SPECIAL WEAKLY RICCI SYMMETRIC LIGHTLIKE HYPERSURFACES IN INDEFINITE KENMOTSU SPACE FORMS

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Abstract

In this paper, we study weakly ϕ-Ricci symmetric and special weakly Ricci symmetric lightlike hypersurfaces of indefinite Kenmotsu space form, tangent to the structure vector field. We obtain sufficient condition for a weakly ϕ-Ricci symmetric lightlike hypersurface to be η-Einstein in indefinite Kenmotsu space form. Later, we give some results. On the other hand, we obtain sufficient conditions for a lightlike hypersurface to be a special weakly Ricci symmetric (SWRS) lightlike hypersurface in indefinite Kenmotsu space form and we show that a special weakly Ricci symmetric (SWRS) lightlike hypersurface is totally geodesic under certain a condition.

Keywords: Weakly ϕ-Ricci symmetric, Special Weakly Ricci symmetric, Indefinite Kenmotsu space form, Lightlike hypersurface .

1. **Introduction**

Notion of ϕ-symmetric was studied by many authors. For example, T.Takahashi introduced the notion of locally ϕ-symmetric Sasakian manifolds as a weaker notion of locally symmetric manifolds [13]. Also, U.C.De studied ϕ-symmetric Kenmotsu manifolds with several examples [1]. Later, U.C.De and A. Sarkar introduced the notion of ϕ-Ricci symmetric Sasakian manifolds and obtained some interesting results of this manifold [2]. S.S. Shukla and M.K. Shukla studied this notion of ϕ-Ricci symmetric in the context of Kenmotsu manifolds [11]. Moreover, as a generalization of Chaki's pseudosymmetric and pseudo Ricci symmetric manifolds, the notion of weakly symmetric and weakly Ricci symmetric manifolds were introduced by L.Tamássy and T.Q. Binh ([15] and [16]). Therefore, a non-flat semi-Riemannian manifold $\bar{M}$ is called weakly Ricci symmetric if the Ricci tensor $\bar{Ric}$ is non-zero

2010 *Mathematics Subject Classification.* 53C15, 53C25, 53C50.

and satisfies the following condition, for any vector fields $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ in $\bar{M}$,

$$(\bar{\nabla}_{\bar{X}}\bar{Ric})(\bar{Y},\bar{Z}) = \bar{\alpha}(\bar{X})\bar{Ric}(\bar{Y},\bar{Z}) + \bar{\beta}(\bar{Y})\bar{Ric}(\bar{X},\bar{Z}) + \bar{\gamma}(\bar{Z})\bar{Ric}(\bar{Y},\bar{X}), \tag{1.1}$$

where $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$ defined respectively by, $\bar{g} = (\bar{X},\bar{\rho}) = \bar{\alpha}(\bar{X})$, $\bar{g} = (\bar{X},\bar{\delta}) = \bar{\beta}(\bar{X})$, $\bar{g} = (\bar{X},\bar{\kappa}) = \bar{\gamma}(\bar{X})$, are 1-forms called the associated 1-forms which sre not zero simultaneously and $\bar{\nabla}$ is the Levi-Civita connection for a semi-Riemannian metric $\bar{g}$. In such case, $\bar{\rho}$, $\bar{\delta}$ and $\bar{\kappa}$ are called associated vector fields corresponding to the 1-forms $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$respectively [7]. In [9], C. Özgür studied weakly symmetric Kenmotsu manifolds. If, in (1.1), the 1-form $\bar{\alpha}$ is replaced by $2\bar{\alpha}$ and $\bar{\beta}$ and $\bar{\gamma}$ are equal to $\bar{\alpha}$, then the semi-Riemannian manifold is called a special weakly Ricci symmetric [7] and investigated by H. Singh and Q. Khan in [6]. Also, in [10], D.G. Prakasha, S.K. Hui and K. Vikas introduced notion weakly ϕ-Ricci symmetric of Kenmotsu manifold. Then, a Kenmotsu manifold M $(n > 2)$ is said to be weakly ϕ-Ricci symmetric if the non-zero Ricci curvature Q of type $(1,1)$ satisfies the condition

$$\phi^2(\nabla_X Q)(Y) = A(X)Q(Y) + B(Y)Q(X) + g(QX,Y)\rho, \tag{1.2}$$

where the vector fields X and Y on M, ρ is a vector field such that $g(\rho,X) = D(X)$, A and B are associated vector fields (not simultaneously zero)[10]. Also, a weakly ϕ-Ricci symmetric Kenmotsu manifold M $(n > 2)$ is said to be locally ϕ-Ricci symmetric, if [10]

$$\phi^2(\nabla Q) = 0. \tag{1.3}$$

On the other hand, lightlike hypersurfaces of a semi-Riemannian manifold have been studied by Duggal-Bejancu and they obtain a transversal bundle for such hypersurfaces to overcome anomaly occured due to degenerate metric. After their book [3], many authors studied lightlike hypersurfaces by using their approach. In [12], Şahin and Yıldız have introduced Chaki type pseudo-symmetric lightlike hypersurfaces of semi-Euclidean space and obtained many results. And, in [7], Massamba has introduced weakly Ricci symmetric lightlike hypersurfaces of indefinite Sasakian manifolds. Therefore,

In this paper, we investigates the effect of weakly ϕ-Ricci symmetric condition on the lightlike geometry of hypersurfaces of an indefinite Kenmotsu space form. In Section 3, we give some general notions about lightlike hypersurfaces of indefinite Kenmotsu manifolds. Later, we study weakly ϕ-Ricci symmetric lightlike hypersurfaces of indefinite Kenmotsu space forms. We find sufficient condition for a weakly ϕ-Ricci symmetric lightlike Einstein hypersurface to be locally ϕ-Ricci symmetric. In Section 4, we study special weakly Ricci symmetric manifold (SWRS) lightlike hypersurfaces.

2. **Preliminaries**

In this section, let us recall some general notions about indefinite Kenmotsu space manifolds:

Let $\bar{M}$ be a (2m+1)-dimensional manifold endowed with an *almost contact structure* $(\bar{\phi}, \xi, \eta)$, i.e. $\bar{\phi}$ is a tensor field of type (1,1), ξ is a vector field and η is a 1-form satisfying

$$\bar{\phi}^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1,\ \eta \circ \phi = 0 \text{ and } \phi\xi = 0. \tag{2.1}$$

Then $(\bar{\phi}, \xi, \eta, \bar{g})$ is called an *indefinite almost contact metric structure* on $\bar{M}$, if $(\bar{\phi}, \xi, \eta)$ is an almost contact structure on $\bar{M}$ and $\bar{g}$ is a semi-Riemannian metric on $\bar{M}$ such that, for any vector field $\bar{X}, \bar{Y}$ on $\bar{M}$,

$$\bar{g}(\bar{\phi}\bar{X}, \bar{\phi}\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) - \eta(\bar{X})\eta(\bar{Y}). \tag{2.2}$$

If follows that, for any vector field $\bar{X}$ on $\bar{M}$, $\eta(\bar{X}) = \bar{g}(\xi, \bar{X})$. If, moreover, $(\bar{\nabla}_{\bar{X}}\bar{\phi})\bar{Y} = \bar{g}(\bar{\phi}\bar{X}, \bar{Y})\xi - \eta(\bar{Y})\bar{\phi}\bar{X}$, where $\bar{\nabla}$ is the Levi-Civita connection for the semi-Riemannian metric $\bar{g}$, we call $\bar{M}$ an *indefinite Kenmotsu manifold*[8].

Since Takahashi [14] shows that it suffices to consider indefinite almost contact manifolds with space-like ξ [5]. In this paper, we will restrict ourselves to the case of ξ a space-like unit vector (that is $\bar{g}(\xi, \xi) = 1$).

A plane section σ in $T_p\bar{M}$ is called a *$\bar{\phi}$-section* if it is spanned by $\bar{X}$ and $\bar{\phi}\bar{X}$, where $\bar{X}$ is a unit tangent vector field orthogonal to ξ. The sectional curvature of a $\bar{\phi}$-section σ is called a *$\bar{\phi}$-sectional curvature*. A Kenmotsu manifold $\bar{M}$ of constant $\bar{\phi}$-sectional curvature c will be called *Kenmotsu space form* and denote by $\bar{M}(c)$. If an indefinite Kenmotsu manifold $\bar{M}$ has a constant $\bar{\phi}$-sectional curvature c, then, $\bar{M}$ is an *Einstein* one and $c = -1$. This means that, it is locally isometric to the pseudo hyperbolic space $H_s^{2n+1}(-1)$, s being the index of its metric [8].

Also, let us recall some general notions about lightlike hypersurfaces:

Theorem 2.1. (Duggal-Bejancu) Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\bar{M}, \bar{g})$. Then there exist a unique vector bundle $tr(TM)$ of rank 1 over M such that for any non-zero section ξ of $T^{\perp}M$ on a coordinate neighborhood $U \subset M$, there exists a unique section N of $tr(TM)$ on U

$$\bar{g}(\xi, N) = 1, \bar{g}(N, N) = \bar{g}(N, X) = 0, \forall X \in \Gamma(S(TM|_U)). \tag{2.3}$$

It follows from (2.3) that $tr(TM)$ is a lightlike vector bundle such that $tr(TM)_x \cap T_xM = \{0\}$ for any $x \in M$. Thus, from Theorem(2.1), we have

$$T\bar{M}|_M = S(TM) \oplus (TM^{\perp} \oplus tr(TM)) \tag{2.4}$$

$$= TM \oplus tr(TM). \tag{2.5}$$

Here, the complementary (non-orthogonal) vector bundle $tr(TM)$ to the tangent bundle TM in $T\bar{M}|_M$ is called *the lightlike transversal bundle* of M with respect to screen distribution $S(TM)$ [3].

Suppose ∇ and $\bar{\nabla}$ are the Levi-Civita connections of M lightlike hypersurface and $\bar{M}$ semi-Riemannian manifold, respectively. According to the (2.5), we have

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \text{ and } \bar{\nabla}_X N = -A_N X + \nabla^t_X N, \tag{2.6}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(tr(TM))$, where $\nabla_X Y, A_N X \in \Gamma(TM)$ and $h(X,Y), \nabla^t_X N \in \Gamma(tr(TM))$. If we set $B(X,Y) = g(h(X,Y),\xi)$ and $\tau(X) = \bar{g}(\nabla^t_X N, \xi)$, then, from (2.6), we have

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N \text{ and } \bar{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(tr(TM))$, A_N and $\mathcal{B}$ are called *the shape operator* and *the second fundamental form* of the lightlike hypersurface M, respectively.

Let P be the projection of $\Gamma(TM)$ on $\Gamma(S(TM))$. Then, we have

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi \text{ and } \nabla_X \xi = -A^*_\xi X + \tau(X)\xi, \tag{2.8}$$

for any $X, Y \in \Gamma(TM)$, where $\nabla^*_X PY, A^*_\xi X \in \Gamma(S(TM))$ and C is a 1-form on U defined by $C(X,PY) = \bar{g}(\nabla_X PY, N)$. $C, A^*_\xi X$ and ∇^* are called *the local second fundamental form*, *the local shape operator* and *the induced connection* on $S(TM)$, respectively. Then, we have the following assertions,

$$g(A_N Y, PW) = C(Y,PW), g(A_N Y, N) = 0, B(X,\xi) = 0, \tag{2.9}$$

$$g(A^*_\xi X, PY) = B(X,PY), g(A^*_\xi X, N) = 0, \tag{2.10}$$

for $X, Y, W \in \Gamma(TM), \xi \in \Gamma(TM^{\perp})$ and $N \in \Gamma(tr(TM))$.

A connection ∇ on a null hypersurface (M, g) is said to be η-conformal if the covariant derivative of g is proportional to $g - \eta \otimes \eta$, that is, there exists a differential 1-form β such that the following

$\nabla g = \beta \otimes g\eta \otimes \eta$, holds. If in addition, ∇ is torsion-free, it is said to be Weyl-connection in the direction of the distribution Ker(η). But on M, such a connection will be called η-Weyl connection [8].

Theorem 2.2. (Massamba) Let $(M, g, S(TM))$ be a null hypersurface of an indefinite Kenmotsu space form $(M(c), g)$ with $\xi \in TM$. Then, the induced connection is an η-Weyl connection if and only if M is totally geodesic. Moreover, the induced connection on a proper totally contact umbilical null hypersurface is never an η-Weyl connection [8].

For the geometry of lightlike hypersurfaces, we refer to [3], [4].

3. Weakly ϕ-Ricci Symmetric Lightlike Hypersurfaces in Indefinite Kenmotsu Space Forms

In this section, we investigate weakly ϕ-Ricci symmetric lightlike hypersurfaces in an indefinite Kenmotsu space form. Firstly, let us recall some general notions about lightlike hypersurfaces of indefinite Kenmotsu manifolds [8]:

Let$(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ be an indefinite Kenmotsu manifold and let (M, g) be a lightlike hypersurface of $(\bar{M}, \bar{g})$, tangent to the structure vector field $\xi \in \Gamma(TM)$. If E is a local section of $TM^{\perp}$, it is easy to check that $\bar{\phi}E \neq 0$ and $\bar{g}(\bar{\phi}E, E) = 0$, then $\bar{\phi}E$ is tangent to M. Thus $\bar{\phi}(TM^{\perp})$ is a distribution on M of rank 1 such that $\bar{\phi}(TM^{\perp}) \cap TM^{\perp} = \{0\}$. This enables us to choose a screen distribution $S(TM)$ such that it contains $\bar{\phi}(TM^{\perp})$ as a vector subbundle. If we consider a local section N of $N(TM)$, we have $\bar{\phi}N \neq 0$. Since $\bar{g}(\bar{\phi}N, E) = -\bar{g}(N, \bar{\phi}E) = 0$, we deduce that $\bar{\phi}E \in \Gamma(S(TM))$ and $\bar{\phi}N$ is also tangent to M. At the same time, $\bar{g}(\bar{\phi}N, N) = 0$ i.e. $\bar{\phi}N$ has no component with respect to E. Thus $\bar{\phi}N \in \Gamma(S(TM))$, that is, $\bar{\phi}(N(TM))$ is also a vector subbundle of $S(TM)$ of rank 1. From (2.1), we have $\bar{g}(\bar{\phi}N, \bar{\phi}E) = 1$. Therefore, $\bar{\phi}(TM^{\perp}) \oplus \bar{\phi}(tr(TM))$ is a non-degenerate vector subbundle of $S(TM)$ of rank 2. If $\xi \in TM$, we may choose $S(TM)$ so that ξ belongs to $S(TM)$. Using this and since $\bar{g}(\bar{\phi}E, \xi) = \bar{g}(\bar{\phi}N, \xi) = 0$, there exists a non-degenerate distribution D_0 of rank $2n-4$ on M such that

$$S(TM) = \{\bar{\phi}(TM^{\perp}) \oplus \bar{\phi}(tr(TM))\} \perp D_0 \perp < \xi > \tag{3.1}$$

where $<\xi>$ is the distribution spanned by ξ. The distribution D_0 is invariant under $\bar{\phi}$, i.e. $\bar{\phi}(D_0) = D_0$. Moreover, from (2.5) and (3.1) , we obtain the decompositions

$$TM = \{\bar{\phi}(TM^{\perp}) \oplus \bar{\phi}(tr(TM))\} \perp D_0 \perp <\xi> \perp TM^{\perp}, \tag{3.2}$$

$$T\bar{M}|_M = \{\bar{\phi}(TM^{\perp}) \oplus \bar{\phi}(tr(TM))\} \perp D_0 \perp <\xi> \perp (TM^{\perp} \oplus tr(TM)). \tag{3.3}$$

Now, we consider the distributions on M, $D := TM^{\perp} \perp \bar{\phi}(TM^{\perp}) \perp D_0$, $D' := \bar{\phi}(tr(TM))$. Then D is invariant under $\bar{\phi}$ and

$$TM = (D \oplus D') \perp <\xi> . \tag{3.4}$$

Let us consider the local lightlike vector fields $U := -\bar{\phi}N$, $V := -\bar{\phi}E$. Then, from (3.4), any $X \in \Gamma(TM)$ is written as $X = RX + QX + \eta(X)\xi$, $QX = u(X)U$, where R and Q are the projection morphisms of TM into D and D', respectively, and u is a differential 1-form locally defined on M by $u(\cdot) = g(V, \cdot)$. Applying $\bar{\phi}$ and (2.1), one obtain $\bar{\phi}X = \phi X + u(X)N$, where $\bar{\phi}$ is a tensor field of type $(1,1)$ defined on M by $\phi X := \bar{\phi}RX$. In addition, we obtain, $\phi^2 X = -X + \eta(X)\xi + u(X)U$ and $\nabla_X \xi = X - \eta(X)\xi$. We have the following identities, for any $X \in \Gamma(TM)$, $\nabla_X \xi = X - \eta(X)\xi$ and

$$B(X, \xi) = 0, \ C(X, \xi) = \theta(X) \tag{3.5}$$

Define the induced Ricci type tensor $R^{(0,2)}$ of M as

$$R^{(0,2)}(X, Y) = trace(Z \to R(Z, X)Y), \ \forall X, Y \in \Gamma(TM). \tag{3.6}$$

Since the induced connection ∇ on M is not a Levi-Civita connection, in general, $R^{(0,2)}$ is not symmetric. Therefore, in general, it is just a tensor quantity and has no geometric or physical meaning similar to the symmetric Ricci tensor of $\bar{M}$. If $\bar{M}$ is an indefinite Kenmotsu space form $(\bar{M}(c), \bar{g})$, then, for any $X, Y, Z \in \Gamma(TM)$,

$$\bar{R}(X, Y)Z = g(X, Z)Y - g(Y, Z)X. \tag{3.7}$$

A direct calculation gives

$$R^{(0,2)}(X, Y) = -(2n-1)g(X, Y) + B(X, Y)trA_N - B(A_N X, Y), \tag{3.8}$$

where tracetr is written with respect to g restricted to $S(TM)$. Note that the Ricci tensor does not depend on the choice of the vector field E of the distribution $TM^{\perp}$. The tensor field $R^{(0,2)}$ of a lightlike hypersurface M of an indefinite Kenmotsu manifold $\bar{M}$ is called induced Ricci tensor if it is symmetric.

Now, we can give the definition of a weakly ϕ-Ricci symmetric lightlike hypersurface :

Definition 3.1. Let $\bar{M}(c)$ be a indefinite Kenmotsu space form and M be a lightlike hypersurface of $\bar{M}(c)$ indefinite Kenmotsu space form with $\xi \in \Gamma(TM)$. We say that M is a weakly ϕ-Ricci symmetric lightlike hypersurface, if the non-zero Ricci curvature Q of type $(1,1)$ satisfies the condition

$$\phi^2(\nabla_X Q)(Y) = A(X)Q(Y) + B(Y)Q(X) + g(QX,Y)\rho, \tag{3.9}$$

where the vector fields X and Y on M, ρ is a vector field such that $g(\rho, X) = D(X)$, A and B $(\neq 0)$ are associated vector fields.

Theorem 3.2. Let $\bar{M}(c)$ be a indefinite Kenmotsu space form and M be a weakly ϕ-Ricci symmetric lightlike hypersurface of $\bar{M}(c)$ indefinite Kenmotsu space form with Killing radical distribution, $\xi \in \Gamma(TM)$ such $B(\xi) - 1 \neq 0$ and A_N be symmetric with respect to the second fundamental form $\mathcal{B}$ of M. If $v(Z) = 0$, for $\forall Z \in \Gamma(TM)$, then M is a η-Einstein lightlike hypersurface.

Proof. Suppose that M is a weakly ϕ-Ricci symmetric lightlike hypersurface of an indefinite Kenmotsu space form. Then, from (3.9), we have

$$-(\nabla_X Q)(Y) + \eta((\nabla_X Q)(Y))\xi + u((\nabla_X Q)(Y))U = A(X)Q(Y) + B(Y)Q(X) + g(QX,Y)\rho.$$

Here, taking inner product with Z, we obtain

$$\begin{aligned} &- g((\nabla_X Q)(Y), Z) + \eta((\nabla_X Q)(Y))\eta(Z) + u((\nabla_X Q)(Y))g(U,Z) \\ &= A(X)S(Y,Z) + B(Y)S(X,Z) + S(X,Y)D(Z), \end{aligned}$$

where A, B and D are associated vector fields. Since radical distribution is killing, we get $(\nabla_X Q)(Y) = \nabla_X QY - Q\nabla_X Y$ in above equation, then we have

$$\begin{aligned} &- g(\nabla_X QY, Z) + g(Q\nabla_X Y, Z) + \eta(\nabla_X QY)\eta(Z) - \eta(Q\nabla_X Y)\eta(Z) + u(\nabla_X QY)g(U,Z) \\ &- u(Q\nabla_X Y)g(U,Z) = A(X)S(Y,Z) + B(Y)S(X,Z) + S(X,Y)D(Z). \end{aligned} \tag{3.10}$$

Taking $Y=\xi$ in (3.10) and using $\nabla_X\xi = X-\eta(X)\xi$, $Q\xi=-(2n-1)\xi$ and $S(X,\xi)=-(2n-1)\eta(X)$, we obtain

$$(B(\xi)-1)S(X,Z) = (2n-1)[g(X,Z)-u(X)v(Z)+A(X)\eta(Z) + \eta(X)D(Z)] - S(X,V)v(Z), \tag{3.11}$$

where $u(\cdot)=g(\cdot,V)$ and $v(\cdot)=g(\cdot,U)$. Here, taking $X=\xi$, using A_N be symmetric with respect to $\mathcal{B}$ and $A(\xi)+B(\xi)+D(\xi)=0$ (taking $X=Z=\xi$ in(3.11)), we obtain $D(Z)=D(\xi)\eta(Z)$ and again taking $Z=\xi$, we obtain $A(X)=A(\xi)\eta(X)$. Thus, from (3.11), we have

$$(B(\xi)-1)S(X,Z) = (2n-1)[g(X,Z)-u(X)v(Z)-B(\xi)\eta(X)\eta(Z)] - S(X,V)v(Z). \tag{3.12}$$

Then, from the hypothesis, we obtain

$$S(X,Z)=\alpha g(X,Z)+\beta\eta(X)\eta(Z),$$

where $\alpha=\frac{2n-1}{B(\xi)-1}$ and $\beta=\frac{(2n-1)B(\xi)}{B(\xi)-1}$. Thus, proof is complete. □

Then, we have the following result:

Corollary 3.3. Let $\bar{M}(c)$ be a indefinite Kenmotsu space form and M be a weakly ϕ-Ricci symmetric lightlike hypersurface of $\bar{M}(c)$ indefinite Kenmotsu space form with Killing radical distribution, $\xi \in \Gamma(TM)$ such $B(\xi)=0$ and A_N be symmetric with respect to the second fundamental form $\mathcal{B}$ of M. If $v(Z)=0$, for $\forall Z\in\Gamma(TM)$, then M is a Einstein lightlike hypersurface.

Proof. The proof is obvious from (3.12). □

Definition 3.4. Let $\bar{M}(c)$ be a indefinite Kenmotsu space form and M be a lightlike hypersurface of $\bar{M}(c)$ indefinite Kenmotsu space form with $\xi\in\Gamma(TM)$. We say that M is a locally ϕ-Ricci symmetric lightlike hypersurface, if the following condition is satisfied

$$\phi^2(\nabla Q)=0, \tag{3.13}$$

Theorem 3.5. Let $\bar{M}(c)$ be a indefinite Kenmotsu space form and M be a weakly ϕ-Ricci symmetric lightlike hypersurface of $\bar{M}(c)$ indefinite Kenmotsu space form with $\xi\in\Gamma(TM)$ such that M and $S(TM)$ are totally umbilical. If $\alpha\beta=-(2n-1)+\alpha trA_N$ such that α and β are smooth functions, M is a locally ϕ-Ricci symmetric lightlike hypersurface.

Proof. Suppose that M is a weakly ϕ-Ricci symmetric lightlike hypersurface of an indefinite Kenmotsu space form. Then, taking inner product with Z in (3.9), we have

$$g(\phi^2(\nabla_X Q)Y,Z) = A(X)S(Y,Z) + B(Y)S(X,Z) + S(X,Y)D(Z). \tag{3.14}$$

From (3.8), we obtain

$$\begin{aligned} g(\phi^2(\nabla_X Q)Y,Z) &= A(X)\{-(2n-1)g(Y,Z) + B(Y,Z)trA_N - B(A_N Y,Z)\} \\ &+ B(Y)\{-(2n-1)g(X,Z) + B(X,Z)trA_N - B(A_N X,Z)\} \\ &+ D(Z)\{-(2n-1)g(X,Y) + B(X,Y)trA_N - B(A_N X,Y)\}. \end{aligned}$$

Since M and $S(TM)$ are totally umbilical, $\mathcal{B}(X,Y) = \alpha g(X,Y)$ and $C(X,Y) = \beta g(X,Y)$, for $X,Y \in \Gamma(TM)$, where α and β are smooth functions. The above equation then becomes

$$g(\phi^2(\nabla_X Q)Y,Z) = [-(2n-1) + \alpha trA_N - \alpha\beta]g(A(X)Y + B(Y)X + g(X,Y)\rho, Z).$$

Then, for $\forall Z \in \Gamma(TM)$, we obtain

$$\phi^2(\nabla_X Q)Y = [-(2n-1) + \alpha trA_N - \alpha\beta][A(X)Y + B(Y)X + g(X,Y)\rho].$$

Hence, from the hypothesis, the proof is complete. □

Corollary 3.6. Let $\bar{M}(c)$ be a indefinite Kenmotsu space form and M be a weakly ϕ-Ricci symmetric lightlike Einstein hypersurface of $\bar{M}(c)$ indefinite Kenmotsu space form with $\xi \in \Gamma(TM)$. If M is a locally ϕ-Ricci symmetric, then the sum of the associated 1-forms A, B and D is zero everywhere.

Proof. Suppose that M is a weakly ϕ-Ricci symmetric lightlike Einstein hypersurface of an indefinite Kenmotsu space form. Since M is a locally ϕ-Ricci symmetric, form (3.9), we get

$$A(X)Q(Y) + B(Y)Q(X) + g(QX,Y)\rho = 0. \tag{3.15}$$

Taking inner product with Z in (3.15), we obtain

$$A(X)S(Y,Z) + B(Y)S(X,Z) + S(X,Y)D(Z) = 0.$$

Since M is a weakly ϕ-Ricci symmetric lightlike Einstein hypersurface, i.e. $S(X,Y) = \lambda g(X,Y)$, we have

$$A(X)\lambda g(Y,Z) + B(Y)\lambda g(X,Z) + \lambda g(X,Y)D(Z) = 0. \tag{3.16}$$

Here, putting $X = Z = \xi$, we obtain

$$B(Y) = -A(\xi)\eta(Y) - D(\xi)\eta(Y).$$

Again, getting $Y = Z = \xi$ in (3.16), we have

$$A(X) = -B(\xi)\eta(X) - D(\xi)\eta(X).$$

And, taking $X = Y = \xi$ in (3.16), we obtain

$$D(Z) = -A(\xi)\eta(Z) - B(\xi)\eta(Z).$$

Since $A(\xi) + B(\xi) = D(\xi)$, adding above equations by taking $X = Y = Z$, we have

$$A(X) + B(X) + D(X) = 0,$$

for any vector field X on M so that $A + B + D = 0$. □

4. Special Weakly Ricci Symmetric Lightlike Hypersurfaces in Indefinite Kenmotsu Space Forms

In this section, we investigate special weakly Ricci symmetric lightlike hypersurfaces in a indefinite Kenmotsu space form and give some characterizations about this hypersurfaces. Then, we can give main definition:

Definition 4.1. Let $\bar{M}(c)$ be a indefinite Kenmotsu space form and M be a lightlike hypersurface of $\bar{M}(c)$ indefinite Kenmotsu space form with $\xi \in \Gamma(TM)$. We say that M is a special weakly Ricci symmetric (SWRS) lightlike hypersurface, if

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + B(Y)S(X,Z) + D(Z)S(Y,X) \tag{4.1}$$

where A, B and D are defined respectively by, for any $X \in \Gamma(TM)$, $g(X,\rho) = A(X)$, $g(Y,\gamma) = B(Y)$, $g(Z,\kappa) = D(Z)$, are 1-forms called the associated 1-forms which are not zero simultaneously and S is Ricci tensor.

Proposition 4.2. Let $\bar{M}(c)$ be a indefinite Kenmotsu space form and M be lightlike hypersurface of $\bar{M}(c)$ indefinite Kenmotsu space form. Then M is a special weakly Ricci symmetric (SWRS) lightlike hypersurface if the conditions

$$\mathcal{B}(X,Y)\theta(Z) - \mathcal{B}(X,Z)\theta(Y) = 2A(X)g(Y,Z) + B(Y)g(X,Z) + D(Z)g(Y,X)$$
$$+ g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \tag{4.2}$$

$$(\nabla_X \mathcal{B})(Y,Z) = 2A(X)\mathcal{B}(Y,Z) + B(Y)\mathcal{B}(X,Z) + D(Z)\mathcal{B}(Y,X), \tag{4.3}$$

and

$$\mathcal{B}(A_N \nabla_X Y, Z) + \mathcal{B}(A_N Y, \nabla_X Z) = \nabla_X \mathcal{B}(A_N Y, Z) - 2A(X)\mathcal{B}(A_N Y, Z)$$
$$- B(Y)\mathcal{B}(A_N X, Z) - D(Z)\mathcal{B}(A_N Y, X) \tag{4.4}$$

are satisfied, where $X, Y, Z \in \Gamma(TM)$

Proof. Let $\bar{M}(c)$ be a indefinite Kenmotsu space form and M be lightlike hypersurface of $\bar{M}(c)$ indefinite Kenmotsu space form. From (3.8) and (4.1), we obtain

$$(\nabla_X S)(Y,Z) = -(2n-1)[\mathcal{B}(X,Y)\theta(Z) - \mathcal{B}(X,Z)\theta(Y) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)]$$
$$+ (\nabla_X \mathcal{B})(Y,Z) tr A_N - \nabla_X \mathcal{B}(A_N Y, Z) + \mathcal{B}(A_N \nabla_X Y, Z) + \mathcal{B}(A_N Y, \nabla_X Z), \tag{4.5}$$

for $X, Y, Z \in \Gamma(TM)$. Again, from (3.8) and (4.1), we obtain

$$- (2n-1)[2A(X)g(Y,Z) + B(Y)g(X,Z) + D(Z)g(Y,X)] + [2A(X)\mathcal{B}(Y,Z)$$
$$+ B(Y)\mathcal{B}(X,Z) + D(Z)\mathcal{B}(Y,X)] tr A_N - [2A(X)\mathcal{B}(A_N Y, Z)$$
$$+ B(Y)\mathcal{B}(A_N X, Z) + D(Z)\mathcal{B}(A_N Y, X)]. \tag{4.6}$$

Then, from (4.5) and (4.6), we say that if (4.2), (4.3) and (4.4) are satisfied then equation (4.1) is also satisfied. □

Theorem 4.3. Let $\bar{M}(c)$ be a indefinite Kenmotsu space form and M be a special weakly Ricci symmetric (SWRS) lightlike hypersurface of $\bar{M}(c)$ indefinite Kenmotsu space form with $\xi \in \Gamma(TM)$ such that the Ricci tensor S of M is parallel. If the following condition

$$A_E^* A_N Y = -\frac{(2n-1)}{D(\xi)}[2\eta(Y)\rho + B(Y)\xi + D(\xi)Y], D(\xi) \neq 0$$

M is totally geodesic, where $X, Y \in \Gamma(TM)$, $E \in \Gamma(TM)$.

Proof. For $X, Y, Z \in \Gamma(TM)$, from (4.1), we have

$$\begin{aligned} &-(2n-1)[2A(X)g(Y,Z) + B(Y)g(X,Z) + D(Z)g(Y,X)] \\ &+ [2A(X)\mathcal{B}(Y,Z) + B(Y)\mathcal{B}(X,Z) + D(Z)\mathcal{B}(Y,X)]trA_N \\ &- [2A(X)\mathcal{B}(A_N Y,Z) + B(Y)\mathcal{B}(A_N X,Z) + D(Z)\mathcal{B}(A_N Y,X)] = 0. \end{aligned} \tag{4.7}$$

Taking $Z = \xi$ in (4.7), we obtain

$$-(2n-1)[2A(X)\eta(Y) + B(Y)\eta(X) + D(\xi)g(Y,X)] + D(\xi)[\mathcal{B}(Y,X)trA_N - \mathcal{B}(A_N Y,X)] = 0.$$

Then, from the hypothesis, we have

$$D(\xi)\mathcal{B}(Y,X)trA_N = 0.$$

Therefore, the proof is complete. □

Corollary 4.4. Let $\bar{M}(c)$ be a indefinite Kenmotsu space form and M be a special weakly Ricci symmetric (SWRS) lightlike hypersurface of $\bar{M}(c)$ indefinite Kenmotsu space form with $\xi \in \Gamma(TM)$ such that the Ricci tensor S of M is parallel. If the following condition

$$A_E^* A_N Y = -\frac{(2n-1)}{D(\xi)}[2\eta(Y)\rho + B(Y)\xi + D(\xi)Y], D(\xi) \neq 0$$

the induced connection of M is η-Weyl connection, where $X, Y \in \Gamma(TM)$, $E \in \Gamma(TM)$.

Proof. From the hypothesis, M is totally geodesic. Then, the proof is obvious from Theorem (2.2). □

Proposition 4.5. Let $\bar{M}(c)$ be a indefinite Kenmotsu space form and M be a special weakly Ricci symmetric (SWRS) lightlike hypersurface of $\bar{M}(c)$ indefinite Kenmotsu space form admits a cyclic parallel Ricci tensor and with $\xi \in \Gamma(TM)$ such A_N is non-null. Then $\mathcal{B}(B(A_N E, Y) = 0, \forall Y \in \Gamma(TM)$.

Proof. A non-zero Ricci tensor S of lightlike hypersurface M is said to be cyclic parallel if $C\nabla S = 0$, namely, for any $X, Y, Z \in \Gamma(TM)$ [7],

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0. \tag{4.8}$$

Hence, let M admits a cyclic parallel Ricci tensor. Then, using (4.1) and (4.8), we have

$$\begin{aligned} 0 &= (\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) \\ &= 2A(X)S(Y, Z) + B(Y)S(X, Z) + D(Z)S(Y, X) \\ &+ 2A(Y)S(Z, X) + B(Z)S(Y, X) + D(X)S(Z, Y) \\ &+ 2A(Z)S(X, Y) + B(X)S(Z, Y) + D(Y)S(X, Z). \end{aligned} \tag{4.9}$$

Taking $X = E \in \Gamma(Rad(TM))$ in (4.9), we obtain

$$B(Y)\mathcal{B}(A_N E, Z) + 2A(Z)\mathcal{B}(A_N E, Y) + D(Y)\mathcal{B}(A_N E, Z) = 0,$$

where $\mathcal{B}$ is the second fundamental form of M. Putting $Z = \xi$ and using $B(X, \xi) = 0$, we have

$$2A(\xi)\mathcal{B}(A_N E, Y) = 0,$$

Since A is associated vector field which is not zero, then $\mathcal{B}(A_N E, Y) = 0$. □

References

[1] De U.C, On ϕ-symmetric Kenmotsu manifolds. Int. Electronic J.. Geom. 2008; 1(1): 33-38.

[2] De U.C. and Sarkar A, On ϕ-Ricci symmetric Sasakian manifolds. Proceedings of the Jangjeon Math. Soc. 2008; 11(1): 47-52.

[3] Duggal KL. and Bejancu A, Lightlike submanifolds of Semi-Riemannian manifolds and applications. Dordrecht: Kluwer Acad. Publishers, 1996.

[4] Duggal KL. and Şahin B, Differential geometry of lightlike submanifolds. Birkhäuser Verlag AG, 2010.

[5] Lungiambudila O., Massamba F. and Tossa J, Symmetry properties of lightlike hypersurfaces in indefinite Sasakian manifolds. STU. J. Math 2010; 46(2): 177-204.

[6] Singh H. and Khan Q, On special weakly symmetric Riemannian manifolds, Publ. Math. Debrecen, Hungary. 2001; 58/3: 523-536.

[7] Massamba F. On weakly Ricci symmetric lightlike hypersurfaces of indefinite Sasakian manifolds. STU J. Math 2008; 44(2): 165-185.

[8] Massamba F, Symmetry of null geometry in indefinite Kenmotsu manifolds. Mediterr. J. Math. Springer Basel AG, 2012.

[9] Özgür C, On weakly symmetric Kenmotsu manifolds. Differential Geometry-Dynamical Systems, 2006; Vol. 8, 204-209.

[10] Prakasha D.G., Hui S.K. and Vikas K, On weakly ϕ-Ricci symmetric Kenmotsu manifolds. International Journal of Pure and Applied Mathematics, 2014; 95(4): 515-521.

[11] Shukla S.S. and Shukla M.K, On ϕ-Ricci symmetric Kenmotsu manifolds. Novi Sad J. Math., 2009; 39(2): 89-95.

[12] Şahin B. and Yıldız A, Chaki Type Pseudo-symmetric Lightlike Hypersurfaces. International Journal of Geometric Methods in Modern Physics 2015; 12(4),(doi: 10.1142/S0219887815500516).

[13] Takahashi T, Sasakian ϕ-symmetric spaces. Tôhoku Math. J., 1977; 29: 91-113.

[14] Takahashi T, Sasakian manifolds with pseudo-Riemannian metric,. Tôhoku Math. J 1969; 21: 271-290.

[15] Tamássy L. and Binh T.Q, On weakly symmetric and weakly projective symmetric Riemannian manifolds. Coll. Math. Soc. J. Bolyai. 1992; 56: 663-670.

[16] Tamássy L. and Binh T.Q, On weak symmetrics of Einstein and Sasakian manifolds. Tensor N.S. 1993; 53: 140-148.

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