POSITIVE WEIGHTED PSEUDO ALMOST AUTOMORPHIC SOLUTION FOR A CLASS OF SYSTEMS OF NEUTRAL NONLINEAR DELAY INTEGRAL EQUATIONS

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Abstract. In this work, we shall explain a new result concerning weighted pseudo almost automorphic solutions for more general systems of nonlinear neutral infinite delay integral equations. We establish a new fixed point theorem in the cone, which extend some existing results even in the case of scalar version, and then, we apply it to prove our results.

1. Introduction

Since the work of Bochner in [7], almost automorphy, as a natural generalization of the concept of almost periodicity in the sense of Bohr [6], has been of great interest for many authors to study almost automorphic solutions to various equations including linear and nonlinear evolution equations, integro-differential equations, delay integral equations, functional-differential equations, etc. For more details about this topics we refer to the recent book [22], where the author gave an important overview about the theory of almost automorphic functions and their applications to differential equations.

The concept of weighted pseudo almost automorphic functions with values in a Banach space, was introduced by G.M.N'Guerekata et al. [8] as a generalization of that of pseudo almost automorphic functions, which generalizes that of pseudo almost periodic functions introduced by Diagana [13]. Since then, these functions have generated lot of developments and applications. For more details we refer the reader to [8, 13, 21, 22] and the references therein.

The study of the existence of almost periodic, almost automorphic, pseudo almost periodic, pseudo almost automorphic, weighted pseudo almost periodic and weighted pseudo almost automorphic solutions is one of the most interesting topics in the qualitative theory of differential and integral equations. In [25], we considered the existence and uniqueness of positive almost periodic solution to the following system of nonlinear finite delay integral equations

$$\begin{aligned} x(t) &= \int_{t-\tau_1(t)}^{t} \tilde{f}(s, x(s), y(s))ds \\ y(t) &= \int_{t-\tau_2(t)}^{t} \tilde{g}(s, x(s), y(s))ds \end{aligned} \tag{1.1}$$

which is a model for the evolution in time of two species with interaction. Also, in [9, 10, 23, 24, 27], the existence of positive periodic solutions for other forms of

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(1.1) was studied by using the method of upper and lower solutions or by topological methods.

In this work, we investigate the existence and uniqueness of a positive weighted pseudo almost automorphic solution for the following more general system of non-linear neutral infinite delay integral equations

$$\begin{aligned} x(t) &= \alpha_1(t)x(t-\beta_1) + \int_{-\infty}^{t} c_1(t,t-s)\tilde{f}(s,x(s),y(s))ds + k_1(t,x(t)) \\ y(t) &= \alpha_2(t)y(t-\beta_2) + \int_{-\infty}^{t} c_2(t,t-s)\tilde{g}(s,x(s),y(s))ds + k_2(t,y(t)) \end{aligned} \tag{1.2}$$

Note that the existence of pseudo almost periodic solutions to the scalar version of system (1.2)

$$x(t) = \alpha(t)x(t-\beta) + \int_{-\infty}^{t} a(t,t-s)f(s,x(s))ds + h(t,x(t)) \tag{1.3}$$

was studied in [14]. Also, the existence of almost periodic, almost automorphic and pseudo almost automorphic of various forms of (1.3) was studied by many authors (see, e.g. [1, 2, 3, 16, 17] and references therein)

To the best of our knowledge, there is no work reported in the literature on weighted pseudo almost automorphic solution to the system (1.2). Therefore, motivated by the works in [14, 15], the purpose of this paper is to establish a new fixed point theorem in partially ordered Banach spaces, which extend some existing results even in the scalar cases, and then used to prove the existence of positive weighted pseudo almost automorphic solution for (1.2). This paper is organized as follows. In Section 2, we recall some notations and preliminaries. Namely some basic results for almost automorphy and weighted pseudo almost automorphy. Section 3, is divoted to extend and prove a fixed point theorem in the cone. In section 4, we prove our results for the existence and uniqueness of positive weighted pseudo almost automorphic solution. In the last section, we give an example.

2. SOME DEFINITIONS AND PRELIMINARIES

We denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}^+$ the set of nonnegative real numbers, by Ω a closed subset in $\mathbb{R}^q$ $(q = 1, 2)$ and by $BC(X)$, where X is a metric set, the space of continuous bounded functions defined on X with values in $\mathbb{R}$. we recall some definitions and notation for almost automorphy and weighted pseudo almost automorphy.

2.1. Almost automorphy.

Definition 2.1 ([22]). *A continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is called almost automorphic if for every sequence of real numbers $(S'_m)_m$ there exists a subsequence $(S_n)_n$ such that*

$$\lim_{m\to+\infty}\lim_{n\to+\infty} f(t + S_n - S_m) = f(t), \ \forall t \in \mathbb{R}.$$

This limit means that

$$g(t) = \lim_{n\to+\infty} f(t + S_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$f(t) = \lim_{n\to+\infty} g(t - S_n), \ \forall t \in \mathbb{R}.$$

The collection of all such functions will be denoted by $AA(\mathbb{R})$.

Notice that some fundamental properties of almost periodic functions are not verified by the almost automorphic functions, as exemple the property of uniform continuity. A well known example of almost automorphic function not almost periodic is

$$f(t) = \sin \frac{1}{2 + \cos t + \cos \sqrt{2}t}.$$

Lemma 2.2 ([22]). *Assume that* $f, g \in AA(\mathbb{R})$ *and* λ *is any scalar. Then the following hold true:*

i) $f + g,\ f.g,\ \lambda f,\ f_\tau(t) = f(t + \tau),\ \widetilde{f}(t) = f(-t)$ *are almost automorphic.*
ii) *The range* $R_f = \{f(t) :\ t \in \mathbb{R}\}$ *is precompact in* $\mathbb{R}$, *and so* f *is bounded.*
iii) *If* $\{f_n\}$ *is a sequence of almost automorphic functions and* $f_n \to f$ *uniformly on* $\mathbb{R}$, *then* f *is almost automorphic.*
iii) *Equipped with the sup norm*

$$\|f\| = \sup_{t \in \mathbb{R}} |f(t)|$$

$AA(\mathbb{R})$ *turns out to be a Banach space.*

Definition 2.3 ([22]). *A continuous function* $f : \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ *is called almost automorphic in* t *uniformly for* x *in compact subset of* $\Omega \subset \mathbb{R}$ *(respectively for* (x, y) *in compact subset of* $\Omega \subset \mathbb{R} \times \mathbb{R}$*) if for every compact subset* K *of* Ω *and every real sequence* $(S_m)_m$, *there exists a subsequence* $(S_n)_n$ *such that*

$$g(t, x) = \lim_{n \to +\infty} f(t + S_n, x)\ (resp. g(t, x, y) = \lim_{n \to +\infty} f(t + S_n, x, y))$$

is well defined for each $t \in \mathbb{R},\ x \in K\ (resp. (x, y) \in K)$ *and*

$$f(t, x) = \lim_{n \to +\infty} g(t - S_n, x)\ (resp. f(t, x, y) = \lim_{n \to +\infty} g(t - S_n, x, y)),\ \forall t \in \mathbb{R}.$$

The collection of all such functions will be denoted by $AA(\mathbb{R} \times \Omega)$.

2.2. **Weighted pseudo almost automorphy.** Let U denote the collection of all functions (weights) $\rho : \mathbb{R} \longrightarrow (0, +\infty)$ which are locally integrable over $\mathbb{R}$ such that $\rho(t) > 0$ for almost each $t \in \mathbb{R}$. For $\rho \in U$ and $r > 0$, we set

$$m(r, \rho) = \int_{-r}^{r} \rho(t) dt.$$

Throughout this paper, the set of weights U_∞ stands for

$$U_\infty = \{\rho \in U : \lim_{r \to +\infty} m(r, \rho) = \infty\}.$$

Obviously, $U_\infty \subset U$, with strict inclusions.

Let $\rho \in U_\infty$. Set

$$PAA_0(\mathbb{R}, \rho) = \{f \in BC(\mathbb{R}) : \lim_{r \to +\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} |f(t)| \rho(t) dt = 0\}.$$

In the same way, we define $PAA_0(\mathbb{R} \times \mathbb{R}^+, \rho)$ $(PAA_0(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \rho))$ as the collection of continuous functions f defined on $\mathbb{R} \times \mathbb{R}^+$ $(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$ such that $f(., x)$ $(f(., x, y))$ is bounded for each $x \in \mathbb{R}^+$ $((x, y) \in \mathbb{R}^+ \times \mathbb{R}^+)$ and

$$\lim_{r \to +\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} |f(t, x)| \rho(t) dt = 0\ (\lim_{r \to +\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} |f(t, x, y)| \rho(t) dt = 0)$$

uniformly in $x \in \mathbb{R}^+$ $((x,y) \in \mathbb{R}^+ \times \mathbb{R}^+)$.

Definition 2.4 ([8]). *Let $\rho \in U_\infty$. A functon $f \in BC(\mathbb{R})$ is called weighted pseudo almost automorphic (or ρ-pseudo almost automorphic) if it can be expressed as $f = f^{aa} + f^e$, where $f^{aa} \in AA(\mathbb{R})$ and $f^e \in PAA_0(\mathbb{R}, \rho)$. The collection of such functions is denoted by $WPAA(\mathbb{R}, \rho)$.*

The functions f^{aa} and f^e appearing in definition above are respectively called the almost periodic and the weighted ergodic perturbation components of f.

Example 2.5 ([26]). *Consider the functions*

$$f(t) = \sin \frac{1}{2 + \cos t + \cos \sqrt{2}t} + e^{\alpha t} \text{ and } \rho(t) = \begin{cases} 1 & if\ t < 0 \\ e^{-\beta t} & if\ t \geq 0 \end{cases}$$

If $0 < \alpha \leq \beta$, we have $f \in WPAA(\mathbb{R}, \rho)$ and f does not belongs to $PAA(\mathbb{R})$, the space of all pseudo almost automorphic functions.

In the followng lemma we give some properties of the space $WPAA(\mathbb{R}, \rho)$.

Lemma 2.6 ([8, 20]). *Let $\rho \in U_\infty$.*

(i) *$WPAA(\mathbb{R}, \rho)$ equipped with the sup norm is a Banach space .*
(ii) *If $f = f^{aa} + f^e \in WPAA(\mathbb{R}, \rho)$ with $f^{aa} \in AA(\mathbb{R})$ and $f^e \in PAA_0(\mathbb{R}, \rho)$, then $f^{aa}(\mathbb{R}) \subset \overline{f(\mathbb{R})}$.*
(iii) *If $f \in BC(\mathbb{R})$, then $f \in PAA_0(\mathbb{R}, \rho)$ if and only if for every $\varepsilon > 0$*

$$\lim_{r \to +\infty} \frac{1}{m(r, \rho} \int_{M_{r,\varepsilon}(f)} \rho(t)dt = 0,$$

where $M_{r,\varepsilon}(f) = \{t \in [-r, r] : |f(t)| \geq \varepsilon\}$.
(iv) *If we consider that $\rho \equiv 1$, then we obtain the standard spaces $PAA(\mathbb{R})$.*

Definition 2.7. *A subset B of $BC(\mathbb{R})$ is said to be translation invariant if for any $x \in B$ we have $x(. + \tau) \in B$ for any $\tau \in \mathbb{R}$.*

Lemma 2.8 ([19]). *Let $\rho \in U_\infty$. Assume that $PAA_0(\mathbb{R}, \rho)$ is translation invariant. Then the decomposition of weighted pseudo almost automorphic is unique.*

Lemma 2.9. *Let us fix $\rho \in U_\infty$.*

1) *Following the same reasoning as in the proof of [5] it follows that If $f, g \in WPAA(\mathbb{R}, \rho)$, then $f.g \in WPAA(\mathbb{R}, \rho)$*
2) *We know from Agarwal et al. [4] that if the limits*

$$\limsup_{t \to \infty} \frac{\rho(t+\tau)}{\rho(t)} < \infty \text{ and } \limsup_{t \to \infty} \frac{m(r+\tau, \rho)}{m(r, \rho)} < \infty. \tag{2.1}$$

exist for each $\tau \in \mathbb{R}$. Then the spase $PAA_0(\mathbb{R}, \rho)$ is translation invariant.

Definition 2.10 ([8]). *A functon $f \in BC(\mathbb{R} \times \mathbb{R}^+)$ $(f \in BC(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+))$ is called weighted pseudo almost automorphic (or ρ-pseudo almost automorphic) if it can be expressed as $f = f^{aa} + f^e$, where $f^{aa} \in AA(\mathbb{R} \times \mathbb{R}^+)$ and $f^e \in PAA_0(\mathbb{R} \times \mathbb{R}^+, \rho)$ $(f^{aa} \in AA(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$ and $f^e \in PAA_0(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \rho))$. The collection of such functions is denoted by $WPAA(\mathbb{R} \times \mathbb{R}^+, \rho)$ $(WPAA(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \rho))$.*

Theorem 2.11 ([8, 18]). *Fix $\rho \in U_\infty$. Let $\sigma, \tau \in WPAA(\mathbb{R}, \rho)$ and $f = f^{aa} + f^e \in WPAA(\mathbb{R} \times \mathbb{R}^+, \rho)$ ($f = f^{aa} + f^e \in WPAA(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \rho)$). Assume both f and f^{aa} are uniformly continuous in any bounded subset $K \in \mathbb{R}^+$ ($K \in \mathbb{R}^+ \times \mathbb{R}^+$) uniformly in $t \in \mathbb{R}$. Then, $f(., \sigma(.)) \in WPAA(\mathbb{R}, \rho)$ ($f(., \sigma(.), \tau(.)) \in WPAA(\mathbb{R}, \rho)$).*

Corollary 2.12 ([20]). *Fix $\rho \in U_\infty$. Let $\sigma, \tau \in WPAA(\mathbb{R}, \rho)$ and $f = f^{aa} + f^e \in WPAA(\mathbb{R} \times \mathbb{R}^+, \rho)$ ($f = f^{aa} + f^e \in WPAA(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \rho)$). Assume both f and f^{aa} are lipschitzian in $x \in \mathbb{R}^+$ ($(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+$) uniformly in $t \in \mathbb{R}$. Then, $f(., \sigma(.)) \in WPAA(\mathbb{R}, \rho)$ ($f(., \sigma(.), \tau(.)) \in WPAA(\mathbb{R}, \rho)$).*

3. Fixed point theorem

Definition 3.1 ([12]). *Let E be a real Banach space. A closed convex set P in E is called a convex cone if the following conditions are satisfied*

(1) *If $x \in P$, then $\lambda x \in P$ for any $\lambda \in \mathbb{R}^+$;*
(2) *If $x \in P$ and $-x \in P$, then $x = 0$.*

A cone P induces a partial ordering $\leq$ in E by $x \leq y$ if and only if $y - x \in P$. A cone P is called normal if there exists a constant $N > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq N\|y\|$, where $\|.\|$ is the norm on E. We denote by $\overset{\circ}{P}$ the interior set of P. A cone P is called a solid cone if $\overset{\circ}{P} \neq \emptyset$.

In the following theorem, we extend the results obtained in [14, Theorem 3.1] and [15, Theorem 2.1], used in the scalar case, to other used in the case of systems.

Theorem 3.2. *Let P be a normal solid cone in a real Banach space X. D_1, D_2 are linear operators from P to P and $A_1, A_2, B_1, B_2 : \overset{\circ}{P} \times \overset{\circ}{P} \times \overset{\circ}{P} \times \overset{\circ}{P} \longrightarrow \overset{\circ}{P}$ are operators with*

$$A_1(x, u, y, \xi) = B_1(x, u, y, \xi) + D_1(x),$$
$$A_2(x, u, y, \xi) = B_2(x, u, y, \xi) + D_2(y)$$

such that

(S$_1$) *$B_1(., u, y, \xi)$ is increasing and $B_1(x, ., y, \xi), B_1(x, u, ., \xi), B_1(x, u, y, .)$ are decreasing;*
$B_2(x, u, ., \xi)$ is increasing and $B_2(., u, y, \xi), B_2(x, ., y, \xi), B_2(x, u, y, .)$ are deccreasing.

(S$_2$) *There exist positive functions ϕ_1, ϕ_2 defined on $(0, 1) \times \overset{\circ}{P} \times \overset{\circ}{P} \times \overset{\circ}{P}$ such that for each $x, u, y, \xi \in \overset{\circ}{P}$ and $\alpha \in (0, 1)$, $\phi_i(\alpha, x, u, y) > \alpha$ $(i = 1, 2)$and*

$$B_1(\alpha x, \frac{1}{\alpha} u, \frac{1}{\alpha} y, \xi) \geq \phi_1(\alpha, x, u, y) B_1(x, u, y, \xi),$$
$$B_2(\frac{1}{\alpha} x, \frac{1}{\alpha} u, \alpha y, \xi) \geq \phi_2(\alpha, x, u, y) B_2(x, u, y, \xi).$$

(S$_3$) *There exist $x_0, x^0, y_0, y^0 \in \overset{\circ}{P}$ with $x_0 \leq x^0, y_0 \leq y^0$ such that*

$$\begin{aligned} x_0 \leq A_1(x_0, x^0, y^0, x_0), A_1(x^0, x_0, y_0, x^0) \leq x^0, \\ y_0 \leq A_2(x^0, y^0, y_0, y_0), A_2(x_0, y_0, y^0, y^0) \leq y^0 \end{aligned} \tag{3.1}$$

and for each $\alpha \in (0,1)$

$$\begin{aligned} \inf_{y\in[y_0,y^0],x,u\in[x_0,x^0]} \phi_1(\alpha,x,u,y) > \alpha, \\ \inf_{x\in[x_0,x^0],v,y\in[y_0,y^0]} \phi_2(\alpha,x,v,y) > \alpha. \end{aligned} \tag{3.2}$$

(S_4) *There exist constants* $L_1, L_2 > 0$ *such that for all* $x,u,y,\xi_1,\xi_2 \in \overset{\circ}{P}$ *with* $\xi_1 \geq \xi_2$,

$$B_i(x,u,y,\xi_1) - B_i(x,u,y,\xi_2) \geq -L_i(\xi_1 - \xi_2) \ (i = 1,2).$$

Then operator $A : \overset{\circ}{P} \times \overset{\circ}{P} \times \overset{\circ}{P} \times \overset{\circ}{P} \times \overset{\circ}{P} \times \overset{\circ}{P} \longrightarrow \overset{\circ}{P} \times \overset{\circ}{P}$ *defined by*

$$A(x,u,v,y,\xi,\nu) = (A_1(x,u,y,\xi), A_2(x,v,y,\nu))$$

has a unique fixed point $(x^*,y^*) \in [x_0,x^0] \times [y_0,y^0]$*; that is*

$$A(x^*,x^*,y^*,y^*,x^*,y^*) = (x^*,y^*).$$

Moreover, if (3.2) is true for all $u_0,u^0,v_0,v^0 \in \overset{\circ}{P}$ *with* $u_0 \leq u^0$ *and* $v_0 \leq v^0$*:*

$$\begin{aligned} \inf_{y\in[v_0,v^0],x,u\in[u_0,u^0]} \phi_1(\alpha,x,u,y) > \alpha, \\ \inf_{x\in[u_0,u^0],v,y\in[v_0,v^0]} \phi_2(\alpha,x,v,y) > \alpha. \end{aligned}$$

Then (x^*,y^*) *is the unique fixed point of* A *in* $\overset{\circ}{P} \times \overset{\circ}{P}$.

Proof. From (S_1) and the linearity of the operators $D_1, D_2 : P \longrightarrow P$, we obtain $A_1(.,u,y,\xi)$, $A_2(x,u,.,\xi)$ are increasing and $A_1(x,.,y,\xi)$, $A_1(x,u,.,\xi)$, $A_1(x,u,y,.)$, $A_2(.,u,y,\xi)$, $A_2(x,.,y,\xi)$, $A_2(x,u,y,.)$ are decreasing.

Let $\lambda \in (0,1]$. Denote by

$$\begin{aligned} \varepsilon_\lambda = \sup\{r > 0 : rA_1(\tfrac{1}{\lambda}x^0, \lambda x_0, \lambda y_0, \lambda x_0) \leq B_1(\lambda x_0, \tfrac{1}{\lambda}x^0, \tfrac{1}{\lambda}y^0, \tfrac{1}{\lambda}x^0) \text{ and} \\ rA_2(\lambda x_0, \lambda y_0, \tfrac{1}{\lambda}y^0, \lambda y_0) \leq B_2(\tfrac{1}{\lambda}x^0, \tfrac{1}{\lambda}y^0, \lambda y_0, \tfrac{1}{\lambda}y^0)\} \end{aligned}$$

and by

$$\begin{aligned} \phi_{1,\lambda}(r,x,u,y) = r + \varepsilon_\lambda[\phi_1(r,x,u,y) - r], \\ \phi_{2,\lambda}(r,x,v,y) = r + \varepsilon_\lambda[\phi_2(r,x,v,y) - r], \end{aligned}$$

for all $r \in (0,1), x,u \in [\lambda x_0, \frac{1}{\lambda}x^0], v,y \in [\lambda y_0, \frac{1}{\lambda}y^0]$.

It is clear that $0 < \varepsilon_\lambda \leq 1$, and for each $\lambda \in (0,1]$, $r \in (0,1)$

$$\begin{aligned} \inf_{y\in[y_0,y^0],x,u\in[x_0,x^0]} \phi_{1,\lambda}(r,x,u,y) > r, \\ \inf_{x\in[x_0,x^0],v,y\in[y_0,y^0]} \phi_{2,\lambda}(r,x,v,y) > r. \end{aligned}$$

Also, we have

$$\begin{aligned} \varepsilon_\lambda A_1(\frac{1}{\lambda}x^0, \lambda x_0, \lambda y_0, \lambda x_0) \leq B_1(\lambda x_0, \frac{1}{\lambda}x^0, \frac{1}{\lambda}y^0, \frac{1}{\lambda}x^0), \\ \varepsilon_\lambda A_2(\lambda x_0, \lambda y_0, \frac{1}{\lambda}y^0, \lambda y_0) \leq B_2(\frac{1}{\lambda}x^0, \frac{1}{\lambda}y^0, \lambda y_0, \frac{1}{\lambda}y^0). \end{aligned}$$

It follows that for all $y \in [\lambda y_0, \frac{1}{\lambda}y^0], x, u, \xi \in [\lambda x_0, \frac{1}{\lambda}x^0]$

$$\begin{aligned}\varepsilon_\lambda A_1(x,u,y,\xi) \le \varepsilon_\lambda A_1(\frac{1}{\lambda}x^0, \lambda x_0, \lambda y_0, \lambda x_0) &\le B_1(\lambda x_0, \frac{1}{\lambda}x^0, \frac{1}{\lambda}y^0, \frac{1}{\lambda}x^0)\\ &\le B_1(x,u,y,\xi)\end{aligned}$$

and for all $v, y, \nu \in [\lambda y_0, \frac{1}{\lambda}y^0], x \in [\lambda x_0, \frac{1}{\lambda}x^0]$

$$\begin{aligned}\varepsilon_\lambda A_2(x,v,y,\nu) \le \varepsilon_\lambda A_2(\lambda x_0, \lambda y_0, \frac{1}{\lambda}y^0, \lambda y_0) &\le B_2(\frac{1}{\lambda}x^0, \frac{1}{\lambda}y^0, \lambda y_0, \frac{1}{\lambda}y^0)\\ &\le B_2(x,v,y,\nu).\end{aligned}$$

Therefore, for all $r \in (0,1),\ x, u, \xi \in [\lambda x_0, \frac{1}{\lambda}x^0], y \in [\lambda y_0, \frac{1}{\lambda}y^0]$

$$\begin{aligned}A_1(rx, \frac{1}{r}u, \frac{1}{r}y, \xi) &= B_1(rx, \frac{1}{r}u, \frac{1}{r}y, \xi) + D_1(rx)\\ &\ge \phi_1(r,x,u,y)B_1(x,u,y,\xi) + rD_1(x)\\ &= rA_1(x,u,y,\xi) + [\phi_1(r,x,u,y) - r]B_1(x,u,y,\xi)\\ &\ge rA_1(x,u,y,\xi) + \varepsilon_\lambda[\phi_1(r,x,u,y) - r]A_1(x,u,y,\xi)\\ &= \phi_{1,\lambda}(r,x,u,y)A_1(x,u,y,\xi).\end{aligned}$$

Similarly, fo all $r \in (0,1),\ x \in [\lambda x_0, \frac{1}{\lambda}x^0], v, y, \nu \in [\lambda y_0, \frac{1}{\lambda}y^0]$, we obtain

$$A_2(\frac{1}{r}x, \frac{1}{r}v, ry, \nu) \ge \phi_{2,\lambda}(r,x,v,y)A_2(x,v,y,\nu).$$

Now, we prove that for each $i = 1,2$ and $x, u, y \in \overset{\circ}{P}$, there exists a unique point in $\overset{\circ}{P}$, which we denote by $\Phi_i(x,u,y)$, such that

$$A_i(x,u,y,\Phi_i(x,u,y)) = \Phi_i(x,u,y),\ i = 1,2.$$

Fix $x, u, v, y \in \overset{\circ}{P}$.Then, there exists $\lambda \in (0,1]$ such that $x, u \in [\lambda x_0, \frac{1}{\lambda}x^0]$ and $v, y \in [\lambda y_0, \frac{1}{\lambda}y^0]$. Let

$$\Psi_1(x,u,y)(\xi) = \frac{A_1(x,u,y,\xi) + L_1\xi}{1+L_1},\ \Psi_2(x,v,y)(\xi) = \frac{A_2(x,v,y,\xi) + L_2\xi}{1+L_2},\ \xi \in \overset{\circ}{P}.$$

It is clear that $\Psi_1(x,u,y)(.), \Psi_2(x,v,y)(.)$ are operators from $\overset{\circ}{P}$ to $\overset{\circ}{P}$ and by (S_4) they are increasing in $\overset{\circ}{P}$. From (S_1)-(S_3) we have for every $\lambda \in (0,1)$

$$\begin{aligned}A_1(x,u,y,\lambda x_0) &\ge A_1(\lambda x_0, \frac{1}{\lambda}x^0, \frac{1}{\lambda}y^0, x_0)\\ &\ge \phi_1(\lambda, x_0, x^0, y^0)A_1(x_0, x^0, y^0, x_0) \ge \lambda x_0.\end{aligned}$$

And if $\lambda = 1$, we have $A_1(x,u,y,x_0) \ge x_0$. Thus, $A_1(x,u,y,\lambda x_0) \ge \lambda x_0,\ \forall \lambda \in (0,1]$. Similarly, we obtain

$$A_1(x,u,y,\frac{1}{\lambda}x^0) \le \frac{1}{\lambda}x^0.$$

And analogously, one can show that

$$A_2(x,v,y,\lambda y_0) \ge \lambda y_0,\ A_2(x,v,y,\frac{1}{\lambda}y^0) \le \frac{1}{\lambda}y^0.$$

It follows that

$$\begin{aligned}\Psi_1(x,u,y)(\lambda x_0) \geq \lambda x_0,\ \Psi_1(x,u,y)(\frac{1}{\lambda}x^0) \leq \frac{1}{\lambda}x^0,\\ \Psi_2(x,v,y)(\lambda y_0) \geq \lambda y_0,\ \Psi_2(x,v,y)(\frac{1}{\lambda}y^0) \leq \frac{1}{\lambda}y^0.\end{aligned} \tag{3.3}$$

Set

$$X^n_{xuy} = \psi_1(x,u,y)(X^{n-1}_{xuy}),\ U^n_{xuy} = \psi_1(x,u,y)(U^{n-1}_{xuy}),$$

$$X^0_{xuy} = \lambda x_0,\ U^0_{xuy} = \frac{1}{\lambda}x^0,$$

$$Y^n_{xvy} = \psi_2(x,v,y)(Y^{n-1}_{xvy}),\ V^n_{xvy} = \psi_2(x,u,y)(V^{n-1}_{xvy}),$$

$$Y^0_{xvy} = \lambda y_0,\ V^0_{xvy} = \frac{1}{\lambda}y^0.$$

Next, a similar proof to [28, Theorem 2.1] yields that $\Psi_1(x,u,y)(.)$ has a unique fixed point $\Phi_1(x,u,y) \in [\lambda x_0, \frac{1}{\lambda}x^0]$ and $\Psi_2(x,v,y)(.)$ has a unique fixed point $\Phi_2(x,v,y) \in [\lambda y_0, \frac{1}{\lambda}y^0]$, that is

$$A_1(x,u,y,\Phi_1(x,u,y)) = \Phi_1(x,u,y) \text{ and } A_2(x,v,y,\Phi_2(x,v,y)) = \Phi_2(x,v,y),$$

with

$$\begin{aligned}X^n_{xuy} \longrightarrow \Phi_1(x,u,y),\ U^n_{xuy} \longrightarrow \Phi_1(x,u,y)\ \textit{as}\ n \longrightarrow +\infty,\\ Y^n_{xvy} \longrightarrow \Phi_2(x,v,y),\ V^n_{xvy} \longrightarrow \Phi_2(x,v,y)\ \textit{as}\ n \longrightarrow +\infty.\end{aligned}$$

Now, if $\Phi_1'(x,u,y)$ is a fixed point of $\Psi_1(x,u,y)(.)$ in $\overset{\circ}{P}$, then there exists $\beta \in (0,\lambda)$ such that $\Phi_1'(x,u,y) \in [\beta x_0, \frac{1}{\beta}x^0]$. Since, by the above proof, $\psi_1(x,u,y)$ has a unique fixed point in $[\beta x_0, \frac{1}{\beta}x^0]$, it follows that $\Phi_1'(x,u,y) = \Phi_1(x,u,y)$. Also, we have the uniqueness of $\Phi_2(x,v,y)$ in $\overset{\circ}{P}$. Moreover, by a similar proof to step 2 and step 3 of [15, Theorem 2.1], one can show that $\Phi_1(.,u,y)$, $\Phi_2(x,u,.)$ are increasing and $\Phi_1(x,.,y)$, $\Phi_1(x,u,.)$, $\Phi_2(.,u,y)$, $\Phi_1(x,.,y)$ are decreasing. Therefore,

$$\begin{aligned}\Phi_1(\alpha x, \frac{1}{\alpha}u, \frac{1}{\alpha}y) &= A_1(\alpha x, \frac{1}{\alpha}u, \frac{1}{\alpha}y, \Phi_1(\alpha x, \frac{1}{\alpha}u, \frac{1}{\alpha}y))\\ &\geq A_1(\alpha x, \frac{1}{\alpha}u, \frac{1}{\alpha}y, \Phi_1(x,u,y))\\ &\geq \phi_{1,\lambda}(\alpha,x,u,y)\Phi_1(x,u,y),\end{aligned}$$

for all $\alpha \in (0,1),\ x,u \in [\lambda x_0, \frac{1}{\lambda}x^0],\ y \in [\lambda y_0, \frac{1}{\lambda}y^0]$. Also, we obtain

$$\Phi_2(\frac{1}{\alpha}x, \frac{1}{\alpha}u, \alpha y) \geq \phi_{2,\lambda}(\alpha,x,u,y)\Phi_2(x,u,y),$$

for all $\alpha \in (0,1),\ x \in [\lambda x_0, \frac{1}{\lambda}x^0],\ u,y \in [\lambda y_0, \frac{1}{\lambda}y^0]$.

Finally, Φ_1, Φ_2 satisfy all hypotheses of [25, Theorem 2.7]. Thus, the operator $\Phi : \overset{\circ}{P} \times \overset{\circ}{P} \times \overset{\circ}{P} \times \overset{\circ}{P} \longrightarrow \overset{\circ}{P} \times \overset{\circ}{P}$ defined by $\Phi(x,u,v,y) = (\Phi_1(x,u,y), \Phi_2(x,v,y))$ has a unique fixed point $(x^*,y^*) \in [x_0,x^0] \times [y_0,y^0]$; that is

$$\Phi(x^*,x^*,y^*,y^*) = (x^*,y^*).$$

Hence

$$\begin{aligned}
&A(x^*, x^*, y^*, y^*, \Phi_1(x^*, x^*, y^*), \Phi_2(x^*, y^*, y^*)) \\
&= (A_1(x^*, x^*, y^*, \Phi_1(x^*, x^*, y^*)), A_2(x^*, y^*, y^*, \Phi_2(x^*, y^*, y^*))) \\
&= (\Phi_1(x^*, x^*, y^*), \Phi_2(x^*, y^*, y^*)) \\
&= \Phi(x^*, x^*, y^*, y^*) = (x^*, y^*).
\end{aligned}$$

□

4. WEIGHTED PSEUDO ALMOST AUTOMORPHIC SOLUTION

In this section, our goal is to prove the existence and uniqueness of weighted pseudo almost automorphic solution for (1.2). Throughout the rest of this paper, we consider $\rho \in U_\infty$ such that $PAA_0(\mathbb{R}, \rho)$ is translation invariant and that the functions $\tilde{f}$ and $\tilde{g}$ in (1.2) admit the decomposition

$$\tilde{f}(t, x, y) = h_1(t, x) f(t, x, y) \quad \text{and} \quad \tilde{g}(t, x, y) = h_2(t, y) g(t, x, y).$$

As in [14, Lemma 3.2], we have the following lemma.

Lemma 4.1. *Suppose that $f \in WPAA(\mathbb{R} \times \mathbb{R}^+, \rho)$ and one of the following two conditions holds:*

- (i) *$f(t, .)$ is increasing in $\mathbb{R}^+$, and there exists $\varphi : (0, 1) \times (0, +\infty) \to (0, 1]$ such that $\varphi(\alpha, x) > \alpha$ and $f(t, \alpha x) \geq \varphi(\alpha, x) f(t, x)$ for all $x > 0$, $\alpha \in (0, 1)$ and $t \in \mathbb{R}$.*
- (ii) *$f(t, .)$ is decreasing in $\mathbb{R}^+$, and there exists $\psi : (0, 1) \times (0, +\infty) \to (0, 1]$ such that $\psi(\alpha, x) > \alpha$ and $f(t, \frac{1}{\alpha} x) \geq \psi(\alpha, x) f(t, x)$ for all $x > 0$, $\alpha \in (0, 1)$ and $t \in \mathbb{R}$.*

Then, for each $[a, b] \subset (0, +\infty)$, there exists $L > 0$ such that

$$|f(t, u) - f(t, v)| \leq L|u - v|, \ \forall t \in \mathbb{R}, \ \forall u, v \in [a, b].$$

Lemma 4.2. *Suppose that $f \in WPAA(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \rho)$ and one of the following two conditions holds:*

- (i) *$f(t, ., y)$ is increasing in $\mathbb{R}^+$, $f(t, x, .)$ is decreasing in $\mathbb{R}^+$, and there exists $\varphi : (0, 1) \times (0, +\infty) \times (0, +\infty) \to (0, 1]$ such that $\varphi(\alpha, x, y) > \alpha$ and $f(t, \alpha x, \frac{1}{\alpha} y) \geq \varphi(\alpha, x, y) f(t, x, y)$ for all $x, y > 0$, $\alpha \in (0, 1)$ and $t \in \mathbb{R}$.*
- (ii) *$f(t, ., y)$ is decreasing in $\mathbb{R}^+$, $f(t, x, .)$ is increasing in $\mathbb{R}^+$, and there exists $\psi : (0, 1) \times (0, +\infty) \times (0, +\infty) \to (0, 1]$ such that $\psi(\alpha, x, y) > \alpha$ and $f(t, \alpha x, \alpha y) \geq \psi(\alpha, x, y) f(t, x, y)$ for all $x, y > 0$, $\alpha \in (0, 1)$ and $t \in \mathbb{R}$.*

Then, for each $[a, b], [c, d] \subset (0, +\infty)$, there exists $L > 0$ such that

$$\begin{aligned}
|f(t, x_1, y_1) - f(t, x_2, y_2)| &\leq L\|(x_1, y_1) - (x_2, y_2)\| \\
&= L(|x_1 - x_2| + |y_1 - y_2|),
\end{aligned}$$

$\forall t \in \mathbb{R}, \ \forall x_1, x_2 \in [a, b], \ \forall y_1, y_2 \in [c, d].$

Proof. Suppose (i). Let $[a, b], [c, d] \subset (0, +\infty)$. Since $f \in WPAA(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \rho)$, we have

$$L = \frac{\sup_{t \in \mathbb{R}} \sup_{x \in [a,b], y \in [c,d]} f(t, x, y)}{\min(a, c)} < +\infty.$$

Let $x_1, x_2 \in [a,b],\ y_1, y_2 \in [c,d]$. If $x_1 \geq x_2,\ y_1 \leq y_2$, then

$$f(t,x_1,y_1) - f(t,x_2,y_2) \geq 0 \geq -L\|(x_1 - x_2, y_1 - y_2)\|.$$

If $x_1 < x_2,\ y_1 \leq y_2$, then

$$\begin{aligned} f(t,x_1,y_1) - f(t,x_2,y_2) &= f(t, \frac{x_1}{x_2}x_2, y_1) - f(t,x_2,y_2) \\ &\geq f(t, \frac{x_1}{x_2}x_2, \frac{x_2}{x_1}y_1) - f(t,x_2,y_2) \\ &\geq \varphi(\frac{x_1}{x_2}, x_2, y_1) f(t,x_2,y_1) - f(t,x_2,y_2) \\ &\geq \frac{x_1}{x_2} f(t,x_2,y_2) - f(t,x_2,y_2) \\ &= -|x_1 - x_2| \frac{f(t,x_2,y_2)}{x_2} \geq -L|x_1 - x_2| \\ &\geq -L\|(x_1,y_1) - (x_2,y_2)\|. \end{aligned}$$

Similarely if $x_1 \geq x_2,\ y_1 > y_2$. If $x_1 < x_2,\ y_1 > y_2$, then

$$\begin{aligned} f(t,x_1,y_1) - f(t,x_2,y_2) &= f(t, \frac{x_1}{x_2}x_2, \frac{y_1}{y_2}y_2) - f(t,x_2,y_2) \\ &\geq \min(\frac{x_1}{x_2}, \frac{y_2}{y_1}) f(t,x_2,y_2) - f(t,x_2,y_2) \\ &\geq -L\|(x_1,y_1) - (x_2,y_2)\|. \end{aligned}$$

Thus, $f(t,x_1,y_1) - f(t,x_2,y_2) \geq -L\|(x_1 - x_2, y_1 - y_2)\|$ for all $x_1, x_2 \in [a,b], y_1, y_2 \in [c,d]$ and $t \in \mathbb{R}$. Then by changing the role of (x_1,y_1) and (x_2,y_2), we obtain $\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \leq L\|(x_1 - x_2, y_1 - y_2)\|$ for all $x_1, x_2 \in [a,b], y_1, y_2 \in [c,d]$ and $t \in \mathbb{R}$. The proof in the case of f satisfying (ii) is similar. □

Lemma 4.3. *Let $c : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ be such that the function $t \longrightarrow c(t,.)$ is in $WPAA(L^1(\mathbb{R}^+), \rho)$. we assume that there exists $b \in L^1(\mathbb{R}^+)$ such that $|c^{aa}(t,s)| \leq b(s)$ for all $t \in \mathbb{R}$ and almost everywhere in $\mathbb{R}^+$. If $f \in WPAA(\mathbb{R}, \rho)$, then the function*

$$h(t) = \int_{-\infty}^{t} c(t, t-s) f(s) ds$$

is in $WPAA(\mathbb{R}, \rho)$. Furthermore the almost automorphic component of h is given by

$$h^{aa}(t) = \int_{-\infty}^{t} c^{aa}(t, t-s) f^{aa}(s) ds$$

Proof. It is easy to show that $h^{aa} \in AA(\mathbb{R})$. In addition, by a similar proof of [3, Lemma 5.4] and using the fact that $PAA_0(\mathbb{R}, \rho)$ is translation invariant , one can show that $\lim_{r \to +\infty} I(r, \rho) = 0$, where

$$I(r,\rho) = \frac{1}{m(r,\rho)} \int_{-r}^{r} |\int_{-\infty}^{t} (c(t,t-s)f(s) - c^{aa}(t,t-s)f^{aa}(s)) ds| \rho(t) dt.$$

This means that $h \in WPAA(\mathbb{R}, \rho)$ and $h^{aa}(t) = \int_{-\infty}^{t} c^{aa}(t, t-s) f^{aa}(s) ds$. □

Next, we list the following assumptions that we will use them throughout the rest of this paper.

(H_1) $\alpha_1, \alpha_2 \in WPAA(\mathbb{R}, \rho)$ and $h_1,\ h_2,\ k_1,\ k_2 \in WPAA(\mathbb{R}\times\mathbb{R}^+, \rho)$ are nonnegative functions such that $h_1^{aa},\ h_2^{aa},\ k_1^{aa},\ k_2^{aa}$ are uniformly continuous in any bounded subset $K \in \mathbb{R}^+$ uniformly in $s \in \mathbb{R}$. Also, $f, g \in WPAA\left(\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+, \rho\right)$ are nonnegative functions such that f^{aa}, g^{aa} are uniformly continuous in any bounded subset $K \in \mathbb{R}^+\times\mathbb{R}^+$ uniformly in $s \in \mathbb{R}$.

(H_2) For all $s \in \mathbb{R}$, $f(s,.,y), g(s,x,.)$, are increasing and $f(s,x,.)$, $g(s,.,y)$, $h_1(s,.)$, $h_2(s,.)$, $k_1(s,.)$, $k_2(s,.)$ are decreasing. Moreover, there exist constants $L_1, L_2 > 0$ such that for $i = 1, 2$

$$k_i(s,\xi) - k_i(s,\nu) \geq -L_i(\xi-\nu),\ \forall s \in \mathbb{R},\ \forall \xi \geq \nu \geq 0. \tag{4.1}$$

(H_3) There exist $\varphi_1, \varphi_2 : (0,1)\times(0,+\infty) \to (0,1]$, $\psi_1, \psi_2 : (0,1)\times(0,+\infty)\times(0,+\infty) \to (0,1]$ such that

$$h_1(s,\frac{1}{\alpha}x) \geq \varphi_1(\alpha,x)h_1(s,x),\ f(s,\alpha x,\frac{1}{\alpha}y) \geq \psi_1(\alpha,x,y)f(s,x,y),$$
$$h_2(s,\frac{1}{\alpha}x) \geq \varphi_2(\alpha,x)h_2(s,x),\ g(s,\frac{1}{\alpha}x,\alpha y) \geq \psi_2(\alpha,x,y)f(s,x,y),$$

and

$$\varphi_i(\alpha,x) > \alpha,\ \psi_i(\alpha,x,y) > \alpha,\ i = 1,2,$$

$\forall x, y > 0, \forall \alpha \in (0,1), \forall s \in \mathbb{R}$. Moreover, for any $0 < a \leq b < +\infty$ and $0 < c \leq d < +\infty$

$$\inf_{x,u\in[a,b],y\in[c,d]} \varphi_1(\alpha,u)\psi_1(\alpha,x,y) > \alpha,$$
$$\inf_{x\in[a,b],u,y\in[c,d]} \varphi_2(\alpha,u)\psi_2(\alpha,x,y) > \alpha,$$

for every $\alpha \in (0,1)$.

(H_4) c_1, c_2 are functions from $\mathbb{R}\times\mathbb{R}^+$ to $\mathbb{R}^+$ and the function $t \longrightarrow c_i(t,.)$ is in $WPAA(L^1(\mathbb{R}^+), \rho)$ for $i = 1,2$. Moreover, there exist $b_1, b_2 \in L^1(\mathbb{R}^+)$ such that $|c_i^{aa}(t,s)| \leq b_i(s),\ i = 1,2$, for all $t \in \mathbb{R}$ and almost everywhere for $s \in \mathbb{R}^+$.

Now, we are ready to present and prove our results for the existence and uniqueness of positve solution.

Theorem 4.4. *Assume that (H_1)-(H_4) hold. Moreover, for each $\tau > 0$ there exist $\sigma_1, \sigma_2 \in (0,\tau]$ and there exist $\tau_1 \geq \sigma_1,\ \tau_2 \geq \sigma_2$ such that*

(i)

$$\begin{aligned}&\inf_{t\in\mathbb{R}} \int_{-\infty}^{t} c_1(t,t-s)h_1(s,\tau_1)f(s,\sigma_1,\tau_2)ds \geq \sigma_1,\\ &\inf_{t\in\mathbb{R}} \int_{-\infty}^{t} c_2(t,t-s)h_2(s,\tau_2)g(s,\tau_1,\sigma_2)ds \geq \sigma_2.\end{aligned} \tag{4.2}$$

(ii)

$$\begin{aligned}&\sup_{t\in\mathbb{R}}\alpha_1(t)\tau_1 + \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} c_1(t,t-s)h_1(s,\sigma_1)f(s,\tau_1,\sigma_2)ds + k_1(t,\tau_1) \leq \tau_1,\\ &\sup_{t\in\mathbb{R}}\alpha_2(t)\tau_2 + \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} c_2(t,t-s)h_2(s,\sigma_2)g(s,\sigma_1,\tau_2)ds + k_2(t,\tau_2) \leq \tau_2,\end{aligned} \tag{4.3}$$

Then, system (1.2) has exactly one weighted pseudo almost automorphic solution $(x^*, y^*) \in \overset{\circ}{P} \times \overset{\circ}{P}$.

Proof. It consist to prove that all hypotheses of Theorem 3.2 are satisfied for adequate operators A_1 and A_2. Denote by P the following set in the Banach space $WPAA(\mathbb{R}, \rho)$

$$P = \{x \in WPAA(\mathbb{R}, \rho) : x(t) \geq 0,\ \forall t \in \mathbb{R}\}.$$

It is not difficult to verify that P is a normal and solid cone in $WPAA(\mathbb{R}, \rho)$ and

$$\overset{\circ}{P} = \{x \in P : \exists \varepsilon > 0 \text{ such that } x(t) \geq \varepsilon,\ \forall t \in \mathbb{R}\}.$$

Consider the nonlinear operator $A(x, u, v, y, \xi, \nu) = (A_1(x, u, y, \xi), A_2(x, v, y, \nu))$ with

$$A_1(x, u, y, \xi) = B_1(x, u, y, \xi) + D_1(x),\ A_2(x, v, y, \nu) = B_2(x, v, y, \nu) + D_2(y).$$

such that for all $x, u, v, y, \xi, \nu \in \overset{\circ}{P}$ and $t \in \mathbb{R}$

$$B_1(x, u, y, \xi)(t) = \int_{-\infty}^{t} c_1(t, t-s) h_1(s, u(s)) f(s, x(s), y(s)) ds + k_1(t, \xi(t)),$$

$$B_2(x, v, y, \nu)(t) = \int_{-\infty}^{t} c_2(t, t-s) h_2(s, v(s)) f(s, x(s), y(s)) ds + k_2(t, \nu(t)).$$

and

$$D_1(x)(t) = \alpha_1(t) x(t - \beta_1),\ D_2(y)(t) = \alpha_2(t) y(t - \beta_2).$$

By Lemma 2.9, it is easy to show that D_1, D_2 are linear operators from P to P. In addition, it follows from (H_1)-(H_3), Lemma 4.1, Lemma 4.2 and Theorem 2.11 that

$$h_1(., x(.)), h_2(., x(.)), f(., x(.), y(.)), g(., x(.), y(.)) \in WPAA(\mathbb{R}, \rho),\ \forall x, y \in \overset{\circ}{P}.$$

Also, since $k_1(t, .)$ and $k_2(t, .)$ are decreasing in $\mathbb{R}^+$ and satisfying (4.1), one can obtain for $i = 1, 2$

$$|k_i(t, \xi) - k_i(t, \nu)| \leq L_i |\xi - \nu|, \forall t \in \mathbb{R}, \forall \xi, \nu \geq 0,$$

which, with (H_1) and Theorem 2.11 , yields that $k_1(., \xi(.)), k_2(., \xi(.)) \in WPAA(\mathbb{R}, \rho)$, $\forall \xi \in \overset{\circ}{P}$. Combining this with Lemma 2.6, Lemma 2.9, Lemma 4.3 and (H_4) we obtain $A_1(x, u, y, \xi)$, $A_2(x, u, y, \xi) \in WPAA(\mathbb{R}, \rho)$ for all $x, u, y, \xi \in \overset{\circ}{P}$.

Next, we prove that $B_1, B_2 : \overset{\circ}{P} \times \overset{\circ}{P} \times \overset{\circ}{P} \times \overset{\circ}{P} \longrightarrow \overset{\circ}{P}$. Let $x, u, y, \xi \in \overset{\circ}{P}$. Denote

$$\varepsilon = \inf_{t \in \mathbb{R}} x(t) \text{ and } \tau = \max\{\sup_{t \in \mathbb{R}} u(t), \sup_{t \in \mathbb{R}} y(t)\}.$$

Then,

$$\begin{aligned} \inf_{t \in \mathbb{R}} B_1(x, u, y, \xi)(t) &\geq \inf_{t \in \mathbb{R}} \int_{-\infty}^{t} c_1(t, t-s) h_1(s, u(s)) f(s, x(s), y(s)) ds \\ &\geq \inf_{t \in \mathbb{R}} \int_{-\infty}^{t} c_1(t, t-s) h_1(s, \tau) f(s, \varepsilon, \tau) ds. \end{aligned}$$

By (4.2), there exist $\sigma_1, \sigma_2 \in (0, \tau]$ and there exist $\tau_1 \geq \sigma_1,\ \tau_2 \geq \sigma_2$ such that

$$\inf_{t \in \mathbb{R}} \int_{-\infty}^{t} c_1(t, t-s) h_1(s, \tau_1) f(s, \sigma_1, \tau_2) ds \geq \sigma_1.$$

Suppose that $\varepsilon < \sigma_1,\ \tau > \tau_1,\ \tau > \tau_2$ (the other cases are similar and easier to prove), then

$$\begin{aligned}\inf_{t\in\mathbb{R}} B_1(x,u,y,\xi)(t) &\geq \inf_{t\in\mathbb{R}}\int_{-\infty}^{t} c_1(t,t-s)h_1(s,\frac{\tau}{\tau_1}\tau_1)f(s,\frac{\varepsilon}{\sigma_1}\sigma_1,\frac{\tau}{\tau_2}\tau_2)ds\\ &\geq \frac{\tau_1}{\tau}\min(\frac{\varepsilon}{\sigma_1},\frac{\tau_2}{\tau})\inf_{t\in\mathbb{R}}\int_{-\infty}^{t} c_1(t,t-s)h_1(s,\tau_1)f(s,\sigma_1,\tau_2)ds\\ &\geq \frac{\tau_1}{\tau}\min(\frac{\varepsilon}{\sigma_1},\frac{\tau_2}{\tau})\sigma_1 > 0.\end{aligned}$$

Thus, $B_1(x,u,y,\xi) \in \overset{\circ}{P}$ and hence $A_1(x,u,y,\xi) \in \overset{\circ}{P}$. Analogously, one can show that $A_2(x,u,y,\xi) \in \overset{\circ}{P}$ for all $x,u,y,\xi \in \overset{\circ}{P}$.

Now, let us prove (S_1)-(S_4) of Theorem 3.2. It is easy to see that (S_1) and (S_4) follow from (H_2). To prove (S_2), suppose $x,u,y,\xi \in \overset{\circ}{P}$ and $\alpha \in (0,1)$. Set

$$\begin{aligned}a(x,u,y) &= \min\{\inf_{s\in\mathbb{R}} x(s), \inf_{s\in\mathbb{R}} u(s), \inf_{s\in\mathbb{R}} y(s)\},\\ b(x,u,y) &= \max\{\sup_{s\in\mathbb{R}} x(s), \sup_{s\in\mathbb{R}} u(s), \sup_{s\in\mathbb{R}} y(s)\}.\end{aligned}$$

Then, $0 < a(x,u,y) \leq b(x,u,y) < +\infty$ and $x(s),u(s),y(s) \in [a(x,u,y),b(x,u,y)]$ for all $s \in \mathbb{R}$. We difine

$$\phi_i(\alpha,x,u,y) = \inf_{\beta,\gamma,\eta\in[a(x,u,y),b(x,u,y)]} \varphi_i(\alpha,\gamma)\psi_i(\alpha,\beta,\eta), \quad i = 1,2.$$

By (H_3), we have

$$\begin{aligned}B_1(\alpha x,\frac{1}{\alpha}u,\frac{1}{\alpha}y,\xi)(t) &= \int_{-\infty}^{t} h_1(s,\frac{1}{\alpha}u(s))f(s,\alpha x(s),\frac{1}{\alpha}y(s))ds + k_1(t,\xi(t))\\ &\geq \phi_1(\alpha,x,u,y)\int_{-\infty}^{t} h_1(s,u(s))f(s,x(s),y(s))ds + k_1(t,\xi(t)),\end{aligned}$$

which means that

$$B_1(\alpha x,\frac{1}{\alpha}u,\frac{1}{\alpha}y,\xi) \geq \phi_1(\alpha,x,u,y)B_1(x,u,y,\xi)$$

for each $x,u,y,\xi \in \overset{\circ}{P}$ and $\alpha \in (0,1)$. Similarly, we obtain

$$B_2(\frac{1}{\alpha}x,\frac{1}{\alpha}u,\alpha y,\xi) \geq \phi_2(\alpha,x,u,y)B_2(x,u,y,\xi)$$

for each $x,u,y,\xi \in \overset{\circ}{P}$ and $\alpha \in (0,1)$. Finally, (S_3) follows from (4.2), (4.3) and (H_3). The proof is completed. □

In the following corollary, we give a concrete way to obtain the constants $\tau_1,\tau_2,\sigma_1,\sigma_2$ of the previous theorem. First, let us introduce some notations. We set uniformly in $t \in \mathbb{R}$ and $p, q \in [0,1]$:

$$\begin{aligned}
\liminf_{(x,y)\longrightarrow(0^+,+\infty)} \frac{f(t,x,y)}{x^p} &= f_{p,(0^+,+\infty)}(t), \quad \liminf_{u\longrightarrow+\infty} \frac{h_1(t,u)}{u^q} = h_{1,q,+\infty}(t),\\
\limsup_{(x,y)\longrightarrow(+\infty,0^+)} \frac{f(t,x,y)}{x^p} &= f^{p,(+\infty,0^+)}(t), \quad \limsup_{u\longrightarrow0^+} \frac{h_1(t,u)}{u^q} = h_1^{q,0^+}(t),\\
\liminf_{(x,y)\longrightarrow(+\infty,0^+)} \frac{g(t,x,y)}{y^p} &= g_{p,(+\infty,0^+)}(t), \quad \liminf_{u\longrightarrow+\infty} \frac{h_2(t,u)}{u^q} = h_{2,q,+\infty}(t),\\
\limsup_{(x,y)\longrightarrow(0^+,+\infty)} \frac{g(t,x,y)}{y^p} &= g^{p,(0^+,+\infty)}(t), \quad \limsup_{u\longrightarrow0^+} \frac{h_2(t,u)}{u^q} = h_2^{q,0^+}(t).
\end{aligned}$$

Corollary 4.5. *Assume that (H_1)-(H_4) hold. Moreover there exist $p, q \in [0,1]$ such that the following conditions hold:*

(i)$'$

$$\begin{aligned}
&\inf_{t\in\mathbb{R}} \int_{-\infty}^{t} c_1(t,t-s)h_{1,q,+\infty}(s)f_{p,(0^+,+\infty)}(s)ds > 1,\\
&\inf_{t\in\mathbb{R}} \int_{-\infty}^{t} c_2(t,t-s)h_{2,q,+\infty}(s)g_{p,(+\infty,0^+)}(s)ds > 1.
\end{aligned} \tag{4.4}$$

(ii)$'$

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} c_1(t,t-s)h_1^{q,0^+}(s)f^{p,(+\infty,0^+)}(s)ds < 1,\\
&\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} c_2(t,t-s)h_2^{q,0^+}(s)g^{p,(0^+,+\infty)}(s)ds < 1,\\
&\qquad \overline{\alpha}_i = \sup_{t\in\mathbb{R}} \alpha_i(t) < 1, \ i = 1,2.
\end{aligned} \tag{4.5}$$

Then system (1.2) has exactly one weighted pseudo almost automorphic solution $(x^*, y^*) \in \overset{\circ}{P} \times \overset{\circ}{P}$.

Proof. We prove that hypotheses (i) and (ii) of Theorem 4.4 are satisfied.

From (i)$'$ and (ii)$'$, there exist $\varepsilon > 0$ verifying

$$\begin{aligned}
&\inf_{t\in\mathbb{R}} \int_{-\infty}^{t} c_1(t,t-s)(h_{1,q,,+\infty}(s)-\varepsilon)(f_{p,(0^+,+\infty)}(s)-\varepsilon)ds > 1,\\
&\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} c_1(t,t-s)(h_1^{q,0^+}(s)+\varepsilon)(f^{p,(+\infty,0^+)}(s)+\varepsilon)ds < 1.
\end{aligned}$$

It follows that there exist numbers δ, M with $0 < \delta < 1 < M$ such that

$$\begin{aligned}
&h(s,u) \geq (h_{1,q,+\infty}(s)-\varepsilon)u^q, \ \forall u \geq M, \ \forall s \in \mathbb{R},\\
&f(s,x,y) \geq (f_{p,(0^+,+\infty)}(s)-\varepsilon)x^p, \ \forall x \leq \delta, \ \forall y \geq M, \ \forall s \in \mathbb{R}
\end{aligned}$$

and

$$\begin{aligned}
&h(s,u) \leq (h_1^{q,0^+}(s)+\varepsilon)u^q, \ \forall u \leq \delta, \ \forall s \in \mathbb{R},\\
&f(s,x,y) \leq (f^{p,(+\infty,0^+)}(s)+\varepsilon)x^p, \ \forall x \geq M, \ \forall y \leq \delta, \ \forall s \in \mathbb{R}.
\end{aligned}$$

Let $\tau_1, \tau_2 \geq M$, $\sigma_1, \sigma_2 \in (0, \delta]$ with $\sigma_1^q < 1 - \overline{\alpha}_1$ and $\sigma_2^q < 1 - \overline{\alpha}_2$. Then,

$$\begin{aligned}
&\inf_{t\in\mathbb{R}} \int_{-\infty}^{t} c_1(t, t-s) h_1(s, \tau_1) f(s, \sigma_1, \tau_2) ds \\
&\geq \inf_{t\in\mathbb{R}} \int_{-\infty}^{t} c_1(t, t-s)(h_{1,q,+\infty}(s) - \varepsilon)\tau_1^q (f_{p,(0^+,+\infty)}(s) - \varepsilon)\sigma_1^p ds \\
&\geq \tau_1^q \sigma_1^p \geq \sigma_1.
\end{aligned}$$

This prove the first inequalitie of (4.2). We follow the same reasoning to get the second inequalitie. On the other hand,

$$\begin{aligned}
&\overline{\alpha}_1 \tau_1 + \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} c_1(t, t-s) h_1(s, \sigma_1) f(s, \tau_1, \sigma_2) ds + k_1(t, \tau_1) \\
&\leq \overline{\alpha}_1 \tau_1 + \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} c_1(t, t-s)(h_1^{q,0^+}(s) + \varepsilon)\sigma_1^q (f^{p,(+\infty,0^+)}(s) + \varepsilon)\tau_1^p ds + k_1(t, 0) \\
&\leq \overline{\alpha}_1 \tau_1 + \sigma_1^q \tau_1^p + k_1(t, 0) \leq (\overline{\alpha}_1 + \sigma_1^q)\tau_1 + k_1(t, 0).
\end{aligned}$$

Since $k_1(., 0)$ is bounded, we can choose a sufficiently large constant τ_1 such that $(\overline{\alpha}_1 + \sigma_1^q)\tau_1 + k_1(t, 0) \leq \tau_1$. This prove the first inequalitie of (4.3). To prove the second inequalitie, one can follows the same reasoning. The proof is ended. □

5. Example

Example 5.1. *Consider system (1.2) by setting, for all $(t, x, y) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$ and $\alpha \in (0, 1)$*

$$\rho(t) = e^t,\ \alpha_1(t) = \alpha_2(t) \equiv \frac{1}{3},\ \beta_1 = \beta_2 = 1,$$

$$\begin{aligned}
f(t, x, y) &= \{1 + \cos^2 \frac{1}{2 + \sin t + \sin \sqrt{2}t} + e^{-t}\} \sqrt[3]{\frac{(y+3)\ln(x+1)}{y+1}} \\
g(t, x, y) &= \{1 + \cos^2 \frac{1}{2 + \sin t + \sin \sqrt{2}t} + e^{-t}\} \sqrt[3]{\frac{(x+3)\ln(y+1)}{x+1}} \\
c_1(t, s) &= c_2(t, s) = \frac{1}{1+s^2},\ h_1(t, x) = h_2(t, x) = \sqrt[3]{\frac{x+3}{x+1}} \ \textit{and} \\
k_1(t, x) &= k_2(t, x) = \frac{1 + \sin t}{1 + x}.
\end{aligned}$$

Take $p = \frac{1}{2}, q = 0$, $L_1 = L_2 = 1$ and define

$$\begin{aligned}
\varphi_1(\alpha, x) &= \sqrt[3]{\frac{(x+1)(x+3\alpha)}{(x+\alpha)(x+3)}},\ \psi_1(\alpha, x, y) = \sqrt[3]{\frac{(y+1)(y+3\alpha)\ln(\alpha x+1)}{(y+\alpha)(y+3)\ln(x+1)}}, \\
\varphi_2(\alpha, y) &= \sqrt[3]{\frac{(y+1)(y+3\alpha)}{(y+\alpha)(y+3)}} \ \textit{and}\ \psi_2(\alpha, x, y) = \sqrt[3]{\frac{(x+1)(x+3\alpha)\ln(\alpha y+1)}{(x+\alpha)(x+3)\ln(y+1)}}.
\end{aligned}$$

It is easy to verify that for $i = 1, 2$

$$\varphi_i(\alpha, x) > \alpha^{\frac{1}{3}},\ \psi_i(\alpha, x, y) > \alpha^{\frac{2}{3}},\ h_{i,0,+\infty}(t) = 1,\ h_i^{0,0^+}(t) = \sqrt[3]{3} \text{ and}$$
$$f_{\frac{1}{2},(0^+,+\infty)}(t) = g_{\frac{1}{2},(+\infty,0^+)}(t) = 1 + \cos^2 \frac{1}{2 + \sin t + \sin \sqrt{2}t} + e^{-t},$$
$$f^{\frac{1}{2},(+\infty,0^+)}(t) = g^{\frac{1}{2},(0^+,+\infty)}(t) = 0.$$

Then,we have all hypotheses of corollary 4.5 are verified. Hence, system (1.2) with the above functions has a unique positive weighted pseudo almost automorphic solution.

References

[1] E. Ait Dads, K. Ezzinbi; *Existence of positive pseudo almost periodic solution for some nonlinear infinite delay integral equations arising in epidemic problems*, Nonlinear Analysis. Theory, Methods and Applications 41 (2000) 1-13.

[2] E. Ait Dads, P. Cieutat, L. Lhachimi; *Positive almost automorphic solutions for some nonlinear infinite delay integral equations*, Dynamic Systems and Applications 17 (2008) 515-538.

[3] E. Ait Dads, P. Cieutat, L. Lhachimi; *Positive pseudo almost periodic solutions for some nonlinear infinite delay integral equations*, Mathematical and Computer Modelling 49 (2009) 721-739.

[4] R.P. Agarwal, T. Diagana, M.E. Hernandez; *Weighted pseudo almost periodic solutions to some partial neutral functional differential equations*, J. Nonlinear Convex Anal. 8 (3) (2007), 397-415.

[5] B. Ammar, F. Chérif and M.A. Alimi; *Existence and Uniqueness of pseudo almost Periodic Solutions of Recurrent Neural Networks with Time-Varying Coefficients and Mixed Delays*, IEEE Transactions on Neural Networks and Learning Systems, Vol. 23,(2012), Issue. 1, 109-118.

[6] S. Bochner; *A new approach to almost-periodicity*, Proc. Nat. Acad. Sci. USA 48 (1962), 2039-2043.

[7] S. Bochner; *Continuous mappings of almost automorphic and almost periodic functions*, Proc. Ncad. Sci. USA 52 (1964) 907-910.

[8] J. Blot, G.M. Mophou, G.M. N'Guérékata and D. Pennequin; *Weighted pseudo-almost automorphic functions and applications*, Nonlinear Anal. 71 (2009) 903-909.

[9] A. Cañada and A. Zertiti; *Fixed point theorems for systems of equations in ordered Banach spaces with applications to differential and integral equations*, Nonlinear Analysis, T.M.A.,Vol.27, N° 4 (1996), 397-411.

[10] A. Cañada, A. Zertiti; *Systems of nonlinear delay integral equations modelling population growth in a periodic environment*, comment. Math. univ. Carolinae 35, 4 (1994), 633 - 644.

[11] F. Chérif; *Existence and global exponential stability of pseudo almost periodic solution for SICNNs with mixed delays*, J Appl Math Comput, Volume 39, Numbers 1-2 (2012), 235-251.

[12] K. Deimling; *Nonlinear Functionl Analysis*, Springer-verlay, New York, (1985).

[13] T. Diagana, Weighted pseudo almost periodic functions and applications, C. R. Acad. Sci. Paris, Ser. I., 343(10)(2006), 643-646.

[14] H. S. Ding, Y. Y. Chen, G. M. N'Guérékata; *Existence of positive pseudo almost periodic solutions to a class of neutral integral equations*,Nonlinear Analysis 74 (2011) 7356-7364.

[15] H.S. Ding, J. Liang, T.J. Xiao; *Fixed point theorems for nonlinear operators with and without monotonicity in partially ordered Banach spaces*, Fixed Point Theory and Applications 2010 (2010) Article ID 108343, 11 pages.

[16] H. S. Ding, J. Liang, G. M. N'Guérékata, T. J. Xiao; *Existence of positive almost automorphic solutions to neutral nonlinear integral equations* , Nonlinear Analysis 69 (2008), 1188-1199.

[17] H. S. Ding, W. Long; *Positive almost automorphic solutions for some nonlinear delay integral equations*, Electronic Journal of Diff. Equations, N° 57 (2008), 1-8.

[18] V. Kavitha, P. Z. Wang, R. Murugesu; *Existence of weighted pseudo almost automorphic mild solution to fractional integro-differential equations*, Journal of fractional calculus and applications, vol. 4(1) (2013), pp. 37-55.

[19] J. Liang, T.J. Xiao and J. Zhang; Decomposition of weighted pseudo almost periodic functions, Nonlinear Analysis: Theory, Methods and Applications, 73 (2010), 3456-3461.
[20] G. M. Mophou; *Weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations, Applied Mathematics and Computation*, 217 (2011) 7579-7587.
[21] G. M. N'Guérékata; *Almost automorphic functions and almost periodic functions in abstract spaces*, Kluwer Academic/Plnum Publishers, New York-Berlin-Moscow, (2001).
[22] G. M. N'Guérékata; *Topics in almost automorphy*, Springer-Verlag, New York, (2005).
[23] A. Sadrati, A. Zertiti; *A study of systems of nonlinear delay integral equations by using the method of upper and lower solutions*, International Journal of Math. and Computation, Vol. 17, N°4 (2012), 93-102
[24] A. Sadrati, A. Zertiti; *A topological methods for Existence and multiplicity of positive solutions for some systems of nonlinear delay integral equations* , International Journal of Mathematics and Statistics, Vol. 13, N° 1 (2013), 47-55.
[25] A. Sadrati, A. Zertiti; *Existence and uniqueness of positive almost periodic solution for systems of nonlinear delay ntegral equations*, Electronic Journal of Diff. Equations, Vol. 2015 (2015), No. 116, pp. 1-12.
[26] Syed Abbas; *Weighted pseudo almost automorphic solutions of fractional functional dfferential equations*, CUBO A Mathematical Journal, Vol.16, N°. 01 (2014), 21-35.
[27] A. Zertiti; *Method of upper and lower solutions applied to a nonlinear delay integral system*, international journal of Mathematics and statistics, spring (2008), vol.2, 508.
[28] Z. Zhang and K. Wang; *On fixed point theorems of mixed monotone operators and applications*, Nonlinear Analysis: Theory, Methods and Applications, vol. 70, no. 9 (2009), 3279-3284.

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