A Short Proof Of The Fermat Composites Problem.

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Abstract and Definitions.

A *Fermat composite* is a non prime number of the form $F_n = 2^{2^n} + 1$, where n is an integer ≥ 1. Original characterizations of Fermat composites via divisibility are given in [7] and [8] and [9] and [10]. It is known (see [1] or [2] or [3] or [4] and [5] and [6]) that F_5 and F_6 are Fermat composites; Fermat composites are known for some integers $> F_6$, and the Fermat composites problem stipulates that there are infinitely many Fermat composites. In this paper, we give the short proof of the Fermat composites problem, by reducing this problem into a trivial equation of four unknowns and by using elementary combinatoric coupled with elementary arithmetic calculus, elementary divisibility, trivial complex calculus and elementary computation. Moreover, our paper clearly shows that divisibility helps to characterize composite numbers as we did in [7], [8], [9] and [10], and elementary arithmetic calculus coupled with elementary divisibility, elementary complex calculus and trivial computation help to give the simple proof of the Fermat composites problem .

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Preliminaries. In Section.**1**, we introduce definitions that are not standard and we present some elementary properties deduced from these definitions. In Section.**2**, we reduce the Fermat composites problem into a trivial equation of four unknowns and we prove properties linked to elementary arithmetic calculus, elementary divisibility, trivial complex calculus and elementary computation. In Section.**3**, using a simple proposition proved in Section.**1**, and some elementary properties of Section.**2**, we give the short proof of the Fermat composites problem.

1. Introduction. In this section, we introduce definitions that are not standard and we present some elementary properties deduced from these definitions.

Definitions 1.1. For every integer $n \geq 2$, we define $\mathcal{FCO}(n)$, o_n, and $o_{n.1}$ as follows: $\mathcal{FCO}(n) = \{x; 1 < x < 2n \, and \, x \, is \, a \, Fermat \, composite\}$, $o_n = \max_{o \in \mathcal{FCO}(n)} o$, and $o_{n.1} = 4o_n^{o_n}$ [observing (see Abstract) that F_5 is a Fermat composite, then it becomes immediate to deduce that for every integer $n \geq F_5$, $F_5 \in \mathcal{FCO}(n)$].

Using the previous definitions and denotations, let us remark.

Remark 1.1.*Let n be an integer $\geq F_5$; look at $\mathcal{FCO}(n)$, o_n, and $o_{n.1}$ introduced in Definitions 1.1. Then we have the following three simple properties.*

(1.1.0.) $-1 + F_5 < o_n < o_{n.1}$; $o_{n.1} = 4o_n^{o_n}$; $o_{n.1} > F_5^{F_5}$; *and* $o_{n.1}$ *is even.*

(1.1.1.) **If** $o_n < n$, *then:* $o_n = o_{n-1}$ *and* $o_{n.1} = o_{n-1.1}$.

(1.1.2.) **If** $o_{n.1} \leq 2n$, *then* $o_n < n$ *and* $o_{n.1} = o_{n-1.1}$.

Proof. Property (1.1.0) is trivial [**Indeed**, it suffices to use the definition of o_n and $o_{n.1}$, and the fact that $F_5 \in \mathcal{FCO}(n)$ (note that F_5 is a Fermat composite (use Abstract), and observe that n is an integer $\geq F_5$)]. Property (1.1.1) is immediate [**Indeed**, **if** $o_n < n$, clearly $n > F_5$ (use the definition of o_n and observe that $F_5 \in \mathcal{FCO}(n)$, since n is an integer $\geq F_5$), and so $o_n < n < 2n - 2$ (since $n > F_5$ (by the previous) and $o_n < n$ (by the hypotheses)); consequently

$$o_n < 2n - 2 \qquad (1.1).$$

Inequality *(1.1)* immediately implies that $\mathcal{FCO}(n) = \mathcal{FCO}(n-1)$ and therefore

$$o_n = o_{n-1} \qquad (1.2).$$

Equality *(1.2)* immediately implies that $o_{n.1} = o_{n-1.1}$. Property (1.1.1) follows]. Property (1.1.2) is trivial [**Indeed**, clearly

$$o_n < n \qquad (1.3);$$

(otherwise

$$o_n \geq n \qquad (1.4).$$

Now look at $o_{n.1}$ and observe (by using property (1.1.0)) that

$$o_{n.1} = 4o_n^{o_n} \qquad (1.5).$$

Noticing (by the hypotheses) that $n \geq F_5$, then, using *(1.4)* and *(1.5)*, it becomes trivial to deduce that $o_{n.1} > -1 + 4n^n > 2n$; so $o_{n.1} > 2n$ and we have a contradiction, since $o_{n.1} \leq 2n$ (by the hypotheses). So $o_n < n$). Clearly $o_{n.1} = o_{n-1.1}$ (use inequality *(1.3)* and property (1.1.1)). Property (1.1.2) follows]. Remark 1.1 follows. □

Using the definition of $o_{n.1}$ (see Definitions 1.1) , then the following remark and proposition become immediate.

Remark 1.2. **If** $\lim_{n \to +\infty} o_{n.1} = +\infty$*, then there are infinitely many Fermat composites.*
Proof. Immediate [indeed, it suffices to use the definition of $o_{n.1}$ (see Definitions 1.1)]. □

Proposition 1.1. **If** *for every integer* $n \geq F_5$*, we have* $o_{n.1} > n$*, then there are infinitely many Fermat composites.*
Proof. Clearly $\lim_{n \to +\infty} o_{n.1} = +\infty$; therefore there are infinitely many Fermat composites [use the previous equality and apply Remark 1.2]. □

Proposition 1.1 clearly says that: **if** for every integer $n \geq F_5$, we have $o_{n.1} > n$, then, there are infinitely many Fermat composites; this is what we will do in Section.**3**, by using Proposition 1.1, elementary combinatoric, elementary complex calculus, elementary divisibility, elementary arithmetic calculus, and reasoning by reduction to absurd. Proposition 1.1 is stronger than all the investigations that have been done on the Fermat composites problem in the past. Morerover, the reader can easily see that Proposition 1.1 does not use divisibility and is completely different from all the investigations that have been done on the Fermat composites problem in the past. So, in Section.**3**, when we will give the analytic simple proof of the Fermat composites problem, we will not need strong investigations that have been done on the previous problem in the past.

2. Simple properties linked to elementary arithmetic calculus, elementary divisibility, trivial complex calculus, and trivial computation. In this section, we reduce the Fermat composites problem into a trivial equation of four unknowns and we prove properties linked to elementary arithmetic calculus, elementary divisibility, trivial complex calculus and trivial computation. Here definitions of $\mathcal{FCO}(n)$, o_n, and $o_{n.1}$ (see Definitions 1.1) are crucial.

Recalls 2.1 (*Real numbers*, *complex numbers*, *relative integers*, $C_n(c, y, k)$, $\mathcal{J}_n(k)$, and $\mathcal{L}_n(u)$). Recall that $\mathcal{R}$ is the set all *real numbers*, and θ is a *complex number* if $\theta = x + iy$, where x and y are real and where i is the complex entity satisfying $i^2 = -1$. We recall that c' is a *relative integer* if c' is an integer ≥ 0 or if c' is an integer ≤ 0 (For example -108 and -13 and -11 and 0 and 7 and 24 are relative integers; $\frac{1}{2}$ is not a relative integer). We recall that $\mathcal{Z}$ is the set of all relative integers (note (see above) that $\mathcal{R}$ is the set all *real numbers*); clearly $\mathcal{R}^2 \times \mathcal{Z}^2 = \{(c, y, k, u); c \in \mathcal{R},\ y \in \mathcal{R},\ k \in \mathcal{Z},\ and\ u \in \mathcal{Z}\}$. Now let n be an integer $\geq F_5$ and let $o_{n.1}$ (see Definitions 1.1), consider $(c, y, k, u) \in \mathcal{R}^2 \times \mathcal{Z}^2$; then

$C_n(c,y,k)$, $\mathcal{J}_n(k)$, and $\mathcal{L}_n(u)$ are defined as follows:

$$C_n(c,y,k) = co_{n.1}^5(\,49 - o_{n.1}^{-28}\,)(\,2i+1\,) + iy(\,7o_{n.1}^{14} - 1\,) + k;$$

$$\mathcal{J}_n(k) = \frac{k(\,6io_{n.1}^{22} - 9io_{n.1}^{8} - i\,)}{3o_{n.1}^4 - i} + \frac{k(\,-70io_{n.1}^{14} - 10i\,)}{2i+4} + \frac{k(\,42io_{n.1}^{36} - 63io_{n.1}^{22}\,)}{(\,1-7o_{n.1}^{14}\,)(\,3o_{n.1}^4 - i\,)};$$

and

$$\mathcal{L}_n(u) = \frac{u}{7o_{n.1}^5}[\,-7i + \frac{14io_{n.1}^{14} + 98o_{n.1}^{14}}{2i+4} + \frac{45io_{n.1}^{22} - 42io_{n.1}^{36}}{3o_{n.1}^4 - i} + \frac{(\,2+7o_{n.1}^{14}\,)(\,63io_{n.1}^{22} - 42io_{n.1}^{36}\,)}{(\,1-7o_{n.1}^{14}\,)(\,3o_{n.1}^4 - i\,)}].$$

Since $n \geq F_5$, then it becomes trivial that for every $(c,y,k,u) \in \mathcal{R}^2 \times \mathcal{Z}^2$, $(\,C_n(c,y,k), \mathcal{J}_n(k), \mathcal{L}_n(u)\,)$ exists, is well defined and gets sense. Example.**0***(Fundamental).* Let n be an integer $\geq F_5$ and let $o_{n.1}$ (see Definitions 1.1); look at $(c,y,k,u) \in \mathcal{R}^2 \times \mathcal{Z}^2$ and let $(\,C_n(c,y,k), \mathcal{J}_n(k), \mathcal{L}_n(u)\,)$ defined above. **If**

$$(c,y,k,u) = (\ \frac{2o_{n.1}^{-5} + 49o_{n.1}^{23}}{49 - o_{n.1}^{-28}}, \frac{21o_{n.1}^{18} + 49o_{n.1}^{14} + 6o_{n.1}^4 + 7}{7o_{n.1}^{14} - 1},\ -7o_{n.1}^{14} - 2,\ -7o_{n.1}^5\ ),$$

then,

$$C_n(c,y,k) - \mathcal{J}_n(k) - \mathcal{L}_n(u) \neq 0.$$

Proof. Indeed, observe (by the hypotheses) that

$$(c,y,k,\,u\,) = (\ \frac{2o_{n.1}^{-5} + 49o_{n.1}^{23}}{49 - o_{n.1}^{-28}}, \frac{21o_{n.1}^{18} + 49o_{n.1}^{14} + 6o_{n.1}^4 + 7}{7o_{n.1}^{14} - 1}, -7o_{n.1}^{14} - 2,\ -7o_{n.1}^5\ ).$$

Using the previous equality and the definition of $(C_n(c,y,k), \mathcal{J}_n(k), \mathcal{L}_n(u)\,)$, then it becomes trivial to check (by elementary computation and elementary divisibility) that

$$C_n(c,y,k) = 49o_{n.1}^{28} - 7o_{n.1}^{14} + 98io_{n.1}^{28} + 21io_{n.1}^{18} + 49io_{n.1}^{14} + 6io_{n.1}^4 + 11i \qquad (2.0);$$

and

$$\mathcal{J}_n(k) + \mathcal{L}_n(u) = \frac{6io_{n.1}^{22} + 18io_{n.1}^8 + 2i + 7io_{n.1}^{14}}{3o_{n.1}^4 - i} + \frac{490io_{n.1}^{28} + 20i}{2i+4} + 49io_{n.1}^{14} + 7i \qquad (2.1).$$

That being so, to prove Example.**0**, it suffices to prove this *Fact.*
Fact: $C_n(c,y,k) \neq \mathcal{J}_n(k) + \mathcal{L}_n(u)$. *Otherwise* (we reason by reduction to absurd),

$$C_n(c,y,k) = \mathcal{J}_n(k) + \mathcal{L}_n(u) \qquad (2.2).$$

Now using equalities *(2.0)* and *(2.1)*, then it becomes trivial to deduce that equality *(2.2)* says that

$$49o_{n.1}^{28} - 7o_{n.1}^{14} + 98io_{n.1}^{28} + 21io_{n.1}^{18} + 49io_{n.1}^{14} + 6io_{n.1}^4 + 11i = \frac{6io_{n.1}^{22} + 18io_{n.1}^8 + 2i + 7io_{n.1}^{14}}{3o_{n.1}^4 - i} + \frac{490io_{n.1}^{28} + 20i}{2i+4} + 49io_{n.1}^{14} + 7i \quad (2.3).$$

It is trivial that equality *(2.3)* clearly implies that

$$49o_{n.1}^{28} - 7o_{n.1}^{14} + 98io_{n.1}^{28} + 21io_{n.1}^{18} + 6io_{n.1}^4 + 4i = \frac{6io_{n.1}^{22} + 18io_{n.1}^8 + 2i + 7io_{n.1}^{14}}{3o_{n.1}^4 - i} + \frac{490io_{n.1}^{28} + 20i}{2i+4} \qquad (2.4).$$

That being so, define $\rho_{n.0}$ and $\rho_{n.1}$ as follows: $\rho_{n.0} = (3o_{n.1}^4 - i)(2i+4)(49o_{n.1}^{28} - 7o_{n.1}^{14} + 98io_{n.1}^{28} + 21io_{n.1}^{18} + 6io_{n.1}^4 + 4i)$ and $\rho_{n.1} = (2i+4)(6io_{n.1}^{22} + 18io_{n.1}^8 + 2i + 7io_{n.1}^{14}) + (3o_{n.1}^4 - i)(490io_{n.1}^{28} + 20i)$. Now using the preceding two equalities, then it becomes very easy to deduce that equality *(2.4)* immediately implies that

$$\rho_{n.0} = \rho_{n.1} \qquad (2.5).$$

Now let $Re[\rho_{n.0}]$ be the real part of $\rho_{n.0}$ and $Re[\rho_{n.1}]$ be the real part of $\rho_{n.1}$; it is trivial that equality *(2.5)* implies that

$$Re[\rho_{n.0}] = Re[\rho_{n.1}] \qquad (2.6).$$

That being so, look at $(\rho_{n.0}, \rho_{n.1})$ defined above; recalling that $i^2 = -1$, then it becomes trivial to check (by elementary computation and the fact that $i^2 = -1$) that

$$Re[\rho_{n.0}] = -126o_{n.1}^{22} - 36o_{n.1}^8 - 14o_{n.1}^{14} + 490o_{n.1}^{28} + 16 \qquad (2.7),$$

and
$$Re[\rho_{n.1}] = -12o_{n.1}^{22} - 36o_{n.1}^{8} - 14o_{n.1}^{14} + 490o_{n.1}^{28} + 16 \qquad (2.8).$$
Now using equalities *(2.8)* and *(2.7)* and *(2.6)*, then immediately deduce that
$$-126o_{n.1}^{22} - 36o_{n.1}^{8} - 14o_{n.1}^{14} + 490o_{n.1}^{28} + 16 = -12o_{n.1}^{22} - 36o_{n.1}^{8} - 14o_{n.1}^{14} + 490o_{n.1}^{28} + 16 \qquad (2.9).$$
Equality *(2.9)* immediately implies that $-126o_{n.1}^{22} = -12o_{n.1}^{22}$ and therefore
$$114o_{n.1}^{22} = 0 \qquad (2.10).$$
Equality *(2.10)* **is clearly impossible** $\Big($ indeed, since $o_{n.1} > F_5^{F_5}$ (use property (1.1.0 of Remark 1.1), then using the previous inequality, it becomes trivial to deduce that $114o_{n.1}^{22} > 0$, and the previous inequality clearly says that equality *(2.10)* is impossible $\Big)$. So assuming that $C_n(c,y,k) = \mathcal{J}_n(k) + \mathcal{L}_n(u)$ gives rise to a serious contradiction. Consequently $C_n(c,y,k) \neq \mathcal{J}_n(k) + \mathcal{L}_n(u)$. The *Fact* follows and Example.**0** immediately follows.

We will use definitions of Recalls 2.1 in Definition 2.1, and Example.**0** of Recall 2.1 will help us in Example.**4** of Definition 2.1 (in Definition 2.1, we will introduce the notion of *tackle*; this notion is fundamental and crucial for the short complete simple proof of the Fermat composites problem).

Definition 2.1 *(tackle).* Let n be an integer $\geq F_5$, and look at $o_{n.1}$ (see Definitions 1.1). We say that Y *tackles* $o_{n.1}$, if there exists $(c,y,k,u) \in \mathcal{R}^2 \times \mathcal{Z}^2$ such that
$$Y = c(\,147 - 3o_{n.1}^{-28}\,) + iy(\,28 - 4o_{n.1}^{-14}\,),$$
and
$$c(\,-49 + o_{n.1}^{-28}\,) + iy(\,-7o_{n.1}^{14} + 1\,) = (\,k + iu\,)(\,7o_{n.1}^{9} + o_{n.1}^{-5} + 3io_{n.1}^{4}\,),$$
and
$$C_n(c,y,k) = \mathcal{J}_n(k) + \mathcal{L}_n(u),$$
where $(\,C_n(c,y,k), \mathcal{J}_n(k), \mathcal{L}_n(u)\,)$ is defined in Recalls 2.1, and where $i^2 = -1$ $\Big($ we will see in Example.**4** that the previous definition helps to reduce the Fermat composites problem into a trivial equation of four unknowns $\Big)$. Example.**1**. Let n be an integer $\geq F_5$ and let $o_{n.1}$. Then the complex number $147o_{n.1}^{23} - 3o_{n.1}^{-5} + i(\,84o_{n.1}^{4} - 12o_{n.1}^{-10}\,)$ tackles $o_{n.1}$. *Proof.* Indeed, it is immediate to check (by elementary computation and elementary divisibility) that
$$147o_{n.1}^{23} - 3o_{n.1}^{-5} + i(\,84o_{n.1}^{4} - 12o_{n.1}^{-10}\,) = c(\,147 - 3o_{n.1}^{-28}\,) + iy(\,28 - 4o_{n.1}^{-14}\,) \qquad (2.11),$$
and
$$c(\,-49 + o_{n.1}^{-28}\,) + iy(\,-7o_{n.1}^{14} + 1\,) = (\,k + iu\,)(\,7o_{n.1}^{9} + o_{n.1}^{-5} + 3io_{n.1}^{4}\,) \qquad (2.12),$$
and
$$C_n(c,y,k) = \mathcal{J}_n(k) + \mathcal{L}_n(u) \qquad (2.13);$$
where
$$c = o_{n.1}^{23} \textit{ and } y = 3o_{n.1}^{4} \textit{ and } k = 1 - 7o_{n.1}^{14} \textit{ and } u = 0 \qquad (2.14).$$
Since it is immediate that
$$(\,o_{n.1}^{23},\ 3o_{n.1}^{4},\ 1 - 7o_{n.1}^{14},\ 0) \in \mathcal{R}^2 \times \mathcal{Z}^2 \qquad (2.15),$$
clearly $147o_{n.1}^{23} - 3o_{n.1}^{-5} + i(\,84o_{n.1}^{4} - 12o_{n.1}^{-10}\,)$ tackles $o_{n.1}$ $\Big($ use *(2.15)* and *(2.11)* and *(2.12)* and *(2.13)* and *(2.14)* and the definition of tackle introduced above $\Big)$. Example.**1** follows. Example.**2**. Let n be an integer $\geq F_5$ and let $o_{n.1}$ (see Definitions 1.1). Now consider equations $\phi_{n.0}$ and $\phi_{n.1}$, where
$$\phi_{n.0} = \frac{(o_{n.1} - 2n - 2)}{2}(\,-9o_{n.1}^{-5} - 196i - 36io_{n.1}^{-10} - 28io_{n.1}^{-14}\,),\ i^2 = -1;$$
and
$$\phi_{n.1} = 147o_{n.1}^{23} - 3o_{n.1}^{-5} + i(\,84o_{n.1}^{4} - 12o_{n.1}^{-10}\,);\ \ i^2 = -1.$$

If $o_{n.1} = 2n$, then

$$\phi_{n.0} + \phi_{n.1} = 147o_{n.1}^{23} + 6o_{n.1}^{-5} + i(84o_{n.1}^{4} + 196 + 24o_{n.1}^{-10} + 28o_{n.1}^{-14}).$$

Proof. Indeed, observing (via the hypotheses) that $o_{n.1} = 2n$, then, using the previous equality and the definition of the couple $(\phi_{n.0}, \phi_{n.1})$ introduced above, it becomes trivial to check (by elementary computation) that $\phi_{n.0} + \phi_{n.1} = 147o_{n.1}^{23} + 6o_{n.1}^{-5} + i(84o_{n.1}^{4} + 196 + 24o_{n.1}^{-10} + 28o_{n.1}^{-14})$. Example.**2** follows. Example.**3**. Let n be an integer $\geq F_5$ and let $o_{n.1}$ (see Definitions 1.1). Now consider equations $\phi_{n.0}$ and $\phi_{n.1}$ introduced in Example.**2**. **If** $o_{n.1} = 2n + 2$, then $\phi_{n.0} + \phi_{n.1}$ tackles $o_{n.1}$. *Proof.* Indeed, look at equations $\phi_{n.0}$ and $\phi_{n.1}$ introduced in Example.**2** ; observing (via the hypotheses) that $o_{n.1} = 2n + 2$ and using the previous equality, then it becomes trivial to deduce that $\phi_{n.0} + \phi_{n.1} = 147o_{n.1}^{23} - 3o_{n.1}^{-5} + i(84o_{n.1}^{4} - 12o_{n.1}^{-10})$. Clearly $\phi_{n.0} + \phi_{n.1}$ tackles $o_{n.1}$ (Use the previous equality and Example.**1**). Example.**3** follows. Example**.4** (**fundamental**: reduction of the Fermat composites problem into a trivial equation of four unknowns) . Let n be an integer $\geq F_5$ and let $o_{n.1}$. Now consider equations $\phi_{n.0}$ and $\phi_{n.1}$ introduced in Example.**2**. **If** $o_{n.1} = 2n$, then

$$\phi_{n.0} + \phi_{n.1} \textit{ does not tackle } o_{n.1}.$$

Proof. Otherwise (we reason by reduction to absurd), let $(c, y, k, u) \in \mathcal{R}^2 \times \mathcal{Z}^2$ such that

$$\phi_{n.0} + \phi_{n.1} = c(147 - 3o_{n.1}^{-28}) + iy(28 - 4o_{n.1}^{-14}) \qquad (2.16),$$

and

$$c(-49 + o_{n.1}^{-28}) + iy(-7o_{n.1}^{14} + 1) = (k + iu)(7o_{n.1}^{9} + o_{n.1}^{-5} + 3io_{n.1}^{4}) \qquad (2.17),$$

and

$$C_n(c, y, k) = \mathcal{J}_n(k) + \mathcal{L}_n(u) \qquad (2.18),$$

where $(C_n(c, y, k), \mathcal{J}_n(k), \mathcal{L}_n(u))$ is defined in Recalls 2.1, and where $i^2 = -1$; such a (c, y, k, u) exists, since $\phi_{n.0} + \phi_{n.1}$ is supposed to tackle $o_{n.1}$. Clearly

$$\phi_{n.0} + \phi_{n.1} = 147o_{n.1}^{23} + 6o_{n.1}^{-5} + i(84o_{n.1}^{4} + 196 + 24o_{n.1}^{-10} + 28o_{n.1}^{-14}) \qquad (2.19)$$

(observe (by the hypotheses) that $o_{n.1} = 2n$ and use Example.**2**). Using equality *(2.19)*, then it becomes trivial to deduce that equality *(2.16)* clearly says that

$$147o_{n.1}^{23} + 6o_{n.1}^{-5} + i(84o_{n.1}^{4} + 196 + 24o_{n.1}^{-10} + 28o_{n.1}^{-14}) = c(147 - 3o_{n.1}^{-28}) + iy(28 - 4o_{n.1}^{-14}) \qquad (2.20).$$

Using the fact that $i^2 = -1$, then it becomes elementary to deduce that equality *(2.20)* says that

$$147o_{n.1}^{23} + 6o_{n.1}^{-5} = c(147 - 3o_{n.1}^{-28}) \textit{ and } 84o_{n.1}^{4} + 196 + 24o_{n.1}^{-10} + 28o_{n.1}^{-14} = y(28 - 4o_{n.1}^{-14}) \qquad (2.21).$$

The two equalities of *(2.21)* trivially imply that

$$c = \frac{49o_{n.1}^{23} + 2o_{n.1}^{-5}}{49 - o_{n.1}^{-28}} \textit{ and } y = \frac{21o_{n.1}^{4} + 49 + 6o_{n.1}^{-10} + 7o_{n.1}^{-14}}{7 - o_{n.1}^{-14}} = \frac{21o_{n.1}^{18} + 49o_{n.1}^{14} + 6o_{n.1}^{4} + 7}{7o_{n.1}^{14} - 1} \qquad (2.22).$$

That being so, observing (by *(2.22)*) that

$$(c, y) = (\frac{49o_{n.1}^{23} + 2o_{n.1}^{-5}}{49 - o_{n.1}^{-28}}, \frac{21o_{n.1}^{18} + 49o_{n.1}^{14} + 6o_{n.1}^{4} + 7}{7o_{n.1}^{14} - 1}) \qquad (2.23),$$

and using equality *(2.23)*, then it becomes trivial to deduce that equality *(2.17)* says that

$$c(-49 + o_{n.1}^{-28}) + iy(-7o_{n.1}^{14} + 1) = (k + iu)(7o_{n.1}^{9} + o_{n.1}^{-5} + 3io_{n.1}^{4}) \qquad (2.24),$$

$$where\ (c,y)=(\frac{49o_{n.1}^{23}+2o_{n.1}^{-5}}{49-o_{n.1}^{-28}},\ \frac{21o_{n.1}^{18}+49o_{n.1}^{14}+6o_{n.1}^{4}+7}{7o_{n.1}^{14}-1}) \qquad (2.25).$$

Using equality of *(2.25)* and the fact that $i^2=-1$ and elementary divisibility, then it becomes elementary to deduce that equality *(2.24)* implies that

$$k=-7o_{n.1}^{14}-2\ and\ u=-7o_{n.1}^{5} \qquad (2.26).$$

Now using equality of *(2.25)* and the two equalities of *(2.26)* and Example.**0** of Recalls 2.1, then we immediatelly deduce that

$$C_n(c,y,k)\neq \mathcal{J}_n(k)+\mathcal{L}_n(u) \qquad (2.27).$$

(2.27) contradicts equality *(2.18)*. So, assuming that $\phi_{n.0}+\phi_{n.1}$ tackles $o_{n.1}$ when $o_{n.1}=2n$ gives rises to a serious contradiction. Consequently, $\phi_{n.0}+\phi_{n.1}$ does not tackle $o_{n.1}$ when $o_{n.1}=2n$. Example.**4** follows.

Example.**4** reduces the Fermat composites problem into a simple equation of four unknowns. Indeed, Example.**4** clearly says that, if $o_{n.1}=2n$, then we will have a simple equation of four unknowns which implies that $\phi_{n.0}+\phi_{n.1}$ does not tackle $o_{n.1}$. We will use Example.**4** in Section.**3** to immediately deduce the Fermat composites problem. Examples of Definition 2.1 will help us in Section.**3**. Now, via Definition 2.1, let us define:

Definitions 2.2 **(Fundamental)**. Let n be an integer $\geq F_5$, and let $o_{n.1}$; then equations $\phi_{n.0}$ and $\phi_{n.1}$ are defined as follows.

$$\phi_{n.0}=\frac{(o_{n.1}-2n-2)}{2}(-9o_{n.1}^{-5}-196i-36io_{n.1}^{-10}-28io_{n.1}^{-14}),\ i^2=-1;$$

and

$$\phi_{n.1}=147o_{n.1}^{23}-3o_{n.1}^{-5}+i(84o_{n.1}^{4}-12o_{n.1}^{-10});\ \ i^2=-1.$$

It is immediate that for every integer $n\geq F_5$, equations $\phi_{n.0}$ and $\phi_{n.1}$ are well defined and get sense (see Example.**2** of Definition 2.1). Now using Definitions 2.2, then we have the following elementary Proposition.

Proposition 2.1.*Let n be an integer $\geq 1+F_5$ and let $o_{n.1}$ (see Definitions* 1.1*); now look at equations $\phi_{n.0}$ and $\phi_{n.1}$ introduced in Definitions* 2.2*, and via* $(\phi_{n.0},\phi_{n.1})$*, consider equations $\phi_{n-1.0}$ and $\phi_{n-1.1}$ (these considerations get sense, since $n\geq 1+F_5$, and therefore $n-1\geq F_5$).* **If** *$o_{n.1}\leq 2n$, then we have the following two simple properties.*
*(*2.1.0.*)* $o_{n.1}=o_{n-1.1}$.
*(*2.1.1.*)* $\phi_{n-1.0}+\phi_{n-1.1}-(\ \phi_{n.0}+\phi_{n.1}\ )$ *tackles* $o_{n.1}$.
Proof. (2.1.0). Indeed, observing (by the hypotheses) that $o_{n.1}\leq 2n$, clearly $o_{n.1}=o_{n-1.1}$ (use the previous inequality and property (1.1.2) of Remark 1.1). Property (2.1.0) follows.
(2.1.1). Indeed, look at $(\phi_{n.0},\phi_{n.1})$ and observe (by using Definitions 2.2) that

$$\phi_{n.0}=\frac{(o_{n.1}-2n-2)}{2}(-9o_{n.1}^{-5}-196i-36io_{n.1}^{-10}-28io_{n.1}^{-14}) \qquad (2.28)$$

and

$$\phi_{n.1}=147o_{n.1}^{23}-3o_{n.1}^{-5}+i(84o_{n.1}^{4}-12o_{n.1}^{-10}) \qquad (2.29).$$

Using equalities *(2.28)* and *(2.29)*, then it becomes trivial to deduce that

$$\phi_{n-1.0}=\frac{(o_{n-1.1}-2(n-1)-2)}{2}(-9o_{n-1.1}^{-5}-196i-36io_{n-1.1}^{-10}-28io_{n-1.1}^{-14}) \qquad (2.30),$$

and

$$\phi_{n-1.1}=147o_{n-1.1}^{23}-3o_{n-1.1}^{-5}+i(84o_{n-1.1}^{4}-12o_{n-1.1}^{-10}) \qquad (2.31).$$

It is trivial to check (by elementary computation) that equality *(2.30)* is of the form

$$\phi_{n-1.0}=\frac{(o_{n-1.1}-2n-2)}{2}(-9o_{n-1.1}^{-5}-196i-36io_{n-1.1}^{-10}-28io_{n-1.1}^{-14})\ -9o_{n-1.1}^{-5}-196i-36io_{n-1.1}^{-10}-28io_{n-1.1}^{-14} \qquad (2.32).$$

Now look at equalities *(2.32)* and *(2.31)*; noticing (by property (2.1.0)) that $o_{n.1} = o_{n-1.1}$, then it becomes trivial to deduce that equalities *(2.32)* and *(2.31)* clearly say that

$$\phi_{n-1.0} = \frac{(o_{n.1} - 2n - 2)}{2}(-9o_{n.1}^{-5} - 196i - 36io_{n.1}^{-10} - 28io_{n.1}^{-14}) - 9o_{n.1}^{-5} - 196i - 36io_{n.1}^{-10} - 28io_{n.1}^{-14} \quad (2.33),$$

and

$$\phi_{n-1.1} = 147o_{n.1}^{23} - 3o_{n.1}^{-5} + i(84o_{n.1}^{4} - 12o_{n.1}^{-10}) \quad (2.34).$$

Clearly

$$\phi_{n-1.0} = \phi_{n.0} - 9o_{n.1}^{-5} - 196i - 36io_{n.1}^{-10} - 28io_{n.1}^{-14} \quad (2.35)$$

$\Big($ use equalities *(2.28)* and *(2.33)* $\Big)$, and clearly

$$\phi_{n-1.1} = \phi_{n.1} \quad (2.36)$$

$\Big($ use equalities *(2.29)* and *(2.34)* $\Big)$. Using *(2.35)*, then it becomes trivial to deduce that

$$\phi_{n-1.0} - \phi_{n.0} = -9o_{n.1}^{-5} - 196i - 36io_{n.1}^{-10} - 28io_{n.1}^{-14} \quad (2.37),$$

and using *(2.36)* then it becomes trivial to deduce that

$$\phi_{n-1.1} - \phi_{n.1} = 0 \quad (2.38).$$

Clearly

$$\phi_{n-1.0} + \phi_{n-1.1} - (\phi_{n.0} + \phi_{n.1}) = -9o_{n.1}^{-5} - 196i - 36io_{n.1}^{-10} - 28io_{n.1}^{-14} \quad (2.39)$$

$\Big($ use equalities *(2.37)* and *(2.38)* $\Big)$. It is trivial to check (by elementary computation) that equality *(2.39)* is of the form

$$\phi_{n-1.0} + \phi_{n-1.1} - (\phi_{n.0} + \phi_{n.1}) = c(147 - 3o_{n.1}^{-28}) + iy(28 - 4o_{n.1}^{-14}) \quad (2.40),$$

and

$$c(-49 + o_{n.1}^{-28}) + iy(-7o_{n.1}^{14} + 1) = (k + iu)(7o_{n.1}^{9} + o_{n.1}^{-5} + 3io_{n.1}^{4}) \quad (2.41),$$

and

$$C_n(c, y, k) = \mathcal{J}_n(k) + \mathcal{L}_n(u) \quad (2.42),$$

where $(C_n(c, y, k), \mathcal{J}_n(k), \mathcal{L}_n(u))$ is defined in Recalls 2.1, and $i^2 = -1$; and where

$$c = \frac{3o_{n.1}^{-5}}{o_{n.1}^{-28} - 49} \text{ and } y = \frac{49o_{n.1}^{14} + 9o_{n.1}^{4} + 7}{-7o_{n.1}^{14} + 1} \text{ and } k = 3 \text{ and } u = 7o_{n.1}^{5} \quad (2.43).$$

Now using *(2.40)* and *(2.41)* and *(2.42)* and *(2.43)*, then it becomes very easy to deduce that

$$\text{there exists } (c, y, k, u) \in \mathcal{R}^2 \times \mathcal{Z}^2 \text{ such that,} \quad (2.44)$$

$$\phi_{n-1.0} + \phi_{n-1.1} - (\phi_{n.0} + \phi_{n.1}) = c(147 - 3o_{n.1}^{-28}) + iy(28 - 4o_{n.1}^{-14}) \quad (2.45),$$

and

$$c(-49 + o_{n.1}^{-28}) + iy(-7o_{n.1}^{14} + 1) = (k + iu)(7o_{n.1}^{9} + o_{n.1}^{-5} + 3io_{n.1}^{4}) \quad (2.46),$$

and

$$C_n(c, y, k) = \mathcal{J}_n(k) + \mathcal{L}_n(u) \quad (2.47).$$

Clearly $\phi_{n-1.0} + \phi_{n-1.1} - (\phi_{n.0} + \phi_{n.1})$ tackles $o_{n.1}$ $\Big($use *(2.44)* and *(2.45)* and *(2.46)* and *(2.47)* and the definition of tackle introduced in Definition 2.1 $\Big)$. Property (2.1.1) follows and Proposition 2.1 immediately follows. □

The previous simple Proposition made, we are now ready to give the analytic simple proof of the Fermat composites problem.

3. The short proof of the Fermat composites problem. In this Section, the definitions of $\mathcal{FCO}(n)$, o_n and $o_{n.1}$ (see Definitions 1.1), the definition of relative integers (see Recalls 2.1), the definition of tackle (see Definition 2.1), and the definition of $(\phi_{n.0}, \phi_{n.1})$ (see Definitions 2.2), are fundamental and crucial.

Now the following Theorem immediately implies the Fermat composites problem.

Theorem 3.1. *Let n be an integer $\geq F_5$ and let $o_{n.1}$ (see Abstract and Definitions for the meaning of F_5, and see Definitions* 1.1 *for the meaning of $o_{n.1}$); look at equations $\phi_{n.0}$ and $\phi_{n.1}$ introduced in Definitions* 2.2. **If** $o_{n.1} \leq 2n+2$, *then*

$$\phi_{n.0} + \phi_{n.1} \textit{ tackles } o_{n.1}.$$

We are going to prove simply Theorem 3.1. But before, let us remark.

Remark 3.1. *Let n be an integer $\geq F_5$ and let $o_{n.1}$. We have the following three trivial properties.*
*(*3.1.0.*)* **If** $o_{n.1} \geq 2n+4$, *then Theorem* 3.1 *is satisfied by* $o_{n.1}$.
*(*3.1.1.*)* **If** $o_{n.1} = 2n+2$, *then Theorem* 3.1 *is satisfied by* $o_{n.1}$.
*(*3.1.2.*)* **If** $n \leq 2+F_5$, *then Theorem* 3.1 *is satisfied by* $o_{n.1}$
Proof. Property (3.1.0) is trivial. Property (3.1.1) is immediate $\Big($**indeed** let n be an integer $\geq F_5$; observing (by the hypotheses) that $o_{n.1} = 2n+2$, then

$$\phi_{n.0} + \phi_{n.1} \textit{ tackles } o_{n.1} \quad (3.1)$$

(use Example.**3** of Definition 2.1). *(3.1)* clearly says that Theorem 3.1 is satisfied by $o_{n.1}$. Property (3.1.1) follows$\Big)$.
Property (3.1.2.) is immediate $\Big($ **indeed**, observing (by using property (1.1.0) of Remark 1.1) that $o_{n.1} > F_5^{F_5}$, and remarking (by the hypotheses) that $n \leq 2+F_5$, then, using the previous two inequalities, it becomes trivial to deduce that

$$o_{n.1} > F_5^{F_5} > 6 + 3F_5 > 2n+4 \quad (3.2);$$

so

$$o_{n.1} > 2n+4 \quad (3.3)$$

(use *(3.2)*) . Clearly Theorem 3.1 is satisfied by $o_{n.1}$ (use inequality *(3.3)* and property (3.1.0))$\Big)$.□

Using Remark 3.1, let us Remark.

Remark 3.2. *Suppose that Theorem* 3.1 *is false; then there exists an integer $n \geq F_5$ such that $o_{n.1}$ does not satisfied Theorem* 3.1. (*Proof.* Immediate.□)

From Remark 3.2, let us define:

Definitions 3.1 *(Fundamental).* **(i).** We say that n is a *counter-example to Theorem* 3.1, if $n \geq F_5$ and if $o_{n.1}$ does not satisfied Theorem 3.1 (If Theorem 3.1 is false, then such a n exists, by using Remark 3.2) .
(ii). We say that n is a *minimum counter-example to Theorem* 3.1, if n is a counter-example to Theorem 3.1 with n minimum (If Theorem 3.1 is false, then such a n exists, by using **(i)**).

The previous simple remarks and definitions made, we now prove simply Theorem 3.1.

Proof of Theorem 3.1. Otherwise (we reason by reduction to absurd), let n be a minimum counter-example to Theorem 3.1 (such a n exists, by using Remark 3.2 and Definitions 3.1). We observe the following.
*Observation.*3.1.*i. Look at n (recall n is a minimum counter-example to Theorem* 3.1*), and let $o_{n.1}$. Then $n > 2+F_5$ and $o_{n.1} \leq 2n+2$.*

Clearly $n > 2+F_5$ $\Big($ Otherwise $n \leq 2+F_5$ and clearly Theorem 3.1 is satisfied by $o_{n.1}$ (use the previous inequality and property (3.1.2) of Remark 3.1); a contradiction, since in particular $o_{n.1}$ does not satisfied Theorem 3.1$\Big)$; and clearly $o_{n.1} \leq 2n+2$ $\Big($ Otherwise $o_{n.1} > 2n+2$; noticing that $o_{n.1}$ and $2n+2$ are even ($o_{n.1}$ is even (use the definition of $o_{n.1}$) and $2n+2$ is trivially even), then it becomes trivial to deduce that the previous inequality implies that $o_{n.1} \geq 2n+2+2$; so $o_{n.1} \geq 2n+4$ and clearly Theorem 3.1 is satisfied by $o_{n.1}$ (use the previous inequality and property (3.1.0) of Remark 3.1); we have a contradiction since $o_{n.1}$ does not clearly satisfied Theorem 3.1. Observation.3.1.i follows.
*Observation.*3.1.*ii. Look at n (recall n is a minimum counter-example to Theorem* 3.1*), and let $o_{n.1}$. Then*

$$\phi_{n.0} + \phi_{n.1} \textit{ does not tackle } o_{n.1}.$$

Immediate, since in particular, n is a counter-example to Theorem 3.1.

Observation.3.1.*iii*. *Look at n, and let $o_{n.1}$. Then*

$$o_{n.1} \leq 2n \ \ and \ \ o_{n.1} = o_{n-1.1}.$$

Firstly, we are going to show that $o_{n.1} \leq 2n$. *Fact:* $o_{n.1} \leq 2n$. Otherwise,

$$o_{n.1} > 2n; \qquad (3.4);$$

remarking that $o_{n.1}$ and $2n$ are even $\big($ $o_{n.1}$ is even (use the definition of $o_{n.1}$) and $2n$ is trivially even$\big)$, then inequality *(3.4)* immediately implies that $o_{n.1} \geq 2n+2$. Note (by using Observation.3.1.*i*) that $o_{n.1} \leq 2n+2$. Now using the previous two inequalities, then it becomes trivial to see that $o_{n.1} = 2n+2$; so Theorem 3.1 is satisfied by $o_{n.1}$ $\big($ use the previous equality and property (3.1.1) of Remark 3.1$\big)$, and we have a contradiction, since $o_{n.1}$ does not clearly satisfied Theorem 3.1. So

$$o_{n.1} \leq 2n \qquad (3.5).$$

Now we show that $o_{n.1} = o_{n-1.1}$. Indeed, using inequality *(3.5)* and property (1.1.2) of Remark 1.1 ,then it becomes trivial to deduce that $o_{n.1} = o_{n-1.1}$. Observation.3.1.*iii* follows.

Observation.3.1.*iv*. *Look at n. Now let $(\phi_{n.0}, \phi_{n.1})$ (see Definitions 2.2), and via $(\phi_{n.0}, \phi_{n.1})$, consider $(\phi_{n-1.0}, \phi_{n-1.1})$ $\big($ this consideration gets sense, since $n > 2 + F_5$ (use Observation.3.1.i), and so $n-1 > 1 + F_5 > F_5\big)$. Then $\phi_{n-1.0} + \phi_{n-1.1} - (\ \phi_{n.0} + \phi_{n.1}\ )$ tackles $o_{n.1}$.*

Indeed, observing (by Observation.3.1.*iii*) that $o_{n.1} \leq 2n$ and noticing (by Observation.3.1.*i*) that $n > 2+F_5$, then using the previous two inequalities, it becomes trivial to deduce that all the hypotheses of Proposition 2.1 are satisfied, therefore, all the conclusions of Proposition 2.1 are satisfied; in particular property (2.1.1) of Proposition 2.1 is satisfied; consequently $\phi_{n-1.0} + \phi_{n-1.1} - (\ \phi_{n.0} + \phi_{n.1}\ )$ tackles $o_{n.1}$. Observation.3.1.*iv* follows.

Observation.3.1.*v*. *Look at n (recall n is a minimum counter-example to Theorem 3.1). Now let $(\phi_{n.0}, \phi_{n.1})$ (see Definitions 2.2), and via $(\phi_{n.0}, \phi_{n.1})$, consider $(\phi_{n-1.0}, \phi_{n-1.1})$ $\big($ this consideration gets sense, since $n > 2+F_5$ (use Observation.3.1.i), and so $n-1 > 1+F_5 > F_5\big)$. Then $\phi_{n-1.0} - \phi_{n-1.1}$ tackles $o_{n.1}$.*

Indeed look at n $\big($recall n is a minimum counter-example to Theorem 3.1$\big)$, and via n, consider $n-1$ $\big($this consideration gets sense, since $n > 2 + F_5$ (use Observation.3.1.*i*), and so $n-1 > 1+F_5 > F_5\big)$. Observing (by Observation.3.1.*iii*) that $o_{n.1} = o_{n-1.1}$ and $o_{n.1} \leq 2n$, then, by the minimality of n, it becomes trivial to deduce that $n-1$ is not a counter-example to Theorem 3.1 and $\phi_{n-1.0} + \phi_{n-1.1}$ tackles $o_{n-1.1}$; the previous clearly says that $\phi_{n-1.0} + \phi_{n-1.1}$ tackles $o_{n.1}$ (since $o_{n.1} = o_{n-1.1}$ (use Observation.3.1.*iii*)). Observation.3.1.*v* follows.

Observation.3.1.*vi*. *Look at n and let let $(\phi_{n.0}, \phi_{n.1})$. Then $\phi_{n.0} + \phi_{n.1}$ tackles $o_{n.1}$.*

Indeed, using Observation.3.1.*iv* and the definition of *tackle* (see Definition 2.1), then it becomes trivial to deduce that

$$there\ exists\ (c, y, k, u) \in \mathcal{R}^2 \times \mathcal{Z}^2\ \ such\ \ that \qquad (3.6)$$

$$\phi_{n-1.0} + \phi_{n-1.1} - (\ \phi_{n.0} + \phi_{n.1}\ ) = c(\ 147 - 3o_{n.1}^{-28}\ ) + iy(\ 28 - 4o_{n.1}^{-14}\ ) \qquad (3.7),$$

and

$$c(\ -49 + o_{n.1}^{-28}\ ) + iy(\ -7o_{n.1}^{14} + 1\ ) = (\ k + iu\ )(\ 7o_{n.1}^{9} + o_{n.1}^{-5} + 3io_{n.1}^{4}\ ) \qquad (3.8),$$

and

$$C_n(c, y, k) = \mathcal{J}_n(k) + \mathcal{L}_n(u) \qquad (3.9),$$

where $(C_n(c,y,k), \mathcal{J}_n(k), \mathcal{L}_n(u))$ is defined in Recalls 2.1, and where $i^2 = -1$. That being so, using Observation.3.1.v and the definition of *tackle* (see Definition 2.1), then it becomes trivial to deduce that

$$\textit{there exists } (c', y', k', u') \in \mathcal{R}^2 \times \mathcal{Z}^2 \textit{ such that} \quad (3.10)$$

$$\phi_{n-1.0} + \phi_{n-1.1} = c'(147 - 3o_{n.1}^{-28}) + iy'(28 - 4o_{n.1}^{-14}) \quad (3.11),$$

and

$$c'(-49 + o_{n.1}^{-28}) + iy'(-7o_{n.1}^{14} + 1) = (k' + iu')(7o_{n.1}^{9} + o_{n.1}^{-5} + 3io_{n.1}^{4}) \quad (3.12),$$

and

$$C_n(c', y', k') = \mathcal{J}_n(k') + \mathcal{L}_n(u') \quad (3.13).$$

Now using *(3.6)* and *(3.7)* and *(3.8)* and *(3.9)* and *(3.10)* and *(3.11)* and *(3.12)* and *(3.13)* , then it becomes trivial to deduce that

$$\phi_{n.0} + \phi_{n.1} = (c' - c)(147 - 3o_{n.1}^{-28}) + i(y' - y)(28 - 4o_{n.1}^{-14}) \quad (3.14),$$

and

$$(c' - c)(-49 + o_{n.1}^{-28}) + i(y' - y)(-7o_{n.1}^{14} + 1) = ((k' - k) + i(u' - u))(7o_{n.1}^{9} + o_{n.1}^{-5} + 3io_{n.1}^{4}) \quad (3.15),$$

and

$$C_n(c', y', k') - C_n(c, y, k) = \mathcal{J}_n(k') + \mathcal{L}_n(u') - (\mathcal{J}_n(k) + \mathcal{L}_n(u)) \quad (3.16),$$

and where

$$(c' - c, y' - y, k' - k, u' - u) \in \mathcal{R}^2 \times \mathcal{Z}^2 \quad (3.17).$$

That being so, look at equality *(3.16)*; it is trivial (by the definition of $(C_n(c,y,k), \mathcal{J}_n(k), \mathcal{L}_n(u))$) that

$$C_n(c', y', k) - C_n(c, y, k) = C_n(c' - c, y' - y, k' - k) \textit{ and } \mathcal{J}_n(k') + \mathcal{L}_n(u') - (\mathcal{J}_n(k) + \mathcal{L}_n(u)) = \mathcal{J}_n(k' - k) + \mathcal{L}_n(u' - u).$$

Now using the preceding two equalities, then it becomes trivial to deduce that *(3.16)* clearly says that

$$C_n(c' - c, y' - y, k' - k) = \mathcal{J}_n(k' - k) + \mathcal{L}_n(u' - u) \quad (3.18).$$

(3.14) and *(3.15)* and *(3.18)* and *(3.17)* clearly say that

$$\textit{there exists } (c'', y'', k'', u'') \in \mathcal{R}^2 \times \mathcal{Z}^2 \textit{ such that} \quad (3.19)$$

$$\phi_{n.0} + \phi_{n.1} = c''(147 - 3o_{n.1}^{-28}) + iy''(28 - 4o_{n.1}^{-14}) \quad (3.20),$$

and

$$c''(-49 + o_{n.1}^{-28}) + iy''(-7o_{n.1}^{14} + 1) = (k'' + iu'')(7o_{n.1}^{9} + o_{n.1}^{-5} + 3io_{n.1}^{4}) \quad (3.21),$$

and

$$C_n(c'', y'', k'') = \mathcal{J}_n(k'') + \mathcal{L}_n(u'') \quad (3.22),$$

and where

$$(c'', y'', k'', u'') = (c' - c, y' - y, k' - k, u' - u) \quad (3.23).$$

Clearly $\phi_{n.0} + \phi_{n.1}$ tackles $o_{n.1}$ (use *(3.19)* and *(3.20)* and *(3.21)* and *(3.22)* and the definition of tackle introduced in Definition 2.1). Observation.3.1.vi follows.

These simple observations made, then it becomes trivial to see that Observation.3.1.vi clearly contradicts Observation.3.1.ii. Theorem 3.1 follows.□

Now the Fermat composites problem directly results from the following Theorem.

Theorem 3.2 (the using of Example.**4** of Definition 2.1 (Section.**2**)). *For every integer* $n \geq F_5$, *we*

have $o_{n.1} > 2n$.
Proof. Otherwise $\Big($ we reason by reduction to absurd$\Big)$, let n be a minimun counter-example and let $o_{n.1}$; then

$$o_{n.1} \leq 2n \qquad (3.24),$$

and we observe the following.
Observation.3.2.1. $n > 2 + F_5$.

Otherwise $n \leq 2 + F_5$; now observing (by using property (1.1.0) of Remark 1.1) that $o_{n.1} > F_5^{F_5}$ and using the previous two inequalities, then it becomes trivial to deduce that $o_{n.1} > F_5^{F_5} > 2n + 4$; so $o_{n.1} > 2n + 4$ and the previous inequality contradicts inequality *(3.24)*. Observation.3.2.1 follows.
Observation.3.2.2. $o_{n.1} = o_{n-1.1}$.

Indeed, remarking (by *(3.24)*) that $o_{n.1} \leq 2n$, then, using the previous inequality and property (1.1.2) of Remark 1.1, it becomes trivial to deduce that $o_{n.1} = o_{n-1.1}$. Observation.3.2.2 follows.
Observation.3.2.3. $o_{n.1} = 2n$.

Indeed look at n, and via n, consider $n - 1$ $\Big($ this consideration gets sense, since $n > 2 + F_5$ (by Observation.3.2.1), and therefore $n - 1 > 1 + F_5 > F_5\Big)$. Then, by the minimality of n, $n - 1$ is not a counter-example to Theorem 3.2; consequently $o_{n-1.1} > 2(n - 1)$ and the previous inequality clearly says that

$$o_{n-1.1} > 2n - 2 \qquad (3.25).$$

Note that

$$o_{n.1} = o_{n-1.1} \qquad (3.26),$$

by Observation.3.2.2. Now using *(3.25)* and *(3.26)*, then it becomes trivial to deduce that

$$o_{n.1} > 2n - 2 \qquad (3.27).$$

Noticing that $o_{n.1}$ and $2n - 2$ are even ($o_{n.1}$ is even (use the definition of $o_{n.1}$) and $2n - 2$ is trivially even), then it becomes trivial to deduce that inequality *(3.27)* implies that $o_{n.1} \geq 2n - 2 + 2$; the previous inequality clearly says that

$$o_{n.1} \geq 2n \qquad (3.28).$$

Clearly $o_{n.1} = 2n$ (use inequalities *(3.24)* and *(3.28)*). Observation.3.2.3 follows.
Observation.3.2.4*(the using of Example.***4** *of Definition* 2.1 *of Section.***2***). Look at* $o_{n.1}$ *and consider* $(\phi_{n.0}, \phi_{n.1})$ *(see Definitions* 2.2*). Then* $\phi_{n.0} + \phi_{n.1}$ *does not tackle* $o_{n.1}$.

Indeed observing (by Observation.3.2.3) that $o_{n.1} = 2n$ and using Example.**4** of Definition 2.1, then it becomes trivial to deduce that $\phi_{n.0} + \phi_{n.1}$ does not tackle $o_{n.1}$. Observation.3.2.4 follows.

These simple observations made, look at $o_{n.1}$ and consider $\phi_{n.0} + \phi_{n.1}$; observing that $o_{n.1} = 2n$ (by Observation.3.2.3) and remarking that $n > 2 + F_5$ (by Observation.3.2.1), then using the previous, it becomes immediate that all the hypotheses of Theorem 3.1 are satisfied, therefore, the conclusion of Theorem 3.1 is satisfied; consequently

$$\phi_{n.0} + \phi_{n.1} \textit{ tackles } o_{n.1} \qquad (3.29).$$

(3.29) clearly contradicts Observation.3.2.4. Theorem 3.2 follows.□

Theorem 3.2 immediately implies the Fermat composites problem.

Theorem 3.3 *(The Proof of the Fermat composites problem). There are infinitely many Fermat composites.* $\big[$*Proof.* Observe [by using Theorem 3.2] that

$$\textit{For every integer } n \geq F_5 \textit{ we have } o_{n.1} > 2n \qquad (3.30);$$

consequently, there are infinitely many Fermat composites $\Big($ use *(3.29)* and Proposition 1.1$\Big)$. □]

Epilogue. *Our simple article clearly shows that divisibility helps to characterize Fermat composites as we did in [7] and [8] and [9] and [10], and elementary arithmetic calculus coupled with trivial complex calculus and elementary computation help to give a simple analytic proof of problem posed by the Fermat composites.*

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