \\
&= a^2 \int_\Omega \nabla v\, \nabla \overline{u}\, dx + \int_\Omega \big(a^2\Delta u + a^2\beta\Delta v - \eta\Delta\theta\big)\, \overline{v}\, dx \\
&\quad + \int_\Omega (\eta\Delta v - \kappa\, div\, q)\, \overline{\theta}\, dx - \int_\Omega (q + \kappa\nabla\theta)\, \overline{q}\, dx \\
&= a^2 \int_\Omega \left[\nabla v\, \nabla\overline{u} - \nabla u\, \nabla\overline{v}\right]\, dx - a^2\beta \int_\Omega |\nabla v|^2 dx \\
&\quad + \eta \int_\Omega \left[\nabla\theta\, \nabla\overline{v}\, dx - \nabla v\, \nabla\overline{\theta}\right] \\
&\quad + \kappa \int_\Omega \left[q\, \nabla\overline{\theta} - \theta\, \nabla\overline{q}\right] dx - \int_\Omega q^2\, dx \\
&= 2\, i\, a^2\, Im \int_\Omega \nabla v\, \nabla\overline{u} - a^2\beta \int_\Omega |\nabla v|^2 dx + 2\, i\eta\, Im \int_\Omega \nabla\theta\, \nabla\overline{v}\, dx \\
&\quad + 2\, i\kappa\, Im \int_\Omega q\, \nabla\overline{\theta} - \int_\Omega q^2\, dx
\end{aligned}$$

Taking the real part, we have

$$Re\, \langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -a^2\beta \int_\Omega |\nabla v|^2 - \int_\Omega q^2 \ \le 0. \tag{2.7}$$

Thus $\mathcal{A}$ is a dissipative operator. Now, we show that $(\lambda I - \mathcal{A})$ is surjective.

Proposition 2.2. $\Re\,(\lambda I - \mathcal{A}) = \mathcal{H}, \quad if \quad \lambda = \dfrac{-1 + \sqrt{1 + 4\tau}}{2}.$

Proof. We show that for all $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}$, there exist a unique $U = (u, v, \theta, q) \in \mathcal{D}(\mathcal{A})$ (either $\mathcal{D}_1$ or $\mathcal{D}_2$) such that $(\lambda I - \mathcal{A})\, U = F$, that is,

$$\lambda u - v = f_1 \quad \text{in} \quad H_0^1(\Omega), \tag{2.8}$$

$$\lambda v - \big(a^2\Delta u + a^2\beta\Delta v - \eta\Delta\theta\big) = f_2 \quad \text{in} \quad L^2(\Omega), \tag{2.9}$$

$$\lambda\theta - (\eta\Delta v - \kappa\, \mathrm{div}\, q) = f_3 \quad \text{in} \quad L^2(\Omega), \tag{2.10}$$

$$\lambda q + \frac{1}{\tau}\,(q + \kappa\nabla\theta) = f_4 \quad \text{in} \quad L^2(\Omega). \tag{2.11}$$

Replacing (2.8) into (2.9), we have

$$\lambda^2 u - a^2(1 + \beta\lambda)\Delta u + \eta\Delta\theta = f_2 + \lambda f_1 - a^2\beta\Delta f_1. \tag{2.12}$$

From (2.11), we have

$$q = \frac{\tau}{\tau\lambda + 1} f_4 - \frac{\kappa}{\tau\lambda + 1}\nabla\theta. \tag{2.13}$$

By using (2.8) and (2.13) into (2.10), we have

$$\lambda(\tau\lambda + 1)\theta - \kappa^2\Delta\theta - \eta\lambda(\tau\lambda + 1)\Delta u = (\tau\lambda + 1) f_3 - \eta(\tau\lambda + 1)\Delta f_1 - \kappa\tau\, \mathrm{div}\, f_4. \tag{2.14}$$

To solve the variational equations corresponding to (2.12) and (2.14), we consider the bilinear form $\mathcal{B} : (H_0^1(\Omega) \times H_0^1(\Omega)) \times (H_0^1(\Omega) \times H_0^1(\Omega)) \longrightarrow \mathbb{R}$ is defined by

$$\begin{aligned}\mathcal{B}\left((u,\theta),(\varphi_1,\varphi_2)\right) &= \lambda^2 \int_\Omega u\,\varphi_1\,dx + a^2(1+\beta\lambda)\int_\Omega \nabla u.\,\nabla\varphi_1 dx - \eta\int_\Omega \nabla\theta.\,\nabla\varphi_1 dx \\ &\quad +\lambda(\tau\lambda+1)\int_\Omega \theta\,\varphi_2 dx + \kappa^2\int_\Omega \nabla\theta.\,\nabla\varphi_2 dx + \eta\lambda(\tau\lambda+1)\int_\Omega \nabla u.\,\nabla\varphi_2\,dx,\end{aligned}$$

and the linear form $\mathcal{J} : (H_0^1(\Omega) \times H_0^1(\Omega)) \longrightarrow \mathbb{R}$ is defined by

$$\begin{aligned}\mathcal{J}(\varphi_1,\varphi_2) &= \int_\Omega f_2\,\varphi_1\,dx + \lambda\int_\Omega f_1\,\varphi_1\,dx + a^2\beta\int_\Omega \nabla f_1.\,\nabla\varphi_1\,dx \\ &\quad +(\tau\lambda+1)\int_\Omega f_3\,\varphi_2\,dx + \eta(\tau\lambda+1)\int_\Omega \nabla f_1.\,\nabla\varphi_2\,dx - \kappa\tau\int_\Omega \operatorname{div} f_4\,\varphi_2\,dx.\end{aligned}$$

By using Green's formula, we have

$$\begin{aligned}\mathcal{B}\left((u,\theta),(u,\theta)\right) &= \lambda^2\int_\Omega u^2\,dx + a^2(1+\beta\lambda)\int_\Omega |\nabla u|^2\,dx - \eta\int_\Omega \nabla u.\,\nabla\theta \\ &\quad +\lambda\int_\Omega \theta^2 dx + \kappa^2\int_\Omega |\nabla\theta|^2 dx + \eta\tau(\tau\lambda+1)\int_\Omega \nabla u.\,\nabla\theta\end{aligned}$$

Thus, for some constant $C > 0$, we have

$$\mathcal{B}\left((u,\theta),(u,\theta)\right) \geq C\left(||u||^2_{H_0^1(\Omega)} + ||\theta||^2_{H_0^1(\Omega)}\right)$$

provided

$$\lambda = \frac{-1+\sqrt{1+4\tau}}{2},$$

where, $H_0^1(\Omega) \times H_0^1(\Omega)$ equipped with the norm $||u,\theta||^2_{H_0^1(\Omega)\times H_0^1(\Omega)} = ||u||^2_{H_0^1(\Omega)} + ||\theta||^2_{H_0^1(\Omega)}$. Hence $\mathcal{B}$ is coercive.

Now, we have from Hölder's inequality

$$\begin{aligned}\mathcal{B}\left((u,\theta),(\varphi_1,\varphi_2)\right) &\leq \left|\lambda^2\right|\,||u||\,||\varphi_1|| + \left|a^2(1+\beta\lambda)\right|\,||\nabla u||\,||\nabla\varphi_1|| + |\eta|\,||\nabla\theta||\,||\nabla\varphi_1|| \\ &\quad + |\lambda(\tau\lambda+1)|\,||\theta||\,||\varphi_2|| + \left|\kappa^2\right|\,||\nabla\theta||\,||\nabla\varphi_2|| + |\eta\,\lambda(\tau\lambda+1)|\,||\nabla u||\,||\nabla\varphi_2|| \\ &\leq C\left(||u||^2_{H_0^1(\Omega)} + ||\theta||^2_{H_0^1(\Omega)}\right)^{\frac{1}{2}} \quad \left(||\varphi_1||^2_{H_0^1(\Omega)} + ||\varphi_2||^2_{H_0^1(\Omega)}\right)^{\frac{1}{2}}.\end{aligned}$$

and

$$\begin{aligned}|\mathcal{J}(\varphi_1,\varphi_2)| &\leq ||f_2||\,||\varphi_1|| + |\lambda|\,||f_1||\,||\varphi_1|| + \left|a^2\beta\right|\,||\nabla f_1||\,||\nabla\varphi_1|| + |\tau\lambda+1|\,||f_3||\,||\varphi_2|| \\ &\quad + |\eta(\tau\lambda+1)|\,||\nabla f_1||\,||\nabla\varphi_2|| + |\kappa\tau|\,||\operatorname{div} f_4||\,||\varphi_2|| \\ &= (||f_2|| + |\lambda|\,||f_1||)\,||\varphi_1|| + \left|a^2\beta\right|\,||\nabla f_1||\,||\nabla\varphi_1|| \\ &\quad + (|\tau\lambda+1|\,||f_3|| + |\kappa\tau|\,||\operatorname{div} f_4||)\,||\varphi_2|| + |\eta(\tau\lambda+1)|\,||\nabla f_1||\,||\nabla\varphi_2|| \\ &\leq C_1\left(||\varphi_1||^2_{H_0^1(\Omega)} + ||\nabla\varphi_1||^2_{H_0^1(\Omega)}\right)^{\frac{1}{2}} + C_2\left(||\varphi_2||^2_{H_0^1(\Omega)} + ||\nabla\varphi_2||^2_{H_0^1(\Omega)}\right)^{\frac{1}{2}}.\end{aligned}$$

Consequently, by using Lax-Milgram lemma, the equations (2.12) and (2.14) have a unique solution $(u,\theta) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$\mathcal{B}\left((u,\theta),(\varphi_1,\varphi_2)\right) = \mathcal{J}(\varphi_1,\varphi_2), \quad \text{for all} \quad (\varphi_1,\varphi_2) \in H_0^1(\Omega) \times H_0^1(\Omega). \tag{2.15}$$

Moreover, if $\varphi_2 \equiv 0 \in H_0^1(\Omega)$, where $\varphi_1 \in H_0^1(\Omega)$, then (2.15) reduces to

$$\begin{aligned}&\lambda^2 \int_\Omega u\,\varphi_1\,dx + a^2(1+\beta\lambda)\int_\Omega \nabla u.\,\nabla\varphi_1 dx - \eta\int_\Omega \nabla\theta.\,\nabla\varphi_1 dx\\ &= \int_\Omega f_2\,\varphi_1\,dx + \lambda\int_\Omega f_1\,\varphi_1\,dx + a^2\beta\int_\Omega \nabla f_1.\,\nabla\varphi_1\,dx\end{aligned}$$

i.e.,

$$\lambda^2 u - a^2(1+\beta\lambda)\Delta u + \eta\Delta\theta = f_2 + \lambda\, f_1 - a^2\beta\Delta f_1. \tag{2.16}$$

We put $v = \lambda u - f_1$. Then $v \in H_0^1(\Omega)$ solves (2.8). Hence, using (2.16) we have

$$\lambda v - \left(a^2\Delta u + a^2\beta\Delta v - \eta\Delta\theta\right) = f_2 \in L^2(\Omega),$$

which guarantees that

$$a^2 u + a^2\beta v - \eta\theta \in H^2(\Omega) \cap H_0^1(\Omega).$$

Similarly, if $\varphi_1 \equiv 0 \in H_0^1(\Omega)$, where $\varphi_2 \in H_0^1(\Omega)$, then using (2.15) we have

$$\begin{aligned}&\lambda(\tau\lambda+1)\int_\Omega \theta\,\varphi_2 dx + \kappa^2\int_\Omega \nabla\theta.\,\nabla\varphi_2 dx + \eta\lambda(\tau\lambda+1)\int_\Omega \nabla u.\,\nabla\varphi_2 dx\\ &= (\tau\lambda+1)\int_\Omega f_3\,\varphi_2\,dx + \eta(\tau\lambda+1)\int_\Omega \nabla f_1.\,\nabla\varphi_2\,dx - \kappa\tau\int_\Omega \operatorname{div} f_4\,\varphi_2\,dx.\end{aligned}$$

i.e.,

$$\lambda(\tau\lambda+1)\theta - \kappa^2\Delta\theta - \eta\lambda(\tau\lambda+1)\Delta u = (\tau\lambda+1)f_3 - \eta(\tau\lambda+1)\Delta f_1 - \kappa\tau\operatorname{div} f_4. \tag{2.17}$$

Proceeding as previous, we have

$$\lambda\theta - \left(\frac{\kappa^2}{\tau\lambda+1}\right)\Delta\theta - \eta\Delta v = f_3 - \frac{\kappa\tau}{\tau\lambda+1}\operatorname{div} f_4.$$

Thus we have

$$\lambda\theta - \eta\Delta v = f_3 + \left(\frac{\kappa^2}{\tau\lambda+1}\right)\Delta\theta - \frac{\kappa\tau}{\tau\lambda+1}\operatorname{div} f_4,$$

i.e.,

$$\lambda\theta - \eta\Delta v = f_3 + \kappa\operatorname{div} q \in L^2(\Omega),$$

which guarantees that

$$\operatorname{div} q \in L^2(\Omega)$$

Finally, by using Lumer-Philips theorem we deduced that $\mathcal{A}$ is an infinitesimal generator of a contraction semigroup in $\mathcal{H}$, thus $\mathcal{A}$ is closed and $\mathcal{D}(\mathcal{A})$ (either $\mathcal{D}_1$ or $\mathcal{D}_2$) is dense in $\mathcal{H}$ and this complete the proof.

3 Stability results

We are now in a position to discuss about stability results of the system (1.8)-(1.10). We expect actually to obtain a better result, that is, an exponential stability. But we did not find the adequate Lyapunove functional associated with the system (1.8)-(1.10) together with boundary conditions (1.12), and it is an ongoing work. At first, we shall discuss the asymptotic stability of the system (1.8)-(1.10) together with boundary conditions (1.12) by constructing Lyapunove like functional associated with that system. Whereas on the other hand, exponential stability of the system (1.8)-(1.10) with boundary conditions (1.13) was achieved through semigroup theory of linear operators. With the help of semigroup theory, the exponential energy decay estimate was studied by several author (cf. Gearhart [4] and Huang [8] and a list of references therein). The main results of the present work are concerned with the asymptotic and exponential behaviour of the system (1.8)-(1.10) with boundary conditions (1.12) and (1.13) respectively and may be stated as in the following theorems,

Theorem 3.1. *For suitable initial data i.e.* $(u_0, u_1, \theta_0, q_0) \in \mathcal{D}_1$*(defined by* (2.5)*), the strong solution of the system* (1.8) – (1.10) *together with boundary conditions* (1.12) *satisfies,*

$$\mathcal{E}_1(t) \leq \frac{c_0(\mathcal{E}_1(0) + \mathcal{E}_2(0))}{t} \qquad \textit{for all} \quad t > 0, \tag{3.1}$$

for a positive constant c_0*, independent of* t *and initial data.*

To prove the theorem we will use energy method, and constructing a suitable Lyapunov like functional.

Theorem 3.2. *For suitable initial data i.e.* $(u_0, u_1, \theta_0, q_0) \in \mathcal{D}_2$ *(defined in* (2.6)*) the semigroup generated by the system* (1.8) – (1.10) *complemented by boundary conditions* (1.13) *is exponentially stable.*

Proof of theorem 3.1. The proof of this theorem will be established through several inequality's and lemmas as follows,

I. For any real number $\alpha > 0$ we have, the Schwarz Inequality (cf. Mitrinović *et al* [11])

$$\int_\Omega \phi\,\psi \leq \int_\Omega \left|\phi\,\psi\right| \leq \frac{1}{2}\left(\alpha \int_\Omega \phi^2 + \frac{1}{\alpha}\int_\Omega \psi^2\right). \tag{3.2}$$

II. For any real number $\gamma \geq 1$ and $\delta > 0$ we have, Poincaré type Scheeffer's inequality (cf. Mitrinović *et al* [11])

$$\int_\Omega u^2\, dx \leq \gamma \int_\Omega |\nabla u|^2\, dx \quad \text{and} \quad \int_\Omega \theta^2\, dx \leq \delta \int_\Omega |\nabla \theta|^2\, dx, \tag{3.3}$$

as $u(x,\, t)$ and $\theta(x, t)$ satisfy boundary conditions (1.12).

Now, we need to establish the following lemmas.

Lemma 3.1. *For every strong solutions* (u, u_t, θ, q) *of the system* (1.8) − (1.10) *together with boundary conditions* (1.12) *with* $(u_0, u_1, \theta_0, q_0) \in \mathcal{D}_1$*, the time derivative of the functional* $G(t)$ *defined by*

$$G(t) = \int_\Omega \left[u\, u_t + \frac{a^2\beta}{2} |\nabla u|^2 \right] dx \tag{3.4}$$

satisfies

$$\frac{dG}{dt} \le -C_1 \mathcal{E}_1 + C_2 \left(\int_\Omega q^2 + \int_\Omega {q_t}^2 + \int_\Omega |\nabla u_t|^2 \right), \tag{3.5}$$

where $C_1, C_2 > 0$ *will be made explicit in the proof.*

Proof. Differentiating (3.4) with respect to t-variable with boundary conditions (1.12) and using the energy estimate (1.14), we get

$$\frac{dG}{dt} = -2\, \mathcal{E}_1(t) + \int_\Omega \theta^2 + \tau \int_\Omega q^2 + 2 \int_\Omega {u_t}^2 - \eta \int_\Omega u\, \nabla\theta. \tag{3.6}$$

Using (3.2) and (3.3) into (3.6), we have

$$\begin{aligned} \frac{dG}{dt} &\le -2\, \mathcal{E}_1(t) + \delta \int_\Omega |\nabla\theta|^2 + \tau \int_\Omega q^2 + 2\gamma \int_\Omega |\nabla u_t|^2 + \frac{\eta}{2} \left[\alpha \int_\Omega u^2 + \frac{1}{\alpha} \int_\Omega |\nabla\theta|^2 \right] \\ &= -2\, \mathcal{E}_1(t) + \left(\delta + \frac{\eta}{2\alpha} \right) \int_\Omega |\nabla\theta|^2 + \tau \int_\Omega q^2 + 2\gamma \int_\Omega |\nabla u_t|^2 + \frac{\eta\alpha}{2\,a^2} \int_\Omega a^2\, |\nabla u|^2 \\ &\le -\left(2 - \frac{\eta\alpha}{a^2} \right) \mathcal{E}_1(t) + \left(\delta + \frac{\eta}{2\alpha} \right) \int_\Omega |\nabla\theta|^2 + \tau \int_\Omega q^2 + 2\gamma \int_\Omega |\nabla u_t|^2, \end{aligned} \tag{3.7}$$

we chose $\alpha > 0$ such that,

$$C_1 := 2 - \frac{\eta\alpha}{a^2} > 0.$$

Now, from equation (1.10) of our system we get

$$|\nabla\theta|^2 = \frac{\tau^2}{\kappa^2} {q_t}^2 + \frac{2\tau}{\kappa^2} q_t\, q + \frac{1}{\kappa^2} q^2.$$

Hence

$$\int_\Omega |\nabla\theta|^2 \le \left(\frac{\tau + 1}{\kappa} \right)^2 \left(\int_\Omega {q_t}^2 + \int_\Omega q^2 \right). \tag{3.8}$$

Therefore, employing (3.8) into (3.7) we get (3.5), where C_1 has already been defined, while

$$C_2 := \max \left[\left(\delta + \frac{\eta}{2\alpha} \right) \left(\frac{\tau + 1}{\kappa} \right)^2 ; \left(\delta + \frac{\eta}{2\alpha} \right) \left(\frac{\tau + 1}{\kappa} \right)^2 + \tau ; 2\,\gamma \right],$$

where $\alpha > 0$ is fixed in C_2. This ends the proof.

Lemma 3.2. *The functional* $G(t)$ *given by* (3.4) *satisfies the inequality*

$$-\frac{\sqrt{\gamma}}{a} \mathcal{E}_1(t) \le G(t) \le \left(\frac{\sqrt{\gamma}}{a} + \beta \right) \mathcal{E}_1(t) \quad \textit{for} \quad t \ge 0. \tag{3.9}$$

Proof. Using (3.2), (3.3) and (1.14), we get

$$\begin{aligned}\left|\int_\Omega u\, u_t\, dx\right| &\leq \frac{1}{2}\left[\frac{a}{\sqrt{\gamma}}\int_\Omega u^2 dx + \frac{\sqrt{\gamma}}{a}\int_\Omega {u_t}^2 dx\right] \\ &\leq \frac{1}{2}\left[\frac{\sqrt{\gamma}}{a}\int_\Omega a^2|\nabla u|^2 dx + \frac{\sqrt{\gamma}}{a}\int_\Omega {u_t}^2 dx\right] \\ &\leq \frac{\sqrt{\gamma}}{a}\mathcal{E}_1(t) \quad \text{for} \quad t \geq 0. \end{aligned} \tag{3.10}$$

Also, from (1.14), we have

$$\frac{a^2\beta}{2}\int_\Omega |\nabla u|^2 dx \leq \beta\mathcal{E}_1(t) \quad \text{for} \quad t \geq 0. \tag{3.11}$$

Thus, from (3.10) and (3.11), we get

$$-\frac{\sqrt{\gamma}}{a}\mathcal{E}_1(t) \leq G(t) \leq \left(\frac{\sqrt{\gamma}}{a} + \beta\right)\mathcal{E}_1(t) \quad \text{for} \quad t \geq 0.$$

Hence the lemma is proved.

Remark 3.1. *One can extend the result of lemma* 3.3 *as*

$$-\frac{\sqrt{\gamma}}{a}\left(\mathcal{E}_1(t) + \mathcal{E}_2(t)\right) \leq G(t) \leq \left(\frac{\sqrt{\gamma}}{a} + \beta\right)\left(\mathcal{E}_1(t) + \mathcal{E}_2(t)\right) \quad \textit{for} \quad t \geq 0. \tag{3.12}$$

This extension will help us to established asymptotic stability of the system (1.8) – (1.10) *together with boundary conditions* (1.12)*.*

Now, we proceed like Gorain [5], Komornik [9]. Let us introduce an energy like Lyapunov functional $V(t)$ defined by

$$V(t) := \mathcal{E}_1(t) + \mathcal{E}_2(t) + \varepsilon G(t) \quad \text{for } t \geq 0, \tag{3.13}$$

where $\varepsilon > 0$ is a non-negative real number, that will be defined later.

Now, taking time derivative of (3.13) and applying (1.15), (1.20), and (3.5), we get

$$\begin{aligned}\frac{dV}{dt} &= \frac{d\mathcal{E}_1}{dt} + \frac{d\mathcal{E}_2}{dt} + \varepsilon\frac{dG}{dt} \\ &\leq -a^2\beta\int_\Omega \nabla {u_t}^2 - \int_\Omega q^2 - a^2\beta\int_\Omega \nabla {u_{tt}}^2 - \int_\Omega q_t^2 - \varepsilon\, C_1\mathcal{E}_1(t) \\ &\quad +\varepsilon\, C_2\int_\Omega q^2 + \varepsilon\, C_2\int_\Omega {q_t}^2 + \varepsilon\, C_2\int_\Omega |\nabla u_t|^2 \\ &= -\varepsilon\, C_1\mathcal{E}_1(t) - (a^2\beta - \varepsilon C_2)\int_\Omega |\nabla u_t|^2 - (1-\varepsilon C_2)\int_\Omega q^2 \\ &\quad -(1-\varepsilon C_2)\int_\Omega {q_t}^2 - a^2\beta\int_\Omega \nabla {u_{tt}}^2. \end{aligned} \tag{3.14}$$

Since C_1 and C_2 are already fixed by lemma 3.2, we assume $\varepsilon > 0$ so that

$$1 - \varepsilon C_2 > 0 \quad \text{and} \quad a^2\beta - \varepsilon C_2 > 0$$

Thus, from (3.14), we get the differential inequality

$$\frac{dV}{dt} \leq -\varepsilon C_1 \mathcal{E}_1(t). \tag{3.15}$$

In order to ensure the positivity of $V(t)$, we use the result (3.12) and we get $1 - \varepsilon \dfrac{\sqrt{\gamma}}{a} > 0$.

Now, we define

$$0 < \varepsilon < \varepsilon_0 := \min \left[\frac{1}{C_2}, \frac{a^2\beta}{C_2}, \frac{a}{\sqrt{\gamma}} \right]. \tag{3.16}$$

Integration of (3.15) over $(0, t)$, recalling that $\mathcal{E}_1(t)$ is non-increasing, yields

$$t\mathcal{E}_1(t) \leq \int_0^t \mathcal{E}_1(s) ds \leq \frac{1}{\varepsilon C_1} (V(0) - V(t)) \leq \frac{V(0)}{\varepsilon C_1}, \quad \text{for all} \quad t > 0,$$

Incorporating the result (3.12), we assume $c_0 = \dfrac{1 + \left(\frac{\sqrt{\gamma}}{a} + \beta \right)}{\varepsilon C_1}$, we have

$$\mathcal{E}_1(t) \leq \frac{c_0(\mathcal{E}_1(0) + \mathcal{E}_2(0))}{t} \qquad \text{for} \quad \text{all} \quad t > 0.$$

Hence the theorem 3.1 is proved.

Proof of theorem 3.2. In order to prove the exponential decay with the help of semigroup theory, we are going to use necessary and sufficient conditions for C_0-semigroups being exponentially stable in a Hilbert space. This result was obtained by Gearhart [4] and Huang [8], independently

Theorem 3.3. *(**Gearhart**) Let $(\mathcal{S}_{\mathcal{A}}(t))_{t\geq 0}$ be a C_0-semigroup of contractions in a Hilbert space. Then $\mathcal{S}_{\mathcal{A}}(t)$ is exponential stable (that is there exist $M \geq 1$, $\mu > 0$ such that $||\mathcal{S}_{\mathcal{A}}(t)|| \leq Me^{-\mu t}$, for all $t \geq 0$) if and only if,*

$$i\mathbb{R} = \{i\mu : \mu \in \mathbb{R}\} \subset \varrho(\mathcal{A})$$

and

$$\limsup_{|\lambda| \to \infty} ||(i\lambda I - \mathcal{A})^{-1}|| < \infty.$$

Now, we will use the theorem 3.3 to prove exponential stability of the system (1.8)-(1.10) with boundary conditions (1.13). Furthermore, we assume that $\displaystyle\int_\Omega \theta_0 = 0$, so that the temperature θ has zero mean value for every time. From now on, we thus assume that

$$\int_\Omega \theta = 0 \quad \text{for} \quad \text{all} \quad t \geq 0. \tag{3.17}$$

According to the theorem 3.3, we will prove

$$i\mathbb{R} = \{i\mu,\ \mu \in \mathbb{R}\} \subset \varrho(\mathcal{A}). \tag{3.18}$$

Suppose the equation (3.18) is false, then there exists $\mu \in \mathbb{R}$ such that $i\mu \in \sigma(\mathcal{A})$.

Since $0 \in \varrho(\mathcal{A})$ and $\mathcal{A}^{-1}$ is compact, that is, the spectral values are eigenvalues.

Let $U = (u, v, \theta, q)^T \in \mathcal{D}_2$, $U \neq 0$, such that

$$(i\mu I - \mathcal{A})U = 0, \quad \text{i.e.} \quad i\mu U = \mathcal{A}U. \tag{3.19}$$

Using the definition of $\mathcal{A}$, we have from (3.19),

$$i\mu u = v, \tag{3.20}$$
$$i\mu v = a^2\Delta u + a^2\beta\Delta v - \eta\Delta\theta, \tag{3.21}$$
$$i\mu\theta = \eta\Delta v - \kappa \operatorname{div} q, \tag{3.22}$$
$$i\mu\tau q = -q - \kappa\nabla\theta. \tag{3.23}$$

Substituting (3.20) into (3.21), we get

$$ia^2\beta\mu\Delta u = \eta\Delta\theta - a^2\Delta u - \mu^2 u. \tag{3.24}$$

From (3.23) we have,

$$q = -\frac{\kappa}{(1 + i\mu\tau)}\nabla\theta. \tag{3.25}$$

Now, multiplying equation (3.24) by u and integrating over Ω, we obtain

$$-ia^2\beta\mu \int_\Omega |\nabla u|^2 = -\eta \int_\Omega \nabla\theta\,\nabla u + a^2 \int_\Omega |\nabla u|^2 - \mu^2 \int_\Omega u^2. \tag{3.26}$$

Then

$$-\eta \int_\Omega \nabla\theta\,\nabla u + a^2 \int_\Omega |\nabla u|^2 - \mu^2 \int_\Omega u^2 = 0$$

and

$$a^2\beta\mu \int_\Omega |\nabla u|^2 = 0. \tag{3.27}$$

Since $\mu > 0$ and $\beta > 0$, so we have $\int_\Omega |\nabla u|^2 = 0$, which implies $|\nabla u| = 0$. Now, by using Poincaré inequality (3.3), we have $u = 0$ on $L^2(\Omega)$. Finally u is continuous, regular and $u = 0$ (by boundary condition (1.13)), we have $u = 0$ and hence from (3.20) $v = 0$.

Since $v = 0$, so equation (3.22) is converted to

$$i\mu\theta = -\kappa \operatorname{div} q. \tag{3.28}$$

Injecting (3.25) into (3.28), we have

$$-\kappa^2\Delta\theta + \mu^2\tau\theta = i\mu\theta. \tag{3.29}$$

Now, multiplying equation (3.29) by u and integrating over Ω, we obtain

$$-\kappa^2 \int_\Omega |\nabla\theta|^2 + \mu^2\tau \int_\Omega \theta^2 = i\mu \int_\Omega \theta^2. \tag{3.30}$$

Then

$$-\kappa^2 \int_\Omega |\nabla\theta|^2 + \mu^2\tau \int_\Omega \theta^2 = 0$$

and

$$\mu \int_\Omega \theta^2 = 0. \tag{3.31}$$

As $\mu > 0$, we have from (3.31) $\displaystyle\int_\Omega \theta^2 = 0$, which implies $\theta = 0$.

Finally, we multiply equation (3.23) by q and integrate over Ω, we have

$$i\mu\tau \int_\Omega q^2 = -\int_\Omega q^2 - \kappa \int_\Omega \nabla\theta\, q. \tag{3.32}$$

Then

$$-\int_\Omega q^2 - \kappa \int_\Omega \nabla\theta\, q = 0$$

and

$$\mu\tau \int_\Omega q^2 = 0. \tag{3.33}$$

As $\mu > 0$ and $\tau > 0$, we have from (3.33) $\displaystyle\int_\Omega q^2 = 0$, which implies $q = 0$.

So, finally we achieve that $u = 0 = v = \theta = q$, which contradicts the fact $U \neq 0$. Therefore, $i\mathbb{R} \subset \varrho(\mathcal{A})$.

Now, we will prove that

$$\limsup_{|\lambda|\to\infty} ||(i\lambda I - \mathcal{A})^{-1}|| < \infty. \tag{3.34}$$

Suppose (3.34) is false, and we assume that

$$\limsup_{|\lambda|\to\infty} ||(i\lambda I - \mathcal{A})^{-1}|| = \infty, \tag{3.35}$$

then there exist a sequence $V_n \in \mathcal{H}$ and $\lambda_n \in \mathbb{R}$ such that $||(i\lambda_n I - \mathcal{A})^{-1}V_n|| \geq n||V_n||$, for all $n > 0$.

Thus $i\lambda_n \in \varrho(\mathcal{A})$, that is, there exist $U_n \in \mathcal{D}_2$ such that $(i\lambda_n I - \mathcal{A})U_n = V_n$ with $||U_n|| = 1$ So we have

$$U_n = (i\lambda_n I - \mathcal{A})^{-1}V_n$$

and

$$||U_n|| \geq n||(i\lambda_n I - \mathcal{A})U_n||.$$

Then $1 = ||U_n|| \geq n||G_n||$, i.e. $\dfrac{1}{n} \geq ||G_n||$, where $G_n := (i\lambda_n I - \mathcal{A})U_n$. As $n \to \infty$, we get $\lim_{n\to\infty} G_n = 0$ on $\mathcal{H}$.

Now, let $U_n := (u_n, v_n, \theta_n, q_n)^T$. Then

$$\begin{aligned} \langle G_n, U_n \rangle &= \langle i\lambda_n U_n - \mathcal{A}U_n, U_n \rangle \\ &= i\lambda_n ||U_n||^2 - \langle \mathcal{A}U_n, U_n \rangle \end{aligned} \tag{3.36}$$

Taking the real part on both side of (3.36), we have

$$-Re\,\langle \mathcal{A}U_n, U_n \rangle = Re\,\langle G_n, U_n \rangle$$

and then

$$a^2\beta \int_\Omega |\nabla v_n|^2 + \int_\Omega {q_n}^2 = Re\,\langle G_n, U_n \rangle \leq ||G_n||\,||U_n|| = ||G_n|| \to 0. \tag{3.37}$$

Thus

$$a^2\beta \int_\Omega |\nabla v_n|^2 + \int_\Omega {q_n}^2 \to 0 \quad \text{as} \quad n \to \infty. \tag{3.38}$$

Now consider the equation (3.36) and multiply it by i, we have

$$-\lambda_n ||U_n||^2 - i\,\langle \mathcal{A}U_n, U_n \rangle = i\,\langle G_n, U_n \rangle$$

Since $|\,\langle G_n, U_n \rangle\,| \leq ||G_n||\,||U_n|| = ||G_n|| \to 0$ and from (3.38), $||U_n||^2 \to 0$ as $n \to 0$, which is a contradiction. Therefore

$$\limsup_{|\lambda|\to\infty} ||(i\lambda I - \mathcal{A})^{-1}|| < \infty.$$

Hence the theorem 3.2 is proved.

4 Conclusion

This study deals the mathematical stability of the vibrations of flexible structures governed by the standard linear model of viscoelasticity together with the thermal effect satisfying the system of differential equations (1.8)-(1.10). The well-possedness of the system is discussed through semi-group theory. The stability of the system by means of explicit forms of exponentially energy decay estimate as well as asymptotical energy decay estimate are achieved.

References

[1] M. S. Alves, C. Buriol, M. V. Ferreira, J. E. M. Rivera, M. Sepúlveda, O. Vera, *Asymptotic behaviour for the vibrations modeled by the standard linear solid model with a thermal effect,* J. Math. Anal. Appl.,399 (2013) 472–479.

[2] S. K. Bose, G. C. Gorain, *Stability of the boundary stabilised internally damped wave equation* $\alpha u_{ttt} + u_{tt} = a^2 (\Delta u + \beta \Delta u_t)$ *in a bounded domain in* $\mathbb{R}^n$, Indian J. Math., 40 (1998) 1-15.

[3] G. Chen, *Energy decay estimate and exact boundary-value controllability for the wave equation in a bounded domain,* Journal de Mathematiques Pures et Appliquees, 58 (1979) 249-273.

[4] L. Gearhart, *Spectral theory for contraction semigroups on Hilbert spaces,* Trans. of the American Mathematical Socity, 236 (1978) 385-394.

[5] G. C. Gorain, *Exponential energy decay estimate for the solution of internally damped wave equation in a bounded domain,* J. Math. Anal. Appl., 216 (1997) 510-520.

[6] M. Grasselli, J. E. Muñoz Rivera, V. Pata, *On the energy decay of the linear thermoelastic plate with memory.* J. Math. Anal. Appl. 309 (2005), 1–14.

[7] J. Lagnese, *Note on Boundary Stabilization of Wave Equations.* SIMA Journal of control and optimization, 26 (1988) 1250-1256.

[8] F.L. Huang, *Characteristic condition for exponential stability of linear dynamical systems in Hilbert spaces,* Ann. of Diff. Eqs, 1 (1985) 43-56.

[9] V. Komornik, *Exact controllability and stabilization. The Multiplier Method,* John Wiley- Masson. Paris, (1994).

[10] V. Komornik, E. Zuazua, *A Direct Method for the Boundary Stabilization of the Wave Equation.* Journal de Mathematiques Pures et Appliquees, 69 (1990) 33-54.

[11] D. S. Mitrinović, J. E. Pečarić, A. M. Fink, *Inequalities involving functions and their integrals and derivatives,* Dordrecht. The Netherlands. Kluwer., (1991).

[12] S. A. Messaoudi, B. Said-Houari, *Exponential stability in one-dimensional non-linear thermoelasticity with second sound.* Math. Meth. Appl. Sci. 28 (2005), 205–232.

[13] A. Pazy, *Semigroup of Linear Operators and Applications to Partial Differential Equations,* Springer-Verlag, New york, 1983.

[14] R. Racke, *Thermoelasticity with second sound — exponential stability in linear and nonlinear 1-d. Math.* Meth. Appl. Sci. 25 (2002), 409–441.

[15] S. Jiang, R. Racke, *Evolution equations in thermoelasticity.* π *Monographs Surveys* Pure Appl. Math. 112, Chapman and Hall/CRC, Boca Raton (2000).