SOME ASPECTS OF PARTIALLY ORDERED MULTISETS

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Abstract

The paper outlines some structural properties of a partially ordered multiset (pomset). A set of necessary and sufficient conditions is provided for characterizing the *width* and *height* of a pomset exploiting set-based partitioning into minimum number of mset chains and antichains, respectively.

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1. Introduction

An mset is an unordered collection of objects in which repetition of elements is significant. For an mset M the *root set* (or support) of M, denoted by M^*, is given by the set $\{x \in M | M(x) > 0\}$. An mset is called finite if the root set is finite and also, multiplicities are finite. In this paper, we shall confine our attention to finite msets. The *cardinality* of an mset is the sum of the multiplicities of all its distinct elements. *Objects* in an mset M represent the elements of the root set of M. An mset can be represented in various forms. For instance, the mset $M = [1,1,1,1,2,4,4,5,5]$ can be denoted by $[1,2,4,5]_{4,1,2,2}$ or $[1^4, 2^1, 4^2, 5^2]$ or $\{4/1, 1/2, 2/4, 2/5\}$. In this paper, we choose to denote an mset M by $[m_1x_1, m_2x_2, \dots, m_nx_n]$, where m_i is the multiplicity of x_i in M, hence m_ix_i will denote a point in M. We will denote the class of all finite mset defined on a set S by $M(S)$. Let $M, N \in M(S)$, then M is a *submset* of N, denoted by $M \subseteq N$, if $M(x) \leq N(x)$ for all $x \in S$, and $M \subset N$ if and only if $M(x) < N(x)$ for at least one x. A submset of a given mset that contains all multiplicities of common elements is called a *whole submset*. A *full submset* contains all objects of the parent mset. The *union* of two msets M and N is the mset given by $(M \cup N)(x) = max\{m, n\}$ such that $mx \in M$ and $nx \in N$ for all $x \in S$. The *intersection* of M and N is the mset given by $(M \cap N)(x) = min\{m, n\}$ such that $mx \in M$ and $nx \in N$ for all $x \in S$ (see [2], [17] and [17] for details on msets). Some works have appeared dealing with infinite multiplicities as well as involving negative multiplicities [3, 22]. In this work, we consider only nonnegative integral multiplicities of objects in an mset.

It is well-known that partially ordered multisets constitute one of the most basic models of concurrency [8, 15, 16]. The problem of extending various mathematical notions and results related to partially ordered sets (posets) (see [20] and [21] for an exposition on posets) to pomsets has attracted serious attention during the last couple of decades [6, 9, 11, 10]. In this paper, we introduce an ordering $\preccurlyeq\leq$ on an mset M and study some properties of the structure $\mathcal{M} = (M, \preccurlyeq\leq)$, in particular, characterization of the width and height of a pomset. In section 2, we define the ordering $\preccurlyeq\leq$ and investigate some properties of the multiset structure $\mathcal{M}$. We discuss mset chains and mset antichains in section 3 and prove some related results. In section 4, we present bounds of pomsets. An extension of Dilworth's decomposition theorem and its dual to pomsets are presented in section 5.

2. Partially ordered multiset (Pomsets)

Let $M = [m_1x_1, m_2x_2, \dots, m_nx_n]$ be an ordered mset. We write $m_ix_i \bowtie m_jx_j$ whenever the two points m_ix_i and m_jx_j in M are *comparable* under the defined order and $m_ix_i || m_jx_j$ whenever m_ix_i and m_jx_j are *incomparable*.

Definition 2.1

For any pair of points m_ix_i *and* m_jx_j in $M \in M(S)$, $m_ix_i \preccurlyeq\leq m_jx_j$ if and only if $x_i \preccurlyeq x_j$, and the points m_ix_i and m_jx_j coincide i.e., $m_ix_i == m_jx_j$ if and only if $x_i = x_j$. Also, $m_ix_i \neq\neq m_jx_j$ if and only if $x_i \neq x_j$. Moreover, $m_ix_i \bowtie m_jx_j$ if and only if $x_i \bowtie x_j$ otherwise $m_ix_i || m_jx_j$.

Note that the condition $m_ix_i == m_jx_j$ if and only if $x_i = x_j$ implies that $m_i = m_j$. This follows from the principle of uniqueness of the multiplicity of an object in an mset.

The strict order associated with $\preccurlyeq\leq$ is the ordering $\prec<$, where $m_ix_i \prec< m_jx_j$ implies that $m_ix_i \preccurlyeq\leq m_jx_j$ and $m_ix_i \neq\neq m_jx_j$.

Definition 2.2

The ordering $\preccurlyeq\leq$ on M is said to be *reflexive* if and only if $m_ix_i \preccurlyeq\leq m_ix_i$ for all $m_ix_i \in M$, *symmetric* if and only if $m_ix_i \preccurlyeq\leq m_jx_j$ implies $m_jx_j \preccurlyeq\leq m_ix_i$, *antisymmetric* if and only if $m_ix_i \preccurlyeq\leq m_jx_j \wedge m_jx_j \preccurlyeq\leq m_ix_i$ implies that $m_ix_i == m_jx_j$, and *transitive* if and only if $m_ix_i \preccurlyeq\leq m_jx_j \wedge m_jx_j \preccurlyeq\leq m_kx_k$ implies $m_ix_i \preccurlyeq\leq m_kx_k$.

Definition 2.3

A relation R is called a *quasi-mset order* (or a *pre-mset order*) if it is reflexive and transitive, and a *strict mset order* if it is irreflexive and transitive. The relation R is called a *partial mset order* (or simply *mset order*) if it is reflexive, antisymmetric and transitive. R is a *linear* (or *total*) mset order if it is a partial mset order and for all pairs of point m_ix_i, m_jx_j in M, we have $m_ix_iRm_jx_j \vee m_jx_jRm_ix_i$.

Definition 2.4

A pomset $\mathcal{M}$ is a pair $(M, \preccurlyeq\leq)$, where $M \in M(S)$, and $\preccurlyeq\leq$ is a partial mset order defined on M.

Theorem 2.1

Let $(S, \preccurlyeq)$ be a poset and $M \in M(S)$. Then $\mathcal{M} = (M, \preccurlyeq\leq)$ is a pomset.

Proof

For any $m_i x_i$ in M, since $x_i \preccurlyeq x_i$ we have $m_i x_i \preccurlyeq\leq m_i x_i$, implying that $(M, \preccurlyeq\leq)$ is reflexive.

Let $m_i x_i \preccurlyeq\leq m_j x_j$ and $m_j x_j \preccurlyeq\leq m_i x_i$ in $\mathcal{M}$. Then, $x_i \preccurlyeq x_j$ and $x_j \preccurlyeq x_i$, and hence $x_i = x_j$.

In particular, $m_i x_i == m_j x_j$, hence $\preccurlyeq\leq$ is antisymmetric.

Let $m_i x_i, m_j x_j, m_k x_k$ be points in M such that $m_i x_i \preccurlyeq\leq m_j x_j$ and $m_j x_j \preccurlyeq\leq m_k x_k$.

We have $x_i \preccurlyeq x_j \preccurlyeq x_k$.

Thus transitivity holds.

Therefore, $(M, \preccurlyeq\leq)$ is a pomset. □

Definition 2.5

For two mset orders $\preccurlyeq_1\leq_1$ and $\preccurlyeq_2\leq_2$ on an mset M, the mset order $\preccurlyeq\leq$ is said to be an intersection of $\preccurlyeq_1\leq_1$ and $\preccurlyeq_2\leq_2$ if and only if $m_i x_i \preccurlyeq\leq m_j x_j \Longrightarrow m_i x_i \preccurlyeq_1\leq_1 m_j x_j \wedge m_i x_i \preccurlyeq_2\leq_2 m_j x_j$, for all $m_i x_i, m_j x_j \in M$.

Theorem 2.2

If $\mathcal{M} = (M, \preccurlyeq_1\leq_1)$ and $\mathcal{N} = (M, \preccurlyeq_2\leq_2)$ are pomsets corresponding to $(S, \preccurlyeq_1)$ and $(S, \preccurlyeq_2)$, then $\mathcal{M} \cap \mathcal{N} = (M, \preccurlyeq\leq)$ is also a pomset, where $\preccurlyeq\leq = \preccurlyeq_1\leq_1 \cap \preccurlyeq_2\leq_2$.

Proof

For any point $m_i x_i$ in M, clearly $m_i x_i \preccurlyeq_1\leq_1 m_i x_i$ and $m_i x_i \preccurlyeq_2\leq_2 m_i x_i$ since $\preccurlyeq_1\leq_1$ and $\preccurlyeq_2\leq_2$ are partial mset orders.

Thus, $m_i x_i \preccurlyeq\leq m_i x_i$ (reflexive property).

Let $m_i x_i$ and $m_j x_j$ be points in M such that

$m_i x_i \preccurlyeq\leq m_j x_j$ and $m_j x_j \preccurlyeq\leq m_i x_i$. (1)

From (1) we have,

$m_i x_i \preccurlyeq_1\leq_1 m_j x_j$ and $m_j x_j \preccurlyeq_1\leq_1 m_i x_i$. (2)

Since $\preccurlyeq_1\leq_1$ is antisymmetric, we have

$m_i x_i == m_j x_j$. (3)

Similarly,

$m_i x_i \preccurlyeq_2 \leq_2 m_j x_j$ and $m_j x_j \preccurlyeq_2 \leq_2 m_i x_i$ imply $m_i x_i == m_j x_j$. (4)

From (2) - (4) we can conclude that,

$m_i x_i \preccurlyeq\leq m_j x_j$ and $m_j x_j \preccurlyeq\leq m_i x_i$ imply $m_i x_i == m_j x_j$.

Therefore, $\preccurlyeq\leq$ is antisymmetric.

For transitivity,

let $m_i x_i, m_j x_j$ and $m_k x_k$ be points in M such that,

$m_i x_i \preccurlyeq\leq m_j x_j$ and $m_j x_j \preccurlyeq\leq m_k x_k$.

We need to show that $m_i x_i \preccurlyeq\leq m_k x_k$.

Now,

$m_i x_i \preccurlyeq\leq m_j x_j$ and $m_j x_j \preccurlyeq\leq m_k x_k$ imply

$m_i x_i \preccurlyeq_1 \leq_1 m_j x_j$ and $m_j x_j \preccurlyeq_1 \leq_1 m_k x_k$.

Since $\preccurlyeq_1 \leq_1$ is transitive, we have $m_i x_i \preccurlyeq_1 \leq_1 m_k x_k$. (5)

Similarly,

$m_i x_i \preccurlyeq_2 \leq_2 m_j x_j$ and $m_j x_j \preccurlyeq_2 \leq_2 m_k x_k$ imply $m_i x_i \preccurlyeq_2 \leq_2 m_k x_k$. (6)

From (5) and (6), we obtain $m_i x_i \preccurlyeq\leq m_k x_k$, hence $\preccurlyeq\leq$ is transitive.

Therefore, $\mathcal{M} \cap \mathcal{N} = (M, \preccurlyeq\leq)$ is a pomset. □

Theorem 2.3

Let $(S, \preccurlyeq)$ be a poset. An mset $M \in M(S)$ is partially ordered if and only if its root set is a subposet of $(S, \preccurlyeq)$.

Proof

Suppose $M \in M(S)$ is partially ordered. Thus, for $m_i x_i \in M$, $m_i x_i \preccurlyeq\leq m_i x_i$ holds. The definition of $\preccurlyeq\leq$ implies that

$x_i \preccurlyeq x_i$ for all $x_i \in M^*$, with $i \in [1, n]$. (1)

Also, for all $m_i x_i, m_j x_j \in M$,

we have $m_i x_i \preccurlyeq\leq m_j x_j \wedge m_j x_j \preccurlyeq\leq m_i x_i \Longrightarrow m_i x_i == m_j x_j$.

Again by the ordering $\preccurlyeq\leq$, it must be the case that

$x_i \preccurlyeq x_j \wedge x_j \preccurlyeq x_i \Longrightarrow x_i = x_j$ for all $x_i, x_j \in M^*$. (2)

Now, let $m_i x_i, m_j x_j, m_k x_k$ be any three points in M. Since M is partially ordered we have

$m_i x_i \preccurlyeq\leq m_j x_j \wedge m_j x_j \preccurlyeq\leq m_k x_k \Longrightarrow m_i x_i \preccurlyeq\leq m_k x_k$, and

$x_i \preccurlyeq x_j \wedge x_j \preccurlyeq x_k \Longrightarrow x_i \preccurlyeq x_k$ for all $x_i \in M^*$. (3)

From (1) through (3), it follows that $(M^*, \preccurlyeq\leq)$ is a subposet of $(S, \preccurlyeq)$.

The converse part is straightforward. Suppose that $(M^* \preccurlyeq)$ is a subposet of $(S, \preccurlyeq)$. Clearly, $x_i \preccurlyeq x_i$ for all $x_i \in M^*$. Let m_i be the multiplicity of x_i in $M \in M(S)$. From the definition of $\preccurlyeq\leq$, we have $m_i x_i \preccurlyeq\leq m_i x_i$ (reflexivity of $\preccurlyeq\leq$). Also, $x_i \preccurlyeq x_j \wedge x_j \preccurlyeq x_i \Longrightarrow x_i = x_j$ for all $x_i, x_j \in M^*$, this in turn gives, $m_i x_i \preccurlyeq\leq m_j x_j \wedge m_j x_j \preccurlyeq\leq m_i x_i \Longrightarrow m_i x_i == m_j x_j$ (antisymmetry of $\preccurlyeq\leq$). And for all $x_i, x_j, x_k \in M^*$, we will have $x_i \preccurlyeq x_j \wedge x_j \preccurlyeq x_k \Longrightarrow x_i \preccurlyeq x_k$. Again, it follows that $m_i x_i \preccurlyeq\leq m_j x_j \wedge m_j x_j \preccurlyeq\leq m_k x_k \Longrightarrow m_i x_i \preccurlyeq\leq m_k x_k$ (transitivity of $\preccurlyeq\leq$). □

3. Mset chains and mset antichains

Definition 3.1

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset. A point $m_i x_i$ in M is *maximal* in $\mathcal{M}$ if for any other point $m_j x_j \in M$ with $m_i x_i \preccurlyeq\leq m_j x_j$ we have $m_i x_i == m_j x_j$. Similarly, a point $m_i x_i$ in M is *minimal* if for any other point $m_j x_j \in M$ with $m_j x_j \preccurlyeq\leq m_i x_i$ we have $m_i x_i == m_j x_j$. If such points are unique, we call them *maximum* and *minimum* respectively.

Theorem 3.1

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset. If $\mathcal{M}$ is totally ordered then maximal and maximum points coincide.

Proof

Let $m_i x_i$ *and* $m_j x_j$ be points in M such that $m_i x_i$ is a maximal point in $\mathcal{M}$ and $m_j x_j$ is a maximum point in $\mathcal{M}$.

Since $\mathcal{M}$ is totally ordered, we will have either $m_i x_i \preccurlyeq\leq m_j x_j$ or $m_j x_j \preccurlyeq\leq m_i x_i$.

Now, suppose that $m_i x_i \preccurlyeq\leq m_j x_j$, then, by definition of a maximal point

$m_i x_i == m_j x_j$.

Similarly, the other case follows. □

A similar argument holds for minimal and minimum points if $\mathcal{M}$ is totally ordered.

Definition 3.2

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset and N, a submset of M. A suborder $\preccurlyeq\leq_{\mathcal{K}}$ is the restriction of

$\preccurlyeq\leq$ to pairs of points in the submset N of M such that

$n_i x_i \preccurlyeq\leq_{\mathcal{K}} n_j x_j \Leftrightarrow m_i x_i \preccurlyeq\leq m_j x_j$, where $n_i x_i, n_j x_j \in N$ and $n_i \leq m_i$. The pair $(N, \preccurlyeq\leq_{\mathcal{K}})$ is called a subpomset of $\mathcal{M}$.

Definition 3.3

A subpomset C of a pomset $\mathcal{M} = (M, \preccurlyeq\leq)$ is called an *mset chain* if C is linearly (or totally) ordered.

A subpomset A of $\mathcal{M}$ is called an *mset antichain* if no two points in A are comparable.

A pomset $\mathcal{M}$ is *connected* (or is an mset chain) if $m_i x_i \bowtie m_j x_j$ for all distinct pairs of points $m_i x_i, m_j x_j \in M$. $\mathcal{M}$ is an mset antichain if $m_i x_i || m_j x_j$ for all distinct pairs of points $m_i x_i, m_j x_j$ in M.

Definition 3.4

An mset chain C in a pomset $\mathcal{M}$ is *maximal* if it is not strictly contained in any other mset chain of $\mathcal{M}$. An mset chain C_i in a pomset $\mathcal{M}$ is a *maximum mset chain* if $|C_i| > |C_j|$ for all other mset chains C_j in the pomset $\mathcal{M}$. A maximal mset antichain is defined analogously. An mset antichain in $\mathcal{M}$ is a *maximum mset antichain* if it contains maximum number of points.

Remark 3.1

A pomset can contain more than one maximal mset chain. Also, in a pomset, maximal and maximum mset chains may coincide. The following example illustrates this.

Example 3.1

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ and let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be the root set for the mset $M = [2x_1, 3x_2, 4x_3, 6x_4, 8x_5, 16x_6]$ where X is partially ordered as follows: $x_1 \preccurlyeq x_3 \preccurlyeq x_5 \preccurlyeq x_6$, $x_1 \preccurlyeq x_4$, and $x_2 \preccurlyeq x_4$.

The following are mset chains in $\mathcal{M}$:

$C_1 = [2x_1, 4x_3, 8x_5, 16x_6]$

$C_2 = [2x_1, 6x_4]$

$C_3 = [3x_2, 6x_4]$

$C_4 = [4x_3, 8x_5]$

Clearly, C_1, C_2 and C_3 are maximal mset chains. Where C_1 is the maximum.

Definition 3.5

A pomset $\mathcal{M} = (M, \preccurlyeq\leq)$ is said to be well-ordered if for any submset N of M, there exists a point $n_i x_i$ in N, such that $n_i x_i$ is the minimum point with respect to the defined order.

Lemma 3.2

Every well-ordered pomset is an mset chain.

Proof

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset and $m_i x_i, m_j x_j$ be any arbitrary pair of distinct points in M. Since $\mathcal{M}$ is well-ordered, the submset $[n_i x_i, n_j x_j]$ has a minimum point.

Thus, either $n_i x_i \prec\prec n_j x_j$ or $n_j x_j \prec\prec n_i x_i$.

Since this condition holds for every pair of distinct points in M, it follows that $\mathcal{M}$ is totally ordered. □

4. Bounds of pomsets

Definition 4.1

Let $\mathcal{K} = (N, \preccurlyeq\leq_{\mathcal{K}})$ be a subpomset of a pomset $\mathcal{M} = (M, \preccurlyeq\leq)$. A point $m_i x_i \in M$ is an upper bound for $\mathcal{K}$ if $m_i x_i \succcurlyeq\geq n_j x_j$ for all points $n_j x_j$ in N. Dually, $m_i x_i \in M$ is a lower bound of $\mathcal{K}$ if $m_i x_i \preccurlyeq\leq n_j x_j$ for all points $n_j x_j$ in N.

Lemma 4.1

If an mset chain C is maximal in a pomset $\mathcal{M}$, then C necessarily contains its upper bound.

Proof

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset and let $C = (N, \preccurlyeq\leq_C)$ be a maximal mset chain in $\mathcal{M}$. Since C is linearly ordered, for some i we will have a point $n_i x_i \in N$ such that $n_i x_i \succ\!> n_j x_j$ for all other points $n_j x_j \in N$. This implies that $n_i x_i$ is a maximum point. Suppose a point $m_k x_k \notin N$ is an upper bound for C. Now C is maximal implies that for any point $m_k x_k \notin N$, we would have either

$m_k x_k || n_i x_i$ or $m_k x_k \preccurlyeq\leq n_i x_i$ since $n_i x_i$ is the maximum point.

If $m_k x_k || n_i x_i$, then $m_k x_k$ cannot be an upper bound for C.

Now, suppose that $m_k x_k \preccurlyeq\leq n_i x_i$, by the definition of upper bound we have a contradiction, hence the result. □

Theorem 4.2

Let $\mathcal{M}$ be a pomset and let $\mathcal{C}$ be a collection of all maximal mset chains in $\mathcal{M}$. If K is an mset containing all upper bounds of the elements of $\mathcal{C}$. Then any two distinct points in K are incomparable.

Proof

Let $C_1, \dots, C_n$ be the maximal mset chains in $\mathcal{M}$. Suppose that $m_1 x_1, m_2 x_2, \dots\ m_n x_n$ are upper bounds for the mset chains $C_1, C_2, \dots, C_n$, then $K = [m_1 x_1, \dots, m_n x_n]$.

Let $m_i x_i\ and\ m_j x_j$ be distinct points in K, then there exists maximal mset chains C_i and C_j in $\mathcal{C}$ such that $m_i x_i$ is an upper bound for C_i and $m_j x_j$ is an upper bound for C_j say.

Now, $C_i \cup [m_j x_j]$ is not an mset chain since C_i is maximal in $\mathcal{M}$. Similarly, $C_j \cup [m_i x_i]$ is not an mset chain.

Assume that $m_i x_i || m_j x_j$, then either $m_i x_i \prec\!< m_j x_j$ or $m_j x_j \prec\!< m_i x_i$ holds.

Suppose $m_i x_i \prec\!< m_j x_j$. Now, $m_i x_i$ is an upper bound for C_i implies that $m_i x_i \succcurlyeq\geq m_k x_k$ for all other points $m_k x_k \in C_i$. By transitivity, it follows that, $m_j x_j \succ\!> m_k x_k$ for all $m_k x_k \in C_i$, which is a contradiction since C_i is maximal in $\mathcal{M}$.

A similar argument holds for the case $m_j x_j \prec\!< m_i x_i$ in C_j.

Hence it must be the case that $m_i x_i || m_j x_j$.

Now $m_i x_i, m_j x_j$ are arbitrary points in K, therefore, no two points in K are comparable. □

5. Height and width of a pomset

Definition 5.1

The *height* of a pomset $\mathcal{M}$ denoted by $\hbar$ is the sum of the multiplicities of all the objects in a maximum mset chain in $\mathcal{M}$. The *width* of a pomset $\mathcal{M}$ denoted by ϖ is the number of points in a maximum mset antichain in $\mathcal{M}$.

Remark 5.1

The number of mset chains in a chain partitioning of $\mathcal{M}$ can be described in relation to the width of $\mathcal{M}$. Likewise, the number of mset antichains in an antichain partitioning of a pomset $\mathcal{M}$ can be described with respect to the height of $\mathcal{M}$. Dilworth's theorem [7], and its dual [14] describe these relationships in the classical setting.

To achieve the desired results for pomsets, it is necessary to exploit set-based partitioning for an antichain partition of the pomset $\mathcal{M}$. Our next result is a necessary and sufficient condition for extending Dilworth's theorem and its dual to pomsets.

Theorem 5.1

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset and let C_i, A_j be mset chains and mset antichains in $\mathcal{M}$ respectively with $i, j \in \{1,2,\dots,n\}$. Then $\left|C_i \cap A_j\right| \leq 1$ for any i, j, if and only if the partitions of the mset antichains are such that each occurrence of the generating object of a point $m_i x_i$ belongs to a different partition i.e. $x_i, x_j \in A_j \Longrightarrow x_i \neq x_j$.

Proof

Assume that $|C_i \cap A_j| \leq 1$. Now, $C_i \cap A_j$ is either empty or has only one point for any i, j. Let the points $l_1 x_1, \dots, l_n x_n$ be in A_j, with $l_i \leq m_i$. The case where $\left|C_i \cap A_j\right| < 1$ is trivial. Suppose $C_i \cap A_j \neq \emptyset$ and let $l_i x_i$ in A_j be a point in $C_i \cap A_j$. Now $|C_i \cap A_j| \leq 1$ implies that $l_i \ngtr 1$. Hence it must be the case that $l_i = 1$. We can apply this process inductively on all points $l_1 x_1, \dots, l_n x_n \in A_j$ since each point $l_i x_i \in A_j$ must belong to a different mset chain C_i. Hence all points in A_j will be of the form $l_i x_i$ with $l_i = 1$. Therefore, $x_i, x_j \in A_j \Longrightarrow x_i \neq x_j$.

Next, assume the converse. Clearly, for each point $l_i x_i \in A_j$, $l_i \ngtr 1$, otherwise we will have a contradiction. If $C_i \cap A_j = \emptyset$, the result follows. Now assume that $C_i \cap A_j$ is not empty and suppose that $\left|C_i \cap A_j\right| > 1$. Then there will be points say $x_1, \dots, x_n$ of A_j, with $n \leq |A_j|$ in $C_i \cap A_j$. This implies that $x_1, \dots, x_n$ are comparable since they are also points in C_i which is a contradiction. Hence $C_i \cap A_j$ is empty or $\left|C_i \cap A_j\right| = 1$. Therefore, $|C_i \cap A_j| \leq 1$. □

Theorem 5.2

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset defined over a partially ordered base set. Then $\mathcal{M}$ can be partitioned into exactly ϖ mset chains where ϖ is the width of the pomset $\mathcal{M}$.

Proof

The case where $\mathcal{M}$ contains only one point $m_i x_i$ is trivial. Suppose the assertion is true for all pomsets $\mathcal{N}_i, i = 1,2, \ldots, k$ with $|\mathcal{N}_i| < |\mathcal{M}|$ for each i and let $\mathcal{M} = \mathcal{N}_k \cup [m_i x_i]$, this implies that $|\mathcal{M}| = |\mathcal{N}_k| + |m_i x_i|$. If A is an mset antichain in $\mathcal{M}$ containing only one point $m_i x_i$, then the assertion is true. Now assume that A contains more than one point and let $\mathcal{C}$ be a maximal mset chain in $\mathcal{M}$, then $\varpi - |A| \leq width(\mathcal{M} \setminus \mathcal{C}) \leq \varpi$. Let F be the subpomset $\mathcal{M} \setminus \mathcal{C}$, if F has width $\varpi - |A|$, by the induction hypothesis F can be partitioned into $\varpi - |A|$ mset chains, together with $\mathcal{C}$ gives a partition into at most ϖ mset chains.

Furthermore, if the pomset $\mathcal{M}$ is partitioned into n mset chains then, $n = \varpi$. Observe that since ϖ is the cardinality of a maximum mset antichain, every point in that mset antichain must belong to a different mset chain. Taking $n < \varpi$ will imply that there exist $m_i x_i, m_j x_j \in C_i$ for some i, j with $m_i x_i || m_j x_j$, which is a contradiction. □

Dually, we present an extension of Mirsky's theorem to pomsets as follows:

Theorem 5.3

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset. Then $\mathcal{M}$ can be partitioned into exactly $\hbar$ mset antichains where $\hbar$ is the height of the pomset $\mathcal{M}$.

Proof

We prove the theorem by induction. If $\mathcal{M}$ is an mset antichain, we have a trivial case. Next, assume that the theorem holds for pomsets of height t where $t < \hbar$. Define $\mathcal{H}$ to be the mset of all maximal points of $\mathcal{M}$. Clearly $\mathcal{H}$ is an mset antichain in $\mathcal{M}$ and every maximal mset chain in $\mathcal{M}$ contains exactly one point $m_i x_i$ from $\mathcal{H}$ which is also the maximum point in that mset chain. Let $\mathcal{B}$ be the pomset $\mathcal{M} \backslash \mathcal{H}$, height of $\mathcal{B}$, denoted height $(\mathcal{B})$, will be $\hbar - (height\ of\ \mathcal{H})$. By the induction hypothesis, height $(\mathcal{B}) < \hbar$ implies that $\mathcal{B}$ is partitioned into $\hbar - (height\ of\ \mathcal{H})$ mset antichains. Therefore the pomset $\mathcal{B}$ together with $\mathcal{H}$ is partitioned into at most $\hbar$ mset antichains.

Theorem 5.1 guarantees that for subpomsets $A_1, A_2, \ldots, A_n$ with $\mathcal{M} = \bigcup_{i=1}^{n} A_i$ (where each A_i is a mset antichain in $\mathcal{M}$) the integer n must be equal to $\hbar$. Using a fewer number of partitions will imply that more than one point in a maximal mset chain belong to some A_i which is a contradiction. □

The following example illustrates theorems 5.2 and 5.3.

Example 5.1

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset and $M = [2x_1, 6x_2, 2x_3, 5x_4, 3x_5, x_6]$. Suppose that

the ordering $\preccurlyeq\leq$ on M is defined as follows:

$2x_1 \preccurlyeq\leq 6x_2,\ 2x_3 \preccurlyeq\leq 5x_4,\ 2x_1 \preccurlyeq\leq 3x_5$.

The pomset $\mathcal{M}$ has $\varpi = 4\ and\ \hbar = 8$.

Observe that, in an mset chain partitioning of $\mathcal{M}$, there are exactly 4 mset chains as follows:

$C_1 = [2x_1, 6x_2], C_2 = [2x_3, 5x_4], C_3 = [3x_5], C_4 = [x_6]$.

In view of theorem 5.1, an mset antichain partitioning of $\mathcal{M}$ gives exactly 8 antichains as follows:

$A_1 = \{x_2, x_4, x_5, x_6\},\ A_2 = \{x_2, x_4, x_5\},\ A_3 = \{x_2, x_4, x_5\},\ A_4 = \{x_2, x_4\},\ A_5 = \{x_2, x_4\},\ A_6 = \{x_2\}, A_7 = \{x_1, x_3\}, A_8 = \{x_1, x_3\}$

6. Concluding remarks

It is known that several characterizations exist for the set of maximal antichains of a poset. An interesting problem will be to characterize the maximal mset antichains of a pomset. In view of wide practical applications of msets, a number of mset orderings have been studied in the literature (see [1, 6, 10, 13], among others). The orderings defined in these literatures are exploited in comparing msets in $M(S)$. With further investigations, the ordering $\preccurlyeq\leq$ can be extended to compare msets.

References

[1] Bachmair, L., Dershowitz, N., and Hsiang, J. (1986). Orderings for equational proofs, in: *Proc. IEEE Symp. on Logic in Computer Science, Cambridge, MA*, 346-357.

[2] Blizard, W. (1989). Multiset theory. *Notre Dame Journal of Formal Logic*, 30: 36-66.

[3] Blizard, W. (1990). Negative membership. *Notre Dame Journal of Formal Logic*, 31: 346-368.

[4] Brandt, J. (1982). Cycles of partitions. *Proc. American Mathematical Society*, 85: 483-486.

[5] Conder, M., Marshall, S., and Slinko, A. (2007). Orders on multisets and discrete cones. *Order*, 24:277-296.

[6] Dershowitz, N., and Manna, Z. (1979). Proving termination with multiset orderings. *Communications of the Journal of Association for Computing Machinery*, 22: 465-476.

[7] Dilworth, R.P. (1950). A decomposition theorem for partially ordered sets. *Annals of Mathematics*, 51(1): 161-166, doi: 10.2307/1969503.

[8] Fanchon, J., and Morin, R. (2002). Regular sets of pomsets with autoconcurrency *Proceedings of the 13th International Conference on Concurrency Theory, 402-417.*

[9] Girish, K. P, and Sunil, J. J. (2009). General relationship between partially ordered multisets and their chains and antichains. *Mathematical communications*, 14(2): 193-205.

[10] Jouannaud, J-P., and Lescanne, P. (1982). On multiset orderings. *Information Processing Letter*, 15: 57-63.

[11] Kilibarda, G., and Jovovic, V. (2004). Antichains of multisets. *Journal of integer sequences*, 7(1), Article 04.1.5

[12] Kilibarda, G. (2015). Enumeration of certain classes of antichains. *Publications De L'Institut Mathematique*, 97(111): 69-87.

[13] Martin, U. (1989). A geometrical approach to multiset orderings. *Theoretical Computer Science*, 67: 37-54.

[14] Mirsky, L. (1971). A dual of Dilworth's decomposition theorem. *American Mathematical Monthly,* 78(8): 876-877.

[15] Pratt, V. R. (1986). Modelling concurrency with partial orders. *International Journal of Parallel Programming*, 15: 33-71.

[16] Rensink, A. (1996). Algebra and theory of order-deterministic pomsets. *Notre Dame Journal of Formal Logic*, 37(2): 283-320.

[17] Singh, D., Ibrahim, A.M., Yohanna, T., and Singh, J.N. (2007). An overview of the applications of multisets. *Novi Sad Journal of Mathematics*, 37(2): 73-92.

[18] Singh, D., and Isah, A. I. (2016). Mathematics of multisets: a unified approach. *Afri. Mat*., 27(1): 1139-1146.

[19] Tella, Y., Singh, D., and Singh, J. N. (2014). A comparative study of multiset orderings. *International Journal of Mathematics and Statistics Invention*, 2(5): 59-71.

[20] Trotter, W. T. (1992). *Combinatorics and partially ordered sets: Dimension Theory*, The Johns Hopkins University Press.

[21] Trotter, W.T. (1995). Partially ordered sets, in : R.L. Graham, M. Grotschel, L. Lovasz (Eds.), *Handbook of combinatorics, Elsevier,* 433-480.

[22] Wildberger, N.J. (2003). A new look at multiset. *School of Mathematics, UNSW Sydney 2052, Australia.*