Series Analysis and Schwartz Algebras of Spherical Convolutions on Semisimple Lie Groups.

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Abstract. We give the exact contributions of *Harish-Chandra transform,* $(\mathcal{H}f)(\lambda)$, of Schwartz functions f to the harmonic analysis of *spherical convolutions* and the corresponding L^p- Schwartz algebras on a connected semisimple Lie group G (with finite center). One of our major results gives the proof of how the *Trombi-Varadarajan Theorem* enters into the spherical convolution transform of L^p- Schwartz functions.

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1 Introduction

Let G be a connected semisimple Lie group with finite center, and denote the Harish-Chandra-type Schwartz spaces of functions on G by $\mathcal{C}^p(G)$, $0 < p \leq 2$. We know that $\mathcal{C}^p(G) \subset L^p(G)$ for every such p, and if K is a maximal compact subgroup of G such that $\mathcal{C}^p(G//K)$ represents the subspace of $\mathcal{C}^p(G)$ consisting of the $K-$bi-invariant functions, Trombi and Varadarajan ([9.]) have shown that the spherical Fourier transform $f \mapsto \widehat{f}$ is a linear topological isomorphism of $\mathcal{C}^p(G//K)$ onto the spaces $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$, $\epsilon = (2/p) - 1$,

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consisting of rapidly decreasing functions on certain sets $\mathfrak{F}^{\epsilon}$ of elementary spherical functions.

We show the existence of a *hyper-function* on both G and $\mathfrak{F}^1$ (here named a *spherical convolution*) whose restriction to the group identity element, e, coincides with the *spherical Fourier transforms,* $f \mapsto \widehat{f}$, of Schwartz functions f on G and which affords us the opportunity of embarking on a more inclusive harmonic analysis on G. Indeed [8*a*.] contains a more general Plancherel formula for the collection of these functions. As a function on G its series expansion is in the present paper studied. We show that, aside from the fact that the spherical Fourier transforms, $\widehat{f}(\lambda)$, is the constant term of this series expansion, there is a region in G where the spherical convolution is *essentially* $\widehat{f}(\lambda)$. Various algebras of these functions are thus studied and ultimately embedded in $L^2(G)$. It is however clear that the results in [8.] and in the present paper may be extended to include what may be termed as *the Harish-Chandra-type Schwartz spaces of Eisenstein Integrals on* G. The author has recently used the idea of a spherical convolution to give an explicit computation of the image of $\mathcal{C}^2(G)$ under the Harish-Chandra transform, [8*b*.] thus giving a concrete realization of the *abstract* results of Arthur, [2.], and showing the direct contribution of the Plancherel formula to Harish-Chandra transform on G.

The following is the breakdown of each of the remaining sections of the paper. §2. contains the preliminaries to the research containing the structure theory, spherical functions and Schwartz algebras on G, while the series analysis of spherical convolutions on G is the subject of §3. The relationship existing among the Schwartz algebras of functions and those of spherical convolutions is considered in §4.

2 Preliminaries

For the connected semisimple Lie group G with finite center, we denote its Lie algebra by $\mathfrak{g}$ whose *Cartan decomposition* is given as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Denote by θ the *Cartan involution* on $\mathfrak{g}$ whose collection of fixed points is $\mathfrak{k}$. We also denote by K the analytic subgroup of G with Lie algebra $\mathfrak{k}$. K is then a maximal compact subgroup of G. Choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$

with algebraic dual $\mathfrak{a}^*$ and set $A = \exp \mathfrak{a}$. For every $\lambda \in \mathfrak{a}^*$ put

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : [H, X] = \lambda(H)X, \forall H \in \mathfrak{a}\},$$

and call λ a restricted root of $(\mathfrak{g}, \mathfrak{a})$ whenever $\mathfrak{g}_\lambda \neq \{0\}$.

Denote by $\mathfrak{a}'$ the open subset of $\mathfrak{a}$ where all restricted roots are $\neq 0$, and call its connected components the *Weyl chambers.* Let $\mathfrak{a}^+$ be one of the Weyl chambers, define the restricted root λ positive whenever it is positive on $\mathfrak{a}^+$ and denote by $\triangle^+$ the set of all restricted positive roots. Members of $\triangle^+$ which form a basis for $\triangle$ and can not be written as a linear combination of other members of $\triangle^+$ are called *simple.* We then have the *Iwasawa decomposition* $G = KAN$, where N is the analytic subgroup of G corresponding to $\mathfrak{n} = \sum_{\lambda \in \triangle^+} \mathfrak{g}_\lambda$, and the *polar decomposition* $G = K \cdot cl(A^+) \cdot K$, with $A^+ = \exp \mathfrak{a}^+$, and $cl(A^+)$ denoting the closure of A^+.

If we set $M = \{k \in K : Ad(k)H = H,\ H \in \mathfrak{a}\}$ and $M' = \{k \in K : Ad(k)\mathfrak{a} \subset \mathfrak{a}\}$ and call them the *centralizer* and *normalizer* of $\mathfrak{a}$ in K, respectively, then (see [5.], p. 284); (i) M and M' are compact and have the same Lie algebra and (ii) the factor $\mathfrak{w} = M'/M$ is a finite group called the *Weyl group.* $\mathfrak{w}$ acts on $\mathfrak{a}^*_{\mathbb{C}}$ as a group of linear transformations by the requirement

$$(s\lambda)(H) = \lambda(s^{-1}H),$$

$H \in \mathfrak{a}$, $s \in \mathfrak{w}$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, the complexification of $\mathfrak{a}^*$. We then have the *Bruhat decomposition*

$$G = \bigsqcup_{s \in \mathfrak{w}} Bm_sB$$

where $B = MAN$ is a closed subgroup of G and $m_s \in M'$ is the representative of s (i.e., $s = m_sM$). The Weyl group invariant members of a space shall be denoted by the superscript $^{\mathfrak{w}}$ while $\mid \mathfrak{w} \mid$ represents the cardinality of $\mathfrak{w}$.

Some of the most important functions on G are the *spherical functions* which we now discuss as follows. A non-zero continuous function φ on G shall be called a *(zonal) spherical function* whenever $\varphi(e) = 1$, $\varphi \in C(G//K) := \{g \in C(G)\colon g(k_1xk_2) = g(x),\ k_1, k_2 \in K,\ x \in G\}$ and $f * \varphi = (f * \varphi)(e) \cdot \varphi$ for every $f \in C_c(G//K)$, where $(f*g)(x) := \int_G f(y)g(y^{-1}x)dy$. This leads to the existence of a homomorphism $\lambda : C_c(G//K) \to \mathbb{C}$ given as $\lambda(f) = (f * \varphi)(e)$. This definition is equivalent to the satisfaction of the functional relation

$$\int_K \varphi(xky)dk = \varphi(x)\varphi(y), \quad x, y \in G.$$

It has been shown by Harish-Chandra [6.] that spherical functions on G can be parametrized by members of $\mathfrak{a}^*_{\mathbb{C}}$. Indeed every spherical function on G is of the form

$$\varphi_\lambda(x) = \int_K e^{(i\lambda - p)H(xk)} dk, \ \lambda \in \mathfrak{a}^*_{\mathbb{C}},$$

$\rho = \frac{1}{2}\sum_{\lambda \in \triangle^+} m_\lambda \cdot \lambda$, where $m_\lambda = dim(\mathfrak{g}_\lambda)$, and that $\varphi_\lambda = \varphi_\mu$ iff $\lambda = s\mu$ for some $s \in \mathfrak{w}$. Some of the well-known properties of spherical functions are $\varphi_{-\lambda}(x^{-1}) = \varphi_\lambda(x)$, $\varphi_{-\lambda}(x) = \bar{\varphi}_{\bar{\lambda}}(x)$, $\mid \varphi_\lambda(x) \mid \leq \varphi_{\Re\lambda}(x)$, $\mid \varphi_\lambda(x) \mid \leq \varphi_{i\Im\lambda}(x)$, $\varphi_{-i\rho}(x) = 1$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, while $\mid \varphi_\lambda(x) \mid \leq \varphi_0(x)$, $\lambda \in i\mathfrak{a}^*$, $x \in G$. Also if Ω is the *Casimir operator* on G then

$$\Omega\varphi_\lambda = -(\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle)\varphi_\lambda,$$

where $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $\langle \lambda, \mu \rangle := tr(adH_\lambda \ adH_\mu)$ for elements $H_\lambda, \ H_\mu \in \mathfrak{a}$. This differential equation may be written simply as $\Omega\varphi_\lambda = \gamma(\Omega)(\lambda)\varphi_\lambda$, where $\lambda \mapsto \gamma(\Omega)(\lambda)$ is the well-known *Harish-Chandra homomorphism.* The elements $H_\lambda, \ H_\mu \in \mathfrak{a}$ are uniquely defined by the requirement that $\lambda(H) = tr(adH \ adH_\lambda)$ and $\mu(H) = tr(adH \ adH_\mu)$ for every $H \in \mathfrak{a}$ ([5.], Theorem 4.2). Clearly $\Omega\varphi_0 = 0$.

Due to a hint dropped by Dixmier [4.] (*cf.* [9.]) in his discussion of some functional calculus, it is necessary to recall the notion of a *'positive-definite'* function and then discuss the situation for positive-definite spherical functions. We call a continuous function $f : G \to \mathbb{C}$ (algebraically) positive-definite whenever, for all $x_1, \ldots, x_m$ in G and all $\alpha_1, \ldots, \alpha_m$ in $\mathbb{C}$, we have

$$\sum_{i,j=1}^{m} \alpha_i \bar{\alpha}_j f(x_i^{-1}x_j) \geq 0.$$

It can be shown (*cf.* [5.]) that $f(e) \geq 0$ and $|f(x)| \leq f(e)$ for every $x \in G$ implying that the space $\mathcal{P}$ of all positive-definite spherical functions on G is a subset of the space $\mathfrak{F}^1$ of all bounded spherical functions on G.

We know, by the Helgason-Johnson theorem ([7.]), that

$$\mathfrak{F}^1 = \mathfrak{a}^* + iC_\rho$$

where C_ρ is the convex hull of $\{s\rho : s \in \mathfrak{w}\}$ in $\mathfrak{a}^*$. Defining the *involution* f^* of f as $f^*(x) = \overline{f(x^{-1})}$, it follows that $f = f^*$ for every $f \in \mathcal{P}$, and if $\varphi_\lambda \in \mathcal{P}$, then λ and $\bar{\lambda}$ are Weyl group conjugate, leading to a realization of $\mathcal{P}$ as a subset of $\mathfrak{w} \setminus \mathfrak{a}^*_{\mathbb{C}}$. $\mathcal{P}$ becomes a locally compact Hausdorff space when endowed with the *weak* $^*-$*topology* as a subset of $L^\infty(G)$.

Let

$$\varphi_0(x) := \int_K \exp(-\rho(H(xk)))dk$$

be denoted as $\Xi(x)$ and define $\sigma : G \to \mathbb{C}$ as

$$\sigma(x) = \|X\|$$

for every $x = k \exp X \in G, \quad k \in K, \; X \in \mathfrak{a}$, where $\| \cdot \|$ is a norm on the finite-dimensional space $\mathfrak{a}$. These two functions are spherical functions on G and there exist numbers c, d such that

$$1 \leq \Xi(a)e^{\rho(\log a)} \leq c(1 + \sigma(a))^d.$$

Also there exists $r > 0$ such that $c =: \int_G \Xi(x)^2(1 + \sigma(x))^r dx < \infty$ ([11.], p. 231). For each $0 \leq p \leq 2$ define $\mathcal{C}^p(G)$ to be the set consisting of functions f in $C^\infty(G)$ for which

$$\mu_{a,b;r}(f) := \sup_G[|f(a; x; b)|\Xi(x)^{-2/p}(1 + \sigma(x))^r] < \infty$$

where $a, b \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$, the *universal enveloping algebra* of $\mathfrak{g}_{\mathbb{C}}$, $r \in \mathbb{Z}^+, x \in G$, $f(x; b) := \frac{d}{dt}\Big|_{t=0} f(x \cdot (\exp tb))$ and $f(a; x) := \frac{d}{dt}\Big|_{t=0} f((\exp ta) \cdot x)$. We call $\mathcal{C}^p(G)$ the Schwartz space on G for each $0 < p \leq 2$ and note that $\mathcal{C}^2(G)$ is the well-known (see [1.]) Harish-Chandra space of *rapidly decreasing functions* on G. The inclusions

$$C_c^\infty(G) \subset \mathcal{C}^p(G) \subset L^p(G)$$

hold and with dense images. It also follows that $\mathcal{C}^p(G) \subseteq \mathcal{C}^q(G)$ whenever $0 \leq p \leq q \leq 2$. Each $\mathcal{C}^p(G)$ is closed under *involution* and the *convolution*, $*$.

Indeed $\mathcal{C}^p(G)$ is a Fréchet algebra ([10.], p. 69). We endow $\mathcal{C}^p(G//K)$ with the relative topology as a subset of $\mathcal{C}^p(G)$.

We shall say a function f on G satisfies a *general strong inequality* if for any $r \geq 0$ there is a constant $c = c_r > 0$ such that

$$\mid f(y) \mid \leq c_r \Xi(y^{-1}x)(1 + \sigma(y^{-1}x))^{-r} \qquad \forall\, x, y \in G.$$

We observe that if $x = e$ then, using the fact that $\Xi(y^{-1}) = \Xi(y)$ and $\sigma(y^{-1}) = \sigma(y)$, $\forall\, y \in G$, such a function satisfies

$$\mid f(y) \mid \leq c_r \Xi(y^{-1})(1 + \sigma(y^{-1}))^{-r} = c_r \Xi(y)(1 + \sigma(y))^{-r},\ \forall\, y \in G,$$

showing that a function on G which satisfies a general strong inequality satisfies in particular a *strong inequality* (in the classical sense of Harish-Chandra, [11.]). Members of $\mathcal{C}^2(G) =: \mathcal{C}(G)$ are those functions f on G for which $f(g_1; \cdot; g_2)$ satisfies the strong inequality, for all $g_1, g_2 \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$. We may then define $\mathcal{C}^{(x)}(G)$ to be those functions f on G for which $f(g_1; \cdot; g_2)$ satisfies the general strong inequality, for all $g_1, g_2 \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ and a fixed $x \in G$. It is clear that $\mathcal{C}^{(e)}(G) = \mathcal{C}(G)$ and that $\bigcup_{x \in G} \mathcal{C}^{(x)}(G)$, which contains $\mathcal{C}(G)$, may be given an inductive limit topology. The seminorms defining this topology will be explicitly given in §4.

For any measurable function f on G we define the *spherical Fourier transform* $\widehat{f}$ as

$$\widehat{f}(\lambda) = \int_G f(x) \varphi_{-\lambda}(x) dx,$$

$\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. It is known (see [3.]) that for $f, g \in L^1(G)$ we have:

(*i.*) $(f * g)^\wedge = \widehat{f} \cdot \widehat{g}$ on $\mathfrak{F}^1$ whenever f (or g) is right - (or left-) K-invariant;

(*ii.*) $(f^*)^\wedge(\varphi) = \overline{\widehat{f}(\varphi^*)}, \varphi \in \mathfrak{F}^1$; hence $(f^*)^\wedge = \overline{\widehat{f}}$ on $\mathcal{P}$: and, if we define $f^\#(g) := \int_{K \times K} f(k_1 x k_2) dk_1 dk_2, x \in G$, then

(*iii.*) $(f^\#)^\wedge = \widehat{f}$ on $\mathfrak{F}^1$.

We shall denote the *spherical Fourier transform* $\widehat{f}(\lambda)$ of $f \in \mathcal{C}(G)$ by $(\mathcal{H}f)(\lambda)$ and refer to it as the *Harish-Chandra transforms* of f. Its major properties are well-known and may be found in [9.]. It should be noted that $(\mathcal{H}f)(\lambda) = \widehat{f}(\lambda) = \int_G f(y)\varphi_{-\lambda}(y)dy = \int_G f(y)\varphi_\lambda(y^{-1})dy = \int_G f(y)\varphi_\lambda(y^{-1}e)dy$ $= (f * \varphi_\lambda)(e)$. That is, the Harish-Chandra transforms of f is the restriction of the function

$$x \mapsto (f * \varphi_\lambda)(x) =: s_{\lambda,f}(x)$$

on G to the identity element. It is therefore worthwhile to explore $s_{\lambda,f}(x)$ in some details for all $x \in G$ in order to put its behaviour at $x = e$ (as the Harish-Chandra transforms of f) in a proper and larger perspective.

The beauty of studying the entirety of the function $s_{\lambda,f}(x)$, for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, $f \in \mathcal{C}^p(G)$, $x \in G$, which we shall explore in this paper, is that it could be viewed as a transformation in six (6) different ways; As

$$(1.) \qquad x \mapsto k_1(\lambda) := s_{\lambda,f}(x), \text{ for any } f \in \mathcal{C}^p(G)$$

and

$$(2.) \qquad x \mapsto k_2(f) := s_{\lambda,f}(x), \text{ for any } \lambda \in \mathfrak{a}^*_{\mathbb{C}},$$

(from where the Plancherel formula for the space of functions $x \mapsto k_2(f)$ has recently been computed in [8*a*.]) both of which are maps on G; or as

$$(3.) \qquad f \mapsto l_1(\lambda) := s_{\lambda,f}(x), \text{ for any } x \in G$$

(which, at $x = e$, led Harish-Chandra to the consideration of $f \mapsto (\mathcal{H}f)(\lambda)$: *cf*. [9.]) and

$$(4.) \qquad f \mapsto l_2(x) := s_{\lambda,f}(x), \text{ for any } \lambda \in \mathfrak{a}^*_{\mathbb{C}},$$

both of which are maps on $\mathcal{C}^p(G)$; or as

$$(5.) \qquad \lambda \mapsto m_1(f) := s_{\lambda,f}(x), \text{ for any } x \in G$$

and

$$(6.) \qquad \lambda \mapsto m_2(x) := s_{\lambda,f}(x), \text{ for any } f \in \mathcal{C}^p(G),$$

both of which are maps on $\mathfrak{a}^*_{\mathbb{C}}$. Hence the function $x \mapsto s_{\lambda,f}(x)$ may rightly be called an *hyper-function* on G whose major contribution to harmonic analysis would be to *absorb* other known functions of the subject and put their results in *proper perspectives,* as we shall establish here for the *Harish-Chandra transform.*

In order to know the image of the spherical Fourier transform when restricted to $\mathcal{C}^p(G//K)$ we need the following spaces that are central to the statement of the well-known result of Trombi and Varadarajan [9.]. Let C_ρ be the closed convex hull of the (finite) set $\{s\rho : s \in \mathfrak{w}\}$ in $\mathfrak{a}^*$, i.e.,

$$C_\rho = \left\{ \sum_{i=1}^{n} \lambda_i(s_i\rho) : \lambda_i \geq 0, \;\; \sum_{i=1}^{n} \lambda_i = 1, \;\; s_i \in \mathfrak{w} \right\}$$

where we recall that, for every $H \in \mathfrak{a}$, $(s\rho)(H) = \frac{1}{2}\sum_{\lambda \in \triangle^+} m_\lambda \cdot \lambda(s^{-1}H)$.

Now for each $\epsilon > 0$ set $\mathfrak{F}^\epsilon = \mathfrak{a}^* + i\epsilon C_\rho$. Each $\mathfrak{F}^\epsilon$ is convex in $\mathfrak{a}^*_{\mathbb{C}}$ and

$$int(\mathfrak{F}^\epsilon) = \bigcup_{0<\epsilon'<\epsilon} \mathfrak{F}^{\epsilon'}$$

([9.], Lemma (3.2.2)). Let us define $\mathcal{Z}(\mathfrak{F}^0) = \mathcal{S}(\mathfrak{a}^*)$ and, for each $\epsilon > 0$, let $\mathcal{Z}(\mathfrak{F}^\epsilon)$ be the space of all $\mathbb{C}$-valued functions Φ such that $(i.)$ Φ is defined and holomorphic on $int(\mathfrak{F}^\epsilon)$, and $(ii.)$ for each holomorphic differential operator D with polynomial coefficients we have $\sup_{int(\mathfrak{F}^\epsilon)} |D\Phi| < \infty$.

The space $\mathcal{Z}(\mathfrak{F}^\epsilon)$ is converted to a Fréchet algebra by equipping it with the topology generated by the collection, $\|\cdot\|_{\mathcal{Z}(\mathfrak{F}^\epsilon)}$, of seminorms given by $\|\Phi\|_{\mathcal{Z}(\mathfrak{F}^\epsilon)} := \sup_{int(\mathfrak{F}^\epsilon)} |D\Phi|$. It is known that $D\Phi$ above extends to a continuous function on all of $\mathfrak{F}^\epsilon$ ([9.], pp. $278-279$). An appropriate subalgebra of $\mathcal{Z}(\mathfrak{F}^\epsilon)$ for our purpose is the closed subalgebra $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$ consisting of $\mathfrak{w}$-invariant elements of $\mathcal{Z}(\mathfrak{F}^\epsilon)$, $\epsilon \geq 0$. The following (known as the *Trombi-Varadarajan Theorem*) is the major result of [9.] : *Let* $0 < p \leq 2$ *and set* $\epsilon = (2/p) - 1$. *Then the spherical Fourier transform* $f \mapsto \widehat{f}$ *is a linear topological algebra isomorphism of* $\mathcal{C}^p(G//K)$ *onto* $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$. That is, the topological algebra $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$ is an isomorphic copy or a realization of $\mathcal{C}^p(G//K)$.

In order to find other isomorphic copies or realizations of $\mathcal{C}^p(G//K)$ under the more inclusive general transformation map

$$f \mapsto l_1(\lambda) := s_{\lambda,f}(x), \text{ for any } x \in G,$$

we shall now introduce a more general algebra, $\bar{\mathcal{Z}}_G(\mathfrak{F}^\epsilon)$, of $\mathbb{C}-$valued functions on $int(\mathfrak{F}^\epsilon) \times G$ which, when restricted to $int(\mathfrak{F}^\epsilon) \times \exp(N_0)$, coincides with $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$. The form of this new algebra is suggested by Theorem 3.5. Set

$\mathcal{Z}_G(\mathfrak{F}^0) = \mathcal{S}(\mathfrak{a}^*) \times G$ and let $\mathcal{Z}_G(\mathfrak{F}^\epsilon)$, $\epsilon > 0$, be the collection of all $\mathbb{C}-$valued functions Ψ $((\lambda, x) \mapsto \Psi(\lambda, x),\ \forall\ (\lambda, x) \in int(\mathfrak{F}^\epsilon) \times G)$ such that

$(i.)$ Ψ is holomorphic in the variable λ, analytic in x and spherical on G;

$(ii.)$ $\sup_{int(\mathfrak{F}^\epsilon)} |D_1\Psi| < \infty$ and $\sup_G |\Psi D_2| < \infty$, for every holomorphic differential operator D_1 with polynomial coefficients and every left-invariant differential operator D_2 on G and

$(iii.)$ the restriction of Ψ to $int(\mathfrak{F}^\epsilon) \times \{e\}$ (or to $int(\mathfrak{F}^\epsilon) \times \exp(N_0(A^+))$, for some zero neighbourhood $N_0(A^+)$ in $\mathfrak{g}$, as will later be seen in Theorem 3.5) is (a non-zero constant multiple of) the Harish-Chandra transform, $(\mathcal{H}f)(\lambda) = \hat{f}$.

It may be shown, in exact manner as for $\mathcal{Z}(\mathfrak{F}^\epsilon)$ above, that the space $\mathcal{Z}_G(\mathfrak{F}^\epsilon)$ is converted to a Fr*é*chet algebra by equipping it with the topology generated by the collection, $\|\cdot\|_{\mathcal{Z}_G(\mathfrak{F}^\epsilon)}$, of seminorms given by

$$\|\Psi\|_{\mathcal{Z}_G(\mathfrak{F}^\epsilon)} := \sup_{int(\mathfrak{F}^\epsilon)\times G} |D_1\Psi D_2|.$$

An appropriate subalgebra of $\mathcal{Z}_G(\mathfrak{F}^\epsilon)$ for our purpose is the closed subalgebra $\bar{\mathcal{Z}}_G(\mathfrak{F}^\epsilon)$ consisting of $\mathfrak{w}$-invariant elements of $\mathcal{Z}_G(\mathfrak{F}^\epsilon)$, $\epsilon \geq 0$. By the time Theorem 3.5 is established it will be clear that $\bar{\mathcal{Z}}_{\{x\}}(\mathfrak{F}^\epsilon) \simeq \bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$, for every x in some zero neighbourhood $N_0(A^+)$ in $\mathfrak{g}$. In particular, $\bar{\mathcal{Z}}_{\{e\}}(\mathfrak{F}^\epsilon) \simeq \bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$.

3 Series Analysis of Spherical Convolutions

Let $f \in \mathcal{C}(G)$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, we recall from [8$a$.] the definition of *spherical convolutions,* $s_{\lambda,f}$, on G corresponding to the pair (λ, f) as

$$s_{\lambda,f}(x) := (f * \varphi_\lambda)(x), \quad x \in G.$$

We already know that $s_{\lambda,f}(e) = (\mathcal{H}f)(\lambda)$, where e is the identity element of G and $\lambda \in i\mathfrak{a}^*$. This relation between a function on G at the identity element and another function on $i\mathfrak{a}^*$ suggests we study the full contribution of the Harish-Chandra transforms, $(\mathcal{H}f)(\lambda)$, of f to the properties of $x \mapsto s_{\lambda,f}(x)$ and to seek other functions on $i\mathfrak{a}^*$ which have not been known in the harmonic analysis of G, but still contribute to a deeper understanding of the structure of G.

In order to explore the nature of this idea we consider opening up the spherical convolutions $x \mapsto s_{\lambda,f}(x)$ via its *Taylor's series expansion.*

Lemma 3.1. *Let* N_0 *be a neighbourhood of origin in* $\mathfrak{g}$ *and* t *be sufficiently small in* $\mathbb{R}$ *(say* $0 \leq t \leq 1$*). Then*

$$s_{\lambda,f}(x \exp tX) = \sum_{n=0}^{\infty} \frac{t^n}{n!}[\tilde{X}^n s_{\lambda,f}](x),$$

where for every $X \in N_0$ *we set* $[\tilde{X}^n s_{\lambda,f}](x) = \frac{d^n}{du^n} s_{\lambda,f}(x \exp uX)_{|u=0}$

Proof. The proof follows from a direct application of *Taylor's series expansion,* [5.], *p.* 105. □

At $x = e$ and $t = 1$ the formula in the Lemma becomes

$$s_{\lambda,f}(\exp X) = \sum_{n=0}^{\infty} \frac{1}{n!}[\tilde{X}^n s_{\lambda,f}](e) = s_{\lambda,f}(e) + \sum_{n=1}^{\infty} \frac{1}{n!}[\tilde{X}^n s_{\lambda,f}](e)$$

$$= (\mathcal{H}f)(\lambda) + \sum_{n=1}^{\infty} \frac{1}{n!}[\tilde{X}^n s_{\lambda,f}](e), \quad X \in N_0.$$

This observation leads quickly to the following result which gives the exact contribution of the Harish-Chandra transforms to the study of spherical convolutions.

Lemma 3.2. *The Harish-Chandra transforms,* $\lambda \mapsto (\mathcal{H}f)(\lambda),\ f \in \mathcal{C}(G)$, *is the constant term in the (Taylor's) series expansion of spherical convolutions,* $x \mapsto s_{\lambda,f}(x)$ *around* $x = e$, *for every* $\lambda \in \mathfrak{a}^*$. □

It may be deduced, from the expansion leading to the proof Lemma 3.2, that the only time the remaining terms in $s_{\lambda,f}(\exp X)$, after the (*non-zero*) constant term $(\mathcal{H}f)(\lambda)$, could vanish is when the differential operator $\tilde{X} = 0$. That is, when $X = 0$. It therefore follows that the well-known (Harish-Chandra) harmonic analysis on G ([1.], [2.], [9.] and [11.]) has always been that of the consideration of the map $X \mapsto s_{\lambda,f}(\exp X)$ at only $X = 0$, which is the origin of $\mathfrak{g}$ or which corresponds to the identity point of $\exp(\mathfrak{g})$. Hence, since the constant term, $(\mathcal{H}f)(\lambda)$, of $s_{\lambda,f}(\exp X)$ corresponds indeed to the consideration of the constant term in the asymptotic expansion of (zonal) spherical functions, φ_λ, it also follows that other terms in the expansion of φ_λ may be needed to completely understand $f \mapsto s_{\lambda,f}(x)$.

The expression for $s_{\lambda,f}(\exp X)$ therefore suggests that a *full* harmonic analysis of G may be attained from a close study of the remaining contributions of the *transform* of f given as

$$\lambda \longmapsto \frac{t^n}{n!}[\tilde{X}^n s_{\lambda,f}](x),$$

for all $X \in N_0,\ n \in \mathbb{N} \cup \{0\},\ x \in G,\ f \in \mathcal{C}(G)$ and sufficiently small values of t, in the same manner that its constant term,

$$\lambda \longmapsto (\mathcal{H}f)(\lambda)$$

had been considered.

However before considering the transformational properties of spherical convolutions we note the following lemmas which lead to a more inclusive view of the Trombi-Varadarajan Theorem and prepares the ground for its generalization.

Lemma 3.3. *Let N_0 be a neighbourhood of origin in $\mathfrak{g}$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and t be sufficiently small in $\mathbb{R}$ (say $0 \leq t \leq 1$). Then*

$$s_{\lambda,f}(x \exp tX) = [\sum_{n=0}^{\infty} \frac{t^n}{n!}\gamma(\frac{d^n}{du^n})(\lambda)_{|u=0}] \cdot s_{\lambda,f}(x),$$

for every $X \in N_0,\ x \in G,\ f \in \mathcal{C}(G)$.

Proof. We note here that

$$[\tilde{X} s_{\lambda,f}](x) = \frac{d}{du} s_{\lambda,f}(x \exp uX)_{|u=0} = \frac{d}{du}(f * \varphi_\lambda)(x \exp uX)_{|u=0}$$

$$= (f * \frac{d}{du}\varphi_\lambda)(x \exp uX)_{|u=0} = \gamma(\frac{d}{du})(\lambda) \cdot (f * \varphi_\lambda)(x \exp uX)_{|u=0}.$$

Hence

$$[\tilde{X}^n s_{\lambda,f}](x) = \gamma(\frac{d^n}{du^n})(\lambda)_{|u=0} \cdot (f * \varphi_\lambda)(x \exp uX)_{|u=0} = \gamma(\frac{d^n}{du^n})(\lambda)_{|u=0} \cdot s_{\lambda,f}(x). \ \square$$

The particular case of setting $x = e$ and $t = 1$ in Lemma 3.3 introduces the Harish-Chandra transforms, $(\mathcal{H}f)(\lambda)$, into the analysis of this series, proving the following.

Lemma 3.4. *Let N_0 be a neighbourhood of origin in $\mathfrak{g}$, $f \in \mathcal{C}(G)$ and $\lambda \in \mathfrak{a}^*$. Then the spherical convolution function, $x \mapsto s_{\lambda,f}(x)$ is a non-zero constant multiple of the Harish-Chandra transforms, $(\mathcal{H}f)(\lambda)$, on* $\exp(N_0)$.

Proof. Set $x = e$ and $t = 1$ into Lemma 3.3 to have

$$s_{\lambda,f}(\exp X) = [\sum_{n=0}^{\infty} \frac{1}{n!}\gamma(\frac{d^n}{du^n})(\lambda)_{|u=0}]\cdot s_{\lambda,f}(e) = [\sum_{n=0}^{\infty} \frac{1}{n!}\gamma(\frac{d^n}{du^n})(\lambda)_{|u=0}]\cdot(\mathcal{H}f)(\lambda),$$

with $\sum_{n=0}^{\infty} \frac{1}{n!}\gamma(\frac{d^n}{du^n})(\lambda)_{|u=0} = 1 + [\sum_{n=1}^{\infty} \frac{1}{n!}\gamma(\frac{d^n}{du^n})(\lambda)_{|u=0}] \neq 0.$ □

Let us denote the non-zero constant in Lemma 3.4 above by κ. The following theorem is a consequence of normalizing the spherical convolutions in Lemma 3.4.

Theorem 3.5. (**Trombi-Varadarajan Theorem for Spherical Convolutions**) *Let $0 < p \leq 2$, set $\epsilon = (2/p) - 1$ and $x \in \exp(N_0)$. Set $\widehat{f}_x(\lambda) = \frac{1}{\kappa}s_{\lambda,f}(x)$ for $f \in \mathcal{C}^p(G//K)$. Then the spherical convolution transforms $f \mapsto \widehat{f}_x$ is a linear topological algebra isomorphism of $\mathcal{C}^p(G//K)$ onto $\bar{\mathcal{Z}}(\mathfrak{F}^{\epsilon})$.* □

We recover the Trombi-Varadarajan Theorem for Harish-Chandra transforms by setting $x = e$ in Theorem 3.5. Indeed, Theorem 3.5 above says that every $x \in \exp(N_0)$ (and not just $x = e$) gives a topological algebra isomorphism between $\mathcal{C}^p(G//K)$ and $\bar{\mathcal{Z}}(\mathfrak{F}^{\epsilon})$. However if $x \in G \setminus \exp(N_0)$, for any neighborhood N_0 of zero in $\mathfrak{g}$, Trombi-Varadarajan Theorem may not be appropriate and it may be necessary to seek a more general realization of $\mathcal{C}^p(G//K)$ under the map $f \mapsto l_1(\lambda) := s_{\lambda,f}(x),$ for any $x \in G$. Before considering another major result of this paper, giving the fine structure of spherical convolution functions, we state a result on the finiteness of a central integral usually used in the estimation of many other integrals of harmonic analysis on semisimple Lie groups.

To this end we define, for every $x \in G$, the function $x \mapsto d(x)$ as

$$d(x) = \int_G \Xi^2(y^{-1}x)(1 + \sigma(y^{-1}x))^{-r} dy.$$

We observe here that

$$d(e) = \int_G \Xi^2(y^{-1})(1 + \sigma(y^{-1}))^{-r} dy = \int_G \Xi^2(y)(1 + \sigma(y))^{-r} dy,$$

which is a constant whose proof of finiteness may be found in [11.], *p.* 231. This constant is crucial to all harmonic analysis of $\mathcal{C}(G)$ and, in particular, to the embedding of $\mathcal{C}(G)$ in $L^2(G)$. It is therefore important to understand the nature of $d(x)$ for all $x \in G$ in order to employ it in a more inclusive harmonic analysis on G. We consider the nature of this integral in the following.

Lemma 3.6. *Let $x \in G$. Then there exist $r \geq 0$ such that*

$$d(x) = \int_G \Xi^2(y^{-1}x)(1 + \sigma(y^{-1}x))^{-r} dy < \infty.$$

Proof. We already know that $\Xi(y^{-1}x) \leq 1$. Also

$$1 + \sigma(y^{-1}x) \leq (1 + \sigma(y^{-1}))(1 + \sigma(x)) = (1 + \sigma(y))(1 + \sigma(x)).$$

It follows therefore that

$$d(x) \leq \int_G (1 + \sigma(y^{-1}x))^{-r} dy \leq (1 + \sigma(x)) \int_G (1 + \sigma(y)) dy.$$

The last integral in the above inequality is finite if we embark on its computation via the polar decomposition, $G = K \cdot cl(A^+) \cdot K$, of G. $\square$

Theorem 3.7. *Let N_0 be a neighbourhood of origin in $\mathfrak{g}$ where f is a measurable function on G which satisfies the general strong inequality. The integral defining the spherical convolution function, $x \mapsto s_{\lambda,f}(x)$, is absolutely and uniformly convergent for all $x \in \exp(N_0)$, $\lambda \in i\mathfrak{a}^*$. Moreover the transforms $\lambda \mapsto s_{\lambda,f}(x)$ of f, with $x \in \exp(N_0)$, is a continuous function on $i\mathfrak{a}^*$. If $r \geq 0$ is such that $d(x) = \int_G \Xi^2(y^{-1}x)(1+\sigma(y^{-1}x))^{-r} dy < \infty$, $x \in G$, then*

$$\mid s_{\lambda,f}(x) \mid \leq d(x) \cdot \mu_{1,1,r}(f), \quad x \in G, \ \lambda \in i\mathfrak{a}^*.$$

Proof. We recall that $\mid \varphi_\lambda(x) \mid \leq \varphi_0(x) = \Xi(x),\ x \in G,\ \lambda \in i\mathfrak{a}^*$. Hence

$$\mid (f*\varphi_\lambda)(x) \mid \leq \int_G \mid f(y)\varphi_\lambda(y^{-1}x) \mid dy \leq \mu_{1,1,r}(f) \int_G \Xi^2(y^{-1}x)(1+\sigma(y^{-1}x))^{-r}dy$$

$= d(x) \cdot \mu_{1,1,r}(f)$. Continuity follows from the use of the Lebesgue's dominated convergence theorem. $\square$

The following well-known result on the foundational properties of the Harish-Chandra transforms, $\lambda \mapsto (\mathcal{H}f)(\lambda),\ \lambda \in i\mathfrak{a}^*$, now follows from the general outlook given by Theorem 3.7.

Corollary 3.8. ([9.]) *Let f be a measurable function on G which satisfies the strong inequality. The integral defining the Harish-Chandra transforms,*

$$(\mathcal{H}f)(\lambda) = \int_G f(x)\varphi_\lambda(x)dx,$$

is absolutely and uniformly convergent for all $\lambda \in i\mathfrak{a}^*$ and is continuous on $i\mathfrak{a}^*$. If $r \geq 0$ is such that $d = \int_G \Xi^2(y)(1+\sigma(y))^{-r}dy < \infty$, then

$$(\mathcal{H}f)(\lambda) \mid \leq d\mu_{1,1,r}(f),\ \ \lambda \in i\mathfrak{a}^*.$$

Proof. Set $X = 0$ in Theorem 3.7 to have the first results. The inequality follows if we set $x = e$ and observe that $d(e) = \int_G \Xi^2(y^{-1})(1+\sigma(y^{-1}))^{-r}dy = d$. $\square$

We now consider the image of $\mathcal{C}^p(G//K)$ under the *full* spherical convolution map, $f \mapsto l_1(\lambda) := s_{\lambda,f}(x)$, for any $x \in G$. In order to discuss this we have two options. One of the options is to introduce wave-packet that will still have its domain as $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$ while using an appropriate Plancherel measure on $\mathfrak{F}^\epsilon$. This option has been explored in [8*a*.], *p*. 34, where the L^2 Plancherel measure, $d\zeta_{x,\lambda}$ on $\mathfrak{F}^1$ for the spherical convolution function (when viewed as a function on G) was defined to absorb the group variable, x. The results therein suggest that the image of $\mathcal{C}^p(G//K)$ under the *full* spherical convolution map is indeed possible.

The second option is to retain the *spherical Bochner measure,* $d\lambda$, on (a subset of) $\mathfrak{F}^\epsilon$ and define the wave-packet as a map on the Fréchet algebra $\bar{\mathcal{Z}}_G(\mathfrak{F}^\epsilon)$. This will reflect the nature of the full spherical convolution map as a transform of members of $\mathcal{C}^p(G//K)$ whose arguments are (generally) taken from $int(\mathfrak{F}^\epsilon) \times G$ (and not just from $int(\mathfrak{F}^\epsilon)$ as in the first option).

To this end recall the Fréchet algebra $\bar{\mathcal{Z}}_G(\mathfrak{F}^\epsilon)$, $\forall\ \epsilon > 0$, let $\Psi \in \bar{\mathcal{Z}}_G(\mathfrak{F}^\epsilon)$ and set

$$N_0(A^+) = N_0 \cap A^+,$$

where N_0 is a zero neighbourhood in $\mathfrak{g}$. It is clear that $N_0(A^+)$ is also a zero neighbourhood in $\mathfrak{g}$ and that $\Psi = \Psi(\lambda, x)$, for all $(\lambda, x) \in int(\mathfrak{F}^\epsilon) \times G$. It follows, from Theorem 3.5, that $\bar{\mathcal{Z}}_{\{x\}}(\mathfrak{F}^\epsilon) \simeq \bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$, for every $x \in \exp(N_0(A^+))$. We then have the following.

Lemma 3.9. *For every $x \in \exp(N_0(A^+))$ and $\Psi \in \bar{\mathcal{Z}}_G(\mathfrak{F}^\epsilon)$, we have that $\Psi(\lambda, x) = \Phi(\lambda)$, for some $\Phi \in \bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$.*

We now employ these remarks to define a map from $\bar{\mathcal{Z}}_G(\mathfrak{F}^\epsilon)$ to $\mathcal{C}^p(G//K)$ as follows. Let $a \in \bar{\mathcal{Z}}_G(\mathfrak{F}^\epsilon)$ and $\lambda \mapsto c(\lambda)$ be the Harish-Chandra $c-$function defined on $\mathfrak{F}_I := i\mathfrak{a}^*$. We associate to every $a \in \bar{\mathcal{Z}}_G(\mathfrak{F}^\epsilon)$ the function φ_a on G defined as

$$\varphi_a(x) =\mid \mathfrak{w} \mid^{-1} \int_{\mathfrak{F}_I} a(-\lambda, x)\varphi_{-\lambda}(x)c(-\lambda)^{-1}c(\lambda)^{-1}d\lambda, \quad x \in G.$$

It should be noted here that

$$\begin{aligned}\varphi_a(x) &=\mid \mathfrak{w} \mid^{-1} \int_{\mathfrak{F}_I} a(-\lambda, x)\varphi_{-\lambda}(x)c(-\lambda)^{-1}c(\lambda)^{-1}d\lambda \\ &=\mid \mathfrak{w} \mid^{-1} \int_{\mathfrak{F}_I} a(\lambda, x)\varphi_{\lambda}(x)c(\lambda)^{-1}c(-\lambda)^{-1}d(-\lambda) \\ &=\mid \mathfrak{w} \mid^{-1} \int_{\mathfrak{F}_I} a(\lambda, x)\varphi_{\lambda}(x)c(\lambda)^{-1}c(-\lambda)^{-1}d\lambda,\end{aligned}$$

which is due to the invarianve of $d\lambda$, and that

$$\varphi_a(k_1 x k_2) = \varphi_a(x),$$

$\forall\ x \in G,\ k_1, k_2 \in K$, being a property inherited from a and φ_λ.

The (extra) requirement of being spherical on G placed on members of $\bar{\mathcal{Z}}_G(\mathfrak{F}^\epsilon)$ may at first be seen as a restriction, when compared to the requirements on members of $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$. It however turns out that this extra requirement is what is needed to assure us of the generalization of the *classical* wave-packets (of Trombi-Varadarajan) on G to all of $x \mapsto \varphi_a(x)$. This is established as follows.

Lemma 3.10. *Let $a \in \bar{\mathcal{Z}}_G(\mathfrak{F}^\epsilon)$ and $N_0(A^+)$ be as defined above. Then, for every $x \in \exp(N_0(A^+))$, the map $x \mapsto \varphi_a(x)$ is the classical wave-packet of G.*

Proof. We observe that, with $\exp tH \in \exp(N_0(A^+))$,

$$a(\lambda, x) = a(\lambda, k_1 \exp tH k_2) = a(\lambda, \exp tH) = \Phi(\lambda),$$

for some $\Phi \in \bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$. Here we have employed the spherical property of a on G in the second equality and Lemma 3.9 in the third equality. $\square$

The above Lemma shows that the definition and properties of the map $x \mapsto \varphi_a(x)$, $x \in G$, is consistent with the relationship (in Lemma 3.4) existing between spherical convolutions, $s_{\lambda,f}(x)$ and the Harish-Chandra transfroms, $(\mathcal{H}f)(\lambda)$. Hence in order to extend Trombi-Varadarajan Theorem (which gives the image of the algebra $\mathcal{C}^p(G//K)$ under $f \mapsto (\mathcal{H}f)(\lambda)$) to all $x \in G$ (under the spherical convolution tranform), it will be necessary to show that $x \mapsto \varphi_a(x)$ is the wave-packet of $f \mapsto s_{\lambda,f}(x)$ for all $x \in G$. According to Lemma 3.10, this needs only be done for those $x = k_1 \exp tH k_2$ in G with $\exp tH \notin \exp(N_0(A^+))$, for any neighbourhood, N_0, of zero in $\mathfrak{g}$. We however give a self-contained discussion of these results, the first of which is given below.

Theorem 3.11. *$\varphi_a \in \mathcal{C}^p(G//K)$ for every $a \in \bar{\mathcal{Z}}_G(\mathfrak{F}^\epsilon)$.*

In order to finish the establishment of this Theorem we need some lemmas which give appropriate background for it. Indeed we derive an appropriate bound for $\mid \varphi_a(h;u) \mid$, where $u \in \mathfrak{U}(\mathfrak{g}_\mathbb{C})$ and h is well-chosen, and the appropriate collection of seminorms are also in place. These will be considered in a forthcoming paper on *Trombi-Varadarajan Theorem* via the eigenfunction expansion of spherical convolution, which includes the extension of Theorem 3.5 to all $x \in G$.

4 Algebras of Spherical Convolutions

We now consider the various algebras of spherical convolutions that have emanated in the course of this research and their relationship with the Harish-Chandra Schwartz algebra, $\mathcal{C}(G)$, on G as well as its distinguished commutative subalgebra, $\mathcal{C}(G//K)$, of (elementary) spherical functions.

Define $\mathcal{C}_\lambda(G) = \{s_{\lambda,f} : f \in \mathcal{C}(G)\}$ and set $\mathcal{C}_{\lambda,0}(G) = \{s_{\lambda,\varphi_\lambda}\}$, for all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. It is clear that $\bigcup_{\lambda\in\mathfrak{a}^*_{\mathbb{C}}} \mathcal{C}_\lambda(G)$ is contained in $\mathcal{C}(G)$. We may therefore topologize $\bigcup_{\lambda\in\mathfrak{a}^*_{\mathbb{C}}} \mathcal{C}_\lambda(G)$ by giving it the *relative topology* from the topology defined on $\mathcal{C}(G)$ by the seminorms, $\mu_{a,b,r}$.

Lemma 4.1. *The inclusions*

$$[\bigcup_{\lambda\in\mathfrak{a}^*_{\mathbb{C}}} \mathcal{C}_{\lambda,0}(G)] \subset \mathcal{C}(G//K) \subset [\bigcup_{\lambda\in\mathfrak{a}^*_{\mathbb{C}}} \mathcal{C}_\lambda(G)] \subset \mathcal{C}(G)$$

are all proper. $\square$

Theorem 4.2. $\bigcup_{\lambda\in\mathfrak{F}^1} \mathcal{C}_\lambda(G)$ *is a closed subalgebra of* $\mathcal{C}(G)$.
Proof. We recall that $\mu_{a,b;r}(f * \varphi_\lambda) \leq c\mu_{1,b;r+r_0}(f) \cdot \mu_{a,1;r}(\varphi_\lambda)$, where $c := \int_G \Xi^2(x)(1+\sigma(x))^{-r_0}dx < \infty$ for some $r_0 \geq 0$. However

$$\mu_{a,1;r}(\varphi_\lambda) = \sup_G[|\varphi_\lambda(1;x;a)| \cdot \Xi(x)^{-1}(1+\sigma(x))^r]$$

$$=| \gamma(a)(\lambda) | \cdot \sup_G[|\varphi_\lambda(x)| \cdot \Xi(x)^{-1}(1+\sigma(x))^r]$$

$$\leq M | \gamma(a)(\lambda) | \cdot \sup_G[\Xi(x)^{-1}(1+\sigma(x))^r] < \infty$$

$$\text{(since } \varphi_\lambda \text{ is bounded for all } \lambda \in \mathfrak{F}^1).$$

Hence $\mu_{a,b;r}(f * \varphi_\lambda) < \infty,\ \forall\ \lambda \in \mathfrak{F}^1$. $\square$

It may be recalled that members of $\mathcal{C}(G)$ are exactly those functions on G whose left and right derivatives satisfy the *strong inequality.* In the light of this observation we define $\mathcal{C}^{(x)}(G)$ as exactly those functions on G whose left and right derivatives satisfy the *general strong inequality,* for each $x \in G$. Explicitly we set $\mathcal{C}^{(x)}(G)$ as

$$\mathcal{C}^{(x)}(G) = \{f : G \mapsto \mathbb{C} : \sup_{y \in G}[|f(a; y; b)| \cdot \Xi(y^{-1}x)^{-1}(1 + \sigma(y^{-1}x))^r] < \infty\},$$

$x \in G$. A collection of seminorms on each of $\mathcal{C}^{(x)}(G)$ may be given by

$$\mu^{(x)}_{a,b;r}(f) := \sup_{y \in G}[|f(a; y; b)| \cdot \Xi(y^{-1}x)^{-1}(1 + \sigma(y^{-1}x))^r].$$

It is however clear that $\mathcal{C}^{(e)}(G) = \mathcal{C}(G)$, so that $\mathcal{C}(G) \subset \bigcup_{x \in G} \mathcal{C}^{(x)}(G)$.

Theorem 4.3. *The natural inclusion* $\bigcup_{x \in G} \mathcal{C}^{(x)}(G) \subset L^2(G)$ *has a dense image.*

Proof. It is known that the natural inclusion of $\mathcal{C}(G)$ in $L^2(G)$ has a dense image, [1.]. The result therefore follows if we recall that, as sets of functions,

$$\mathcal{C}(G) \subset \bigcup_{x \in G} \mathcal{C}^{(x)}(G) \subset L^2(G),$$

where the second inclusion holds from the fact that $d(x) < \infty,\ x \in G.$ $\square$

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