A Short Proof Of The Fermat Composites Problem.

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Abstract and Definitions.

A *Fermat composite* is a non prime number of the form $F_n = 2^{2^n} + 1$, where n is an integer ≥ 1. Original characterizations of Fermat composites via divisibility are given in [7] and [8] and [9] and [10]. It is known (see [1] or [2] or [3] or [4] and [5] and [6]) that F_5 and F_6 are Fermat composites; Fermat composites are known for some integers $> F_6$, and the Fermat composites problem stipulates that there are infinitely many Fermat composites. In this paper, we give the short proof of the Fermat composites problem, by reducing this problem into a simple equation of two unknowns and by using elementary combinatoric coupled with elementary arithmetic calculus, elementary arithmetic congruences, trivial complex calculus and elementary computation. Moreover, our paper clearly shows that divisibility helps to characterize composite numbers as we did in [7] and [8] and [9] and [10], and elementary arithmetic calculus coupled with elementary arithmetic congruences help to give the simple proof of the Fermat composites problem .

AMS **classification** 2000: 05xx and 11xx.

Preliminaries. In Section.**1**, we introduce definitions that are not standard and we present some elementary properties deduced from these definitions. In Section.**2**, we reduce the Fermat composites problem into a simple equation of two unknowns and we prove properties linked to elementary arithmetic claculus, elementary arithmetic congruences, trivial complex calculus and elementary computation. In Section.**3**, using a simple proposition proved in Section.**1**, and some elementary properties of Section.**2**, we give the short proof of the Fermat composites problem.

1. Introduction. In this section, we introduce definitions that are not standard and we present some elementary properties deduced from these definitions.

Definitions 1.1. For every integer $n \geq 2$, we define $\mathcal{FCO}(n)$, o_n, and $o_{n.1}$ as follows: $\mathcal{FCO}(n) = \{x; 1 < x < 2n \, and \, x \, is \, a \, Fermat \, composite\}$, $o_n = \max_{o \in \mathcal{FCO}(n)} o$, and $o_{n.1} = 4o_n^{o_n}$ [observing (see Abstract) that F_5 is a Fermat composite, then it becomes immediate to deduce that for every integer $n \geq F_5$, $F_5 \in \mathcal{FCO}(n)$].

Using the previous definitions and denotations, let us remark.

Remark 1.1.*Let n be an integer $\geq F_5$; look at $\mathcal{FCO}(n)$, o_n, and $o_{n.1}$ introduced in Definitions* 1.1. *Then we have the following three simple properties.*

(1.1.0.) $-1 + F_5 < o_n < o_{n.1}$; $o_{n.1} = 4o_n^{o_n}$; $o_{n.1} > F_5^{F_5}$; *and $o_{n.1}$ is even.*

(1.1.1.) **If** $o_n < n$, *then:* $o_n = o_{n-1}$ *and* $o_{n.1} = o_{n-1.1}$.

(1.1.2.) **If** $o_{n.1} \leq 2n$, *then* $o_n < n$ *and* $o_{n.1} = o_{n-1.1}$.

Proof. Property (1.1.0) is trivial [**Indeed**, it suffices to use the definition of o_n and $o_{n.1}$, and the fact that $F_5 \in \mathcal{FCO}(n)$ (note that F_5 is a Fermat composite (use Abstract), and observe that n is an integer $\geq F_5$)]. Property (1.1.1) is immediate [**Indeed**, **if** $o_n < n$, clearly $n > F_5$ (use the definition of o_n and observe that $F_5 \in \mathcal{FCO}(n)$, since n is an integer $\geq F_5$), and so $o_n < n < 2n - 2$ (since $n > F_5$ (by the previous) and $o_n < n$ (by the hypotheses)); consequently

$$o_n < 2n - 2 \qquad (1.1).$$

Inequality *(1.1)* immediately implies that $\mathcal{FCO}(n) = \mathcal{FCO}(n-1)$ and therefore

$$o_n = o_{n-1} \qquad (1.2).$$

Equality *(1.2)* immediately implies that $o_{n.1} = o_{n-1.1}$. Property (1.1.1) follows]. Property (1.1.2) is trivial [**Indeed**, clearly

$$o_n < n \qquad (1.3);$$

(otherwise

$$o_n \geq n \qquad (1.4).$$

Now look at $o_{n.1}$ and observe (by using property (1.1.0)) that

$$o_{n.1} = 4o_n^{o_n} \qquad (1.5).$$

Noticing (by the hypotheses) that $n \geq F_5$, then, using *(1.4)* and *(1.5)*, it becomes trivial to deduce that $o_{n.1} > -1 + 4n^n > 2n$; so $o_{n.1} > 2n$ and we have a contradiction, since $o_{n.1} \leq 2n$ (by the hypotheses). So $o_n < n$). Clearly $o_{n.1} = o_{n-1.1}$ (use inequality *(1.3)* and property (1.1.1)). Property (1.1.2) follows]. Remark 1.1 follows. □

Using the definition of $o_{n.1}$ (see Definitions 1.1) , then the following remark and proposition become immediate.

Remark 1.2. **If** $\lim_{n\to+\infty} o_{n.1} = +\infty$, *then there are infinitely many Fermat composites.*
Proof. Immediate [indeed, it suffices to use the definition of $o_{n.1}$ (see Definitions 1.1)]. □

Proposition 1.1. **If** *for every integer* $n \geq F_5$*, we have* $o_{n.1} > n$*, then there are infinitely many Fermat composites.*
Proof. Clearly $\lim_{n\to+\infty} o_{n.1} = +\infty$; therefore there are infinitely many Fermat composites [use the previous equality and apply Remark 1.2]. □

Proposition 1.1 clearly says that: **if** for every integer $n \geq F_5$, we have $o_{n.1} > n$, then, there are infinitely many Fermat composites; this is what we will do in Section.**3**, by using Proposition 1.1, elementary combinatoric, elementary arithmetic congruences, elementary complex calculus, and reasoning by reduction to absurd. Proposition 1.1 is stronger than all the investigations that have been done on the Fermat composites problem in the past. Morerover, the reader can easily see that Proposition 1.1 does not use divisibility and is completely different from all the investigations that have been done on the Fermat composites problem in the past. So, in Section.**3**, when we will give the analytic simple proof of the Fermat composites problem, we will not need strong investigations that have been done on the previous problem in the past.

2. Simple properties linked to elementary arithmetic calculus, trivial complex calculus, elementary arithmetic cnogruences and trivial computation. In this section, we reduce the Fermat composites problem into a simple equation of two unknowns and we prove properties linked to elementary arithmetic calculus, elementary arithmetic congruences, trivial complex calculus and trivial computation. Here definitions of $\mathcal{FCO}(n)$, o_n, and $o_{n.1}$ (see Definitions 1.1) are crucial.

Recalls 2.1 (*Real numbers*, *the real number* $z_{n.1}$, *relative integers*, *elementary arithmetic congruences*, and *complex numbers*). Recall that $\mathcal{R}$ is the set all real numbers and for every *real number* y, $sin^2y + cos^2y = 1$. We recall that c is a *relative integer* if c is an integer ≥ 0 or if c is an integer ≤ 0. For example -108 and -13 and -11 and 0 and 7 and 24 are relative integers; $\frac{1}{2}$ is not a relative integer. We recall that $\mathcal{Z}$ is the set of all relative integers. Let n be an integer $\geq F_5$ and let $o_{n.1}$ (see Definitions 1.1); look at the real number

$$z_{n.1} = 4o_{n.1}^5 - 7o_{n.1}^2 + 3o_{n.1} + (3o_{n.1} - 3o_{n.1}^2)sin^2o_{n.1} + (3o_{n.1}^2 - 3o_{n.1})cos^2o_{n.1}.$$

Now let y' be a real number; we recall that $y' \equiv 0 \ mod(z_{n.1})$ *if and only if* there exists a *relative integer* k' such that $y' = k'z_{n.1}$. We recall (see above) that $\mathcal{R}$ is the set all *real numbers* and clearly $\mathcal{Z}\times\mathcal{R} = \{(c,x); c \in \mathcal{Z} \, and \ x \in \mathcal{R}\}$ (in other words, $\mathcal{Z}\times\mathcal{R}$ is the set of all couples (c,x), where $c \in \mathcal{Z}$ and $x \in \mathcal{R}$). Finally, we recall that θ is a complex number, if $\theta = x + iy$, where x and y are real, and where i is the complex entity satisfying $i^2 = -1$.

We will use definitions of Recalls 2.1 in Definition 2.1 (in Definition 2.1, we will introduce the notion of *tackle*; this notion is fundamental and crucial for the short complete simple proof of the Fermat composites problem).

Definition 2.1 *(tackle).* Let n be an integer $\geq F_5$; look at $o_{n.1}$ (see Definitions 1.1) and let the real number $z_{n.1}$ introduced in Recalls 2.1. We say that Y *tackles* tackles $z_{n.1}$, if there exists $(c,y) \in \mathcal{Z}\times\mathcal{R}$ such that

$$Y = c + 5yo_{n.1} + (-c + 4yo_{n.1})i$$

and

$$\frac{c(\ -o_{n.1}^5 + 5o_{n.1}^2 - 4o_{n.1} + (3o_{n.1} - 3o_{n.1}^2)cos^2 o_{n.1} \ )}{o_{n.1}^3 + o_{n.1}^2 + o_{n.1} + 1} + y(o_{n.1} - 1) \ \equiv \ 0 \ mod(z_{n.1})$$

(in Example.**3**, the previous definition will help us to reduce the Fermat composites problem into a simple equation of two unknowns). Example.**0**. Let n be an integer $\geq F_5$ and let $o_{n.1}$ (see Definitions 1.1). Now look at the couple of equations $(\nu_{n.1}, \mu_{n.1})$, where

$$\nu_{n.1} = -1 + 20o_{n.1}^2 - 25o_{n.1}^3 + o_{n.1}^4 + 5o_{n.1}^6 + (15o_{n.1}^3 - 15o_{n.1}^2)cos^2 o_{n.1},$$

and

$$\mu_{n.1} = (\ 1 + 16o_{n.1}^2 - 20o_{n.1}^3 - o_{n.1}^4 + 4o_{n.1}^6 + (12o_{n.1}^3 - 12o_{n.1}^2)cos^2 o_{n.1} \ )i; i^2 = -1.$$

If $o_{n.1} = 2n + 2$, then the complex number $\nu_{n.1} + \mu_{n.1}$ tackles $z_{n.1}$. *Proof.* Indeed, observing (by the hypotheses) that $o_{n.1} = 2n + 2$ and using the definition of the couple $(\nu_{n.1}, \mu_{n.1})$ introduced above, then it becomes immediate to check (by elementary computation) that

$$\nu_{n.1} + \mu_{n.1} = c + 5yo_{n.1} + (-c + 4yo_{n.1})i \qquad (2.1),$$

$$where \ \ c = o_{n.1}^4 - 1 \ \ and \ \ y = 4o_{n.1} - 5o_{n.1}^2 + o_{n.1}^5 + (3o_{n.1}^2 - 3o_{n.1})cos^2 o_{n.1} \qquad (2.2).$$

Now observing that $o_{n.1}^4 - 1 = (o_{n.1} - 1)(o_{n.1}^3 + o_{n.1}^2 + o_{n.1} + 1)$ and using the previous equality along with the two equalities of *(2.2)*, then it becomes trivial to check (by elementary computation) that $\frac{c(-o_{n.1}^5+5o_{n.1}^2-4o_{n.1}+(3o_{n.1}-3o_{n.1}^2)cos^2 o_{n.1})}{o_{n.1}^3+o_{n.1}^2+o_{n.1}+1} + y(o_{n.1} - 1) = 0$; the previous equality clearly says that $\frac{c(-o_{n.1}^5+5o_{n.1}^2-4o_{n.1}+(3o_{n.1}-3o_{n.1}^2)cos^2 o_{n.1})}{o_{n.1}^3+o_{n.1}^2+o_{n.1}+1} + y(o_{n.1} - 1) \equiv 0 \ mod(z_{n.1})$. So $\nu_{n.1} + \mu_{n.1}$ tackles $z_{n.1}$ (use the previous congruence along with *(2.1)* and *(2.2)* and the definition *of tackle* introduced above). Example.**0** follows. Example.**1**. Let n be an integer $\geq F_5$ and let $o_{n.1}$ (see Definitions 1.1). Now consider equations

$$\nu_{n.0} = (\frac{o_{n.1} - 2n - 2}{2})(\, 1 + o_{n.1} - 4o_{n.1}^2 - 14o_{n.1}^3 - 15o_{n.1}^4 - 15o_{n.1}^5 + 15o_{n.1}^2 sin^2 o_{n.1} \,);$$

$$\mu_{n.0} = (\frac{o_{n.1} - 2n - 2}{2})(-1 - o_{n.1} - 5o_{n.1}^2 - 13o_{n.1}^3 - 12o_{n.1}^4 - 12o_{n.1}^5 + 12o_{n.1}^2 sin^2 o_{n.1} \,)i, \ i^2 = -1;$$

$$\nu_{n.1} = -1 + 20o_{n.1}^2 - 25o_{n.1}^3 + o_{n.1}^4 + 5o_{n.1}^6 + (15o_{n.1}^3 - 15o_{n.1}^2)cos^2 o_{n.1};$$

$$\mu_{n.1} = (\ 1 + 16o_{n.1}^2 - 20o_{n.1}^3 - o_{n.1}^4 + 4o_{n.1}^6 + (12o_{n.1}^3 - 12o_{n.1}^2)cos^2 o_{n.1} \ )i, \ i^2 = -1;$$

and

$$\phi_{n.2} = \nu_{n.0} + \mu_{n.0} + \nu_{n.1} + \mu_{n.1}.$$

If $o_{n.1}=2n$, then $\phi_{n.2}=\phi_{n.0}+\phi_{n.1}$, where

$$\phi_{n.0}=-2-o_{n.1}+9o_{n.1}^2-11o_{n.1}^3+16o_{n.1}^4+15o_{n.1}^5+5o_{n.1}^6+15o_{n.1}^3cos^2o_{n.1},$$

and

$$\phi_{n.1}=(2+o_{n.1}+9o_{n.1}^2-7o_{n.1}^3+11o_{n.1}^4+12o_{n.1}^5+4o_{n.1}^6+12o_{n.1}^3cos^2o_{n.1})i;\ \ i^2=-1.$$

Proof. Indeed, observing (via the hypotheses) that $o_{n.1}=2n$ and remarking that $sin^2o_{n.1}+cos^2o_{n.1}=1$, then, using the previous two equalities and the definitions of $(\nu_{n.0},\mu_{n.0},\nu_{n.1},\mu_{n.1},\phi_{n.0},\phi_{n.1})$ given above, it becomes trivial to check (by elementary computation) that $\phi_{n.0}=\nu_{n.0}+\nu_{n.1}$ and $\phi_{n.1}=\mu_{n.0}+\mu_{n.1}$; now using the previous two equalities along with the definition of $\phi_{n.2}$ (recall that $\phi_{n.2}=\nu_{n.0}+\mu_{n.0}+\nu_{n.1}+\mu_{n.1}$), then it becomes trivial to deduce that $\phi_{n.2}=\nu_{n.0}+\mu_{n.0}+\nu_{n.1}+\mu_{n.1}=\phi_{n.0}+\phi_{n.1}$. Example.**1** follows. Example.**2**. Let n be an integer $\geq F_5$ and let $o_{n.1}$ (see Definitions 1.1). Now consider equations $\nu_{n.0}$, $\mu_{n.0}$, $\nu_{n.1}$, $\mu_{n.1}$ and $\phi_{n.2}$ introduced in Example.**1**. **If** $o_{n.1}=2n+2$, then $\phi_{n.2}$ tackles $z_{n.1}$. *Proof.* Indeed, look at equations $\nu_{n.0}$, $\mu_{n.0}$, $\nu_{n.1}$ and $\mu_{n.1}$ introduced in Example.**1**; observing (via the hypotheses) that $o_{n.1}=2n+2$ and using the previous equality, then it becomes trivial to check (by elementary computation) that

$$\nu_{n.0}+\mu_{n.0}+\nu_{n.1}+\mu_{n.1}=0+0+\nu_{n.1}+\mu_{n.1}=\nu_{n.1}+\mu_{n.1} \qquad (2.3).$$

Now consider equation $\phi_{n.2}$ introduced in Example.**1**; then using *(2.3)*, it becomes trivial to deduce that

$$\phi_{n.2}=\nu_{n.1}+\mu_{n.1} \qquad (2.4).$$

Clearly $\phi_{n.2}$ tackles $z_{n.1}$ (use equality *(2.4)* and Example.**0**). Example.**2** follows. Example.**3** (**fundamental**: reduction of the Fermat composites problem into an equation of two unknowns) . Let n be an integer $\geq F_5$ and let $o_{n.1}$. Now consider equations $\nu_{n.0}$, $\mu_{n.0}$, $\nu_{n.1}$, $\mu_{n.1}$ and $\phi_{n.2}$ introduced introduced in Example.**1**. **If** $o_{n.1}=2n$, then

$$\phi_{n.2}\ does\ not\ tackle\ z_{n.1}\ ,$$

where $z_{n.1}$ is introduced in Recalls 2.1. *Proof.* Otherwise (we reason by reduction to absurd), let $(c,y)\in\mathcal{Z}\times\mathcal{R}$ such that

$$\phi_{n.2}=c+5yo_{n.1}+(-c+4yo_{n.1})i \qquad (2.5)$$

and

$$\frac{c(\ -o_{n.1}^5+5o_{n.1}^2-4o_{n.1}+(3o_{n.1}-3o_{n.1}^2)cos^2o_{n.1}\ )}{o_{n.1}^3+o_{n.1}^2+o_{n.1}+1}+y(o_{n.1}-1)\ \equiv\ 0\ mod(z_{n.1}) \qquad (2.5'),$$

such a (c,y) exists, since $\phi_{n.2}$ is supposed to tackle $z_{n.1}$. Observing (under the hypotheses) that $o_{n.1}=2n$ and using Example.**1**, then we immediately deduce that

$$\phi_{n.2}=\phi_{n.0}+\phi_{n.1} \qquad (2.6),$$

where

$$\phi_{n.0}=-2-o_{n.1}+9o_{n.1}^2-11o_{n.1}^3+16o_{n.1}^4+15o_{n.1}^5+5o_{n.1}^6+15o_{n.1}^3cos^2o_{n.1},$$

and

$$\phi_{n.1}=(2+o_{n.1}+9o_{n.1}^2-7o_{n.1}^3+11o_{n.1}^4+12o_{n.1}^5+4o_{n.1}^6+12o_{n.1}^3cos^2o_{n.1})i.$$

Using equality *(2.6)*, then it becomes very easy to deduce that equality *(2.5)* says that

$$\phi_{n.0}+\phi_{n.1}=c+5yo_{n.1}+(-c+4yo_{n.1})i \qquad (2.7).$$

Now using equations $\phi_{n.0}$ and $\phi_{n.1}$ (given above) along with the fact that $i^2=-1$, then it becomes trivial to deduce that *(2.7)* clearly says that

$$c+5yo_{n.1}=\phi_{n.0}\ and\ (-c+4yo_{n.1})i=\phi_{n.1} \qquad (2.8).$$

It is very easy to check that the two equalities of *(2.8)* imply that

$$c=-2-o_{n.1}-o_{n.1}^2-o_{n.1}^3+o_{n.1}^4\ and\ y=2o_{n.1}-2o_{n.1}^2+3o_{n.1}^3+3o_{n.1}^4+o_{n.1}^5+3o_{n.1}^2cos^2o_{n.1} \qquad (2.9).$$

Using the two equalities of *(2.9)*, then it becomes trivial to deduce that

$$\frac{c(\ -o_{n.1}^5+5o_{n.1}^2-4o_{n.1}+(3o_{n.1}-3o_{n.1}^2)cos^2 o_{n.1}\ )}{o_{n.1}^3+o_{n.1}^2+o_{n.1}+1}+y(o_{n.1}-1)\ \not\equiv\ 0\ mod(z_{n.1}),$$

where $z_{n.1}$ is introduced in Recalls 2.1 [[**indeed**, observing (by using the two equalities of *(2.9)*) that

$$(c,y)=(-2-o_{n.1}-o_{n.1}^2-o_{n.1}^3+o_{n.1}^4,2o_{n.1}-2o_{n.1}^2+3o_{n.1}^3+3o_{n.1}^4+o_{n.1}^5+3o_{n.1}^2cos^2 o_{n.1}),$$

then using the previous equality, it becomes trivial to check (by elementary computation) that

$$\frac{c(\ -o_{n.1}^5+5o_{n.1}^2-4o_{n.1}+(3o_{n.1}-3o_{n.1}^2)cos^2 o_{n.1}\ )}{o_{n.1}^3+o_{n.1}^2+o_{n.1}+1}+y(o_{n.1}-1)=-2o_{n.1}+2o_{n.1}^5;$$

the previous equality clearly implies that $\frac{c(\ -o_{n.1}^5+5o_{n.1}^2-4o_{n.1}+(3o_{n.1}-3o_{n.1}^2)cos^2 o_{n.1}\ )}{o_{n.1}^3+o_{n.1}^2+o_{n.1}+1}+y(o_{n.1}-1)\ \not\equiv\ 0\ mod(z_{n.1})$.
Indeed assume otherwise and suppose that $\frac{c(\ -o_{n.1}^5+5o_{n.1}^2-4o_{n.1}+(3o_{n.1}-3o_{n.1}^2)cos^2 o_{n.1}\ )}{o_{n.1}^3+o_{n.1}^2+o_{n.1}+1}+y(o_{n.1}-1)\ \equiv\ 0\ mod(z_{n.1})$, then using the previous congruence along with the previous equality, it becomes trivial to deduce that

$$-2o_{n.1}+2o_{n.1}^5\ \equiv\ 0\ mod(z_{n.1}) \qquad (2.9'),$$

where $z_{n.1}$ is introduced in Recalls 2.1 and is of the form

$$z_{n.1}=4o_{n.1}^5-7o_{n.1}^2+3o_{n.1}+(3o_{n.1}-3o_{n.1}^2)sin^2 o_{n.1}+(3o_{n.1}^2-3o_{n.1})cos^2 o_{n.1}.$$

Using the fact that $sin^2 o_{n.1}=1-cos^2 o_{n.1}$, then it becomes trivial to deduce that the previous equality is of the form

$$z_{n.1}=4o_{n.1}^5-10o_{n.1}^2+6o_{n.1}+(6o_{n.1}^2-6o_{n.1})cos^2 o_{n.1} \qquad (2.9'').$$

Observing that $o_{n.1}>F_5^{F_5}$ (note that $n\geq F_5$ and use property (1.1.0) of Remark 1.1), and using inequality $o_{n.1}>F_5^{F_5}$ previously mentioned, then it becomes trivial to deduce that equality $(2.9'')$ implies that

$$z_{n.1}>2o_{n.1}^5>-2o_{n.1}+2o_{n.1}^5>0. \qquad (2.9''')$$

$(2.9''')$ trivially implies $-2o_{n.1}+2o_{n.1}^5\ \not\equiv\ 0\ mod(z_{n.1})$, and the previous contradicts the congruence $(2.9')$. So

$$\frac{c(\ -o_{n.1}^5+5o_{n.1}^2-4o_{n.1}+(3o_{n.1}-3o_{n.1}^2)cos^2 o_{n.1}\ )}{o_{n.1}^3+o_{n.1}^2+o_{n.1}+1}+y(o_{n.1}-1)\ \not\equiv\ 0\ mod(z_{n.1}).]]$$

Since $\frac{c(\ -o_{n.1}^5+5o_{n.1}^2-4o_{n.1}+(3o_{n.1}-3o_{n.1}^2)cos^2 o_{n.1}\ )}{o_{n.1}^3+o_{n.1}^2+o_{n.1}+1}+y(o_{n.1}-1)\ \not\equiv\ 0\ mod(z_{n.1})$, we have a contradiction (because $\phi_{n.2}$ was supposed tackling $z_{n.1}$; so in particular $\frac{c(\ -o_{n.1}^5+5o_{n.1}^2-4o_{n.1}+(3o_{n.1}-3o_{n.1}^2)cos^2 o_{n.1}\ )}{o_{n.1}^3+o_{n.1}^2+o_{n.1}+1}+y(o_{n.1}-1)\ \equiv\ 0\ mod(z_{n.1})$). Example.**3** follows.

Example.**3** reduces the Fermat composites problem into a simple equation of two unknowns. Indeed, Example.**3** clearly says that, if $o_{n.1}=2n$, then we will have a simple equation of two unknowns which does not tackle $z_{n.1}$. We will use Example.**3** in Section.**3** to immediately deduce the Fermat composites problem. Examples of Definition 2.1 will help us in Section.**3**. Now, via Definition 2.1, let us define:

Definitions 2.2 **(Fundamental)**. Let n be an integer $\geq F_5$, and let $o_{n.1}$; then equations $\nu_{n.0}$, $\mu_{n.0}$, $\nu_{n.1}$, $\mu_{n.1}$ and $\phi_{n.2}$ are defined as follows.

$$\nu_{n.0}=(\frac{o_{n.1}-2n-2}{2})(\ 1+o_{n.1}-4o_{n.1}^2-14o_{n.1}^3-15o_{n.1}^4-15o_{n.1}^5+15o_{n.1}^2sin^2 o_{n.1}\ );$$

$$\mu_{n.0}=(\frac{o_{n.1}-2n-2}{2})(-1-o_{n.1}-5o_{n.1}^2-13o_{n.1}^3-12o_{n.1}^4-12o_{n.1}^5+12o_{n.1}^2sin^2 o_{n.1}\ )i,\ i^2=-1;$$

$$\nu_{n.1}=-1+20o_{n.1}^2-25o_{n.1}^3+o_{n.1}^4+5o_{n.1}^6+(15o_{n.1}^3-15o_{n.1}^2)cos^2 o_{n.1};$$

$$\mu_{n.1}=(\ 1+16o_{n.1}^2-20o_{n.1}^3-o_{n.1}^4+4o_{n.1}^6+(12o_{n.1}^3-12o_{n.1}^2)cos^2 o_{n.1}\ )i,\ i^2=-1;$$

and

$$\phi_{n.2} = \nu_{n.0} + \mu_{n.0} + \nu_{n.1} + \mu_{n.1}.$$

It is immediate that for every integer $n \geq F_5$, equations $\nu_{n.0}$, $\mu_{n.0}$, $\nu_{n.1}$, $\mu_{n.1}$ and $\phi_{n.2}$ are well defined and get sense (see Example.**1** of Definition 2.1). Now using Definitions 2.2, then we have the following elementary Proposition.

Proposition 2.1.*Let n be an integer $\geq 1+F_5$ and let $o_{n.1}$ (see Definitions 1.1); now look at equation $\phi_{n.2}$ introduced in Definitions 2.2, and via $\phi_{n.2}$, consider equation $\phi_{n-1.2}$ (this consideration gets sense, since $n \geq 1+F_5$, and therefore $n-1 \geq F_5$).* **If** *$o_{n.1} \leq 2n$, then we have the following two simple properties.*
(2.1.0.) $o_{n.1} = o_{n-1.1}$.
(2.1.1.) $\phi_{n-1.2} - \phi_{n.2}$ tackles $z_{n.1}$, where $z_{n.1}$ is introduced in Recalls 2.1 and is of the form

$$z_{n.1} = 4o_{n.1}^5 - 7o_{n.1}^2 + 3o_{n.1} + (3o_{n.1} - 3o_{n.1}^2)sin^2 o_{n.1} + (3o_{n.1}^2 - 3o_{n.1})cos^2 o_{n.1}.$$

Proof. (2.1.0). Indeed, observing (by the hypotheses) that $o_{n.1} \leq 2n$, clearly $o_{n.1} = o_{n-1.1}$ (use the previous inequality and property (1.1.2) of Remark 1.1). Property (2.1.0) follows.
(2.1.1). Indeed, look at $\phi_{n.2}$ and observe (by using Definitions 2.2) that

$$\phi_{n.2} = \nu_{n.0} + \mu_{n.0} + \nu_{n.1} + \mu_{n.1} \qquad (2.10),$$

where

$$\nu_{n.0} = (\frac{o_{n.1} - 2n - 2}{2})(1 + o_{n.1} - 4o_{n.1}^2 - 14o_{n.1}^3 - 15o_{n.1}^4 - 15o_{n.1}^5 + 15o_{n.1}^2 sin^2 o_{n.1}) \qquad (2.11),$$

$$\mu_{n.0} = (\frac{o_{n.1} - 2n - 2}{2})(-1 - o_{n.1} - 5o_{n.1}^2 - 13o_{n.1}^3 - 12o_{n.1}^4 - 12o_{n.1}^5 + 12o_{n.1}^2 sin^2 o_{n.1})i \qquad (2.12),$$

$$\nu_{n.1} = -1 + 20o_{n.1}^2 - 25o_{n.1}^3 + o_{n.1}^4 + 5o_{n.1}^6 + (15o_{n.1}^3 - 15o_{n.1}^2)cos^2 o_{n.1} \qquad (2.13),$$

and

$$\mu_{n.1} = (1 + 16o_{n.1}^2 - 20o_{n.1}^3 - o_{n.1}^4 + 4o_{n.1}^6 + (12o_{n.1}^3 - 12o_{n.1}^2)cos^2 o_{n.1})i \qquad (2.14).$$

Using equality (2.10), then it becomes trivial to deduce that

$$\phi_{n-1.2} = \nu_{n-1.0} + \mu_{n-1.0} + \nu_{n-1.1} + \mu_{n-1.1} \qquad (2.15).$$

Look at equality (2.15) and consider $\nu_{n-1.0}$; then using the expression of $\nu_{n.0}$ given by equality (2.11), it becomes trivial to decuce that

$$\nu_{n-1.0} = (\frac{o_{n-1.1} - 2(n-1) - 2}{2})(1 + o_{n-1.1} - 4o_{n-1.1}^2 - 14o_{n-1.1}^3 - 15o_{n-1.1}^4 - 15o_{n-1.1}^5 + 15o_{n-1.1}^2 sin^2 o_{n-1.1}) \qquad (2.16).$$

It is trivial to check (by elementary computation) that equality (2.16) is of the form

$$\nu_{n-1.0} = (\frac{o_{n-1.1} - 2n - 2}{2})(1 + o_{n-1.1} - 4o_{n-1.1}^2 - 14o_{n-1.1}^3 - 15o_{n-1.1}^4 - 15o_{n-1.1}^5 + 15o_{n-1.1}^2 sin^2 o_{n-1.1}) + \nu' \qquad (2.17),$$

where

$$\nu' = (1 + o_{n-1.1} - 4o_{n-1.1}^2 - 14o_{n-1.1}^3 - 15o_{n-1.1}^4 - 15o_{n-1.1}^5 + 15o_{n-1.1}^2 sin^2 o_{n-1.1}) \qquad (2.18).$$

Now look at (2.17) and (2.18); noticing (by property (2.1.0)) that $o_{n.1} = o_{n-1.1}$, then it becomes trivial to deduce that equalities (2.17) and (2.18) clearly say that

$$\nu_{n-1.0} = (\frac{o_{n.1} - 2n - 2}{2})(1 + o_{n.1} - 4o_{n.1}^2 - 14o_{n-1.1}^3 - 15o_{n.1}^4 - 15o_{n.1}^5 + 15o_{n.1}^2 sin^2 o_{n.1}) + \nu' \qquad (2.19),$$

where

$$\nu' = (1 + o_{n.1} - 4o_{n.1}^2 - 14o_{n.1}^3 - 15o_{n.1}^4 - 15o_{n.1}^5 + 15o_{n.1}^2 sin^2 o_{n.1}) \qquad (2.20).$$

Clearly

$$\nu_{n-1.0} = \nu_{n.0} + (1 + o_{n.1} - 4o_{n.1}^2 - 14o_{n.1}^3 - 15o_{n.1}^4 - 15o_{n.1}^5 + 15o_{n.1}^2 sin^2 o_{n.1}) \qquad (2.21),$$

by using equalities (2.11) and (2.19) and (2.20). It is immediate that equality (2.21) says that

$$\nu_{n-1.0} - \nu_{n.0} = (1 + o_{n.1} - 4o_{n.1}^2 - 14o_{n.1}^3 - 15o_{n.1}^4 - 15o_{n.1}^5 + 15o_{n.1}^2 sin^2 o_{n.1}) \qquad (2.22).$$

Look again at equality (2.15) and consider $\mu_{n-1.0}$; then using the expression of $\mu_{n.0}$ given by equality (2.12), it becomes trivial to decuce that

$$\mu_{n-1.0} = (\frac{o_{n-1.1} - 2(n-1) - 2}{2})(-1 - o_{n-1.1} - 5o_{n-1.1}^2 - 13o_{n-1.1}^3 - 12o_{n-1.1}^4 - 12o_{n-1.1}^5 + 12o_{n-1.1}^2 sin^2 o_{n-1.1})i \quad (2.23).$$

It is trivial to check (by elementary computation) that equality (2.23) is of the form

$$\mu_{n-1.0} = (\frac{o_{n-1.1} - 2n - 2}{2})(-1 - o_{n-1.1} - 5o_{n-1.1}^2 - 13o_{n-1.1}^3 - 12o_{n-1.1}^4 - 12o_{n-1.1}^5 + 12o_{n-1.1}^2 sin^2 o_{n-1.1})i + \mu' \quad (2.24),$$

where

$$\mu' = (-1 - o_{n-1.1} - 5o_{n-1.1}^2 - 13o_{n-1.1}^3 - 12o_{n-1.1}^4 - 12o_{n-1.1}^5 + 12o_{n-1.1}^2 sin^2 o_{n-1.1})i \quad (2.25).$$

Now look at (2.24) and (2.25); noticing (by property (2.1.0)) that $o_{n.1} = o_{n-1.1}$, then it becomes trivial to deduce that equalities (2.24) and (2.25) clearly say that

$$\mu_{n-1.0} = (\frac{o_{n.1} - 2n - 2}{2})(-1 - o_{n.1} - 5o_{n.1}^2 - 13o_{n.1}^3 - 12o_{n.1}^4 - 12o_{n.1}^5 + 12o_{n.1}^2 sin^2 o_{n.1})i + \mu' \quad (2.26),$$

where

$$\mu' = (-1 - o_{n.1} - 5o_{n.1}^2 - 13o_{n.1}^3 - 12o_{n.1}^4 - 12o_{n.1}^5 + 12o_{n.1}^2 sin^2 o_{n.1})i \quad (2.27).$$

Clearly

$$\mu_{n-1.0} = \mu_{n.0} + (-1 - o_{n.1} - 5o_{n.1}^2 - 13o_{n.1}^3 - 12o_{n.1}^4 - 12o_{n.1}^5 + 12o_{n.1}^2 sin^2 o_{n.1})i \quad (2.28),$$

by using equalities (2.12) and (2.26) and (2.27). It is immediate that equality (2.28) says that

$$\mu_{n-1.0} - \mu_{n.0} = (-1 - o_{n.1} - 5o_{n.1}^2 - 13o_{n.1}^3 - 12o_{n.1}^4 - 12o_{n.1}^5 + 12o_{n.1}^2 sin^2 o_{n.1})i \quad (2.29).$$

That being so, consider again equality (2.15) and look at $\nu_{n-1.1}$; then using the expression of $\nu_{n.1}$ given by equality (2.13), it becomes trivial to decuce that

$$\nu_{n-1.1} = -1 + 20o_{n-1.1}^2 - 25o_{n-1.1}^3 + o_{n-1.1}^4 + 5o_{n-1.1}^6 + (15o_{n-1.1}^3 - 15o_{n-1.1}^2) cos^2 o_{n-1.1} \quad (2.30).$$

Now look at (2.30); noticing (by property (2.1.0)) that $o_{n.1} = o_{n-1.1}$, then it becomes trivial to deduce that equality (2.30) clearly say that

$$\nu_{n-1.1} = -1 + 20o_{n.1}^2 - 25o_{n.1}^3 + o_{n.1}^4 + 5o_{n.1}^6 + (15o_{n.1}^3 - 15o_{n.1}^2) cos^2 o_{n.1} \quad (2.31).$$

Clearly

$$\nu_{n-1.1} = \nu_{n.1} \quad (2.32),$$

by using equalities (2.13) and (2.31). It is immediate that equality (2.32) says that

$$\nu_{n-1.1} - \nu_{n.1} = 0 \quad (2.33).$$

Finally, consider again equality (2.15) and look at $\mu_{n-1.1}$; then using the expression of equation $\mu_{n.1}$ given by equality (2.14), it becomes trivial to decuce that

$$\mu_{n-1.1} = (1 + 16o_{n-1.1}^2 - 20o_{n-1.1}^3 - o_{n-1.1}^4 + 4o_{n-1.1}^6 + (12o_{n-1.1}^3 - 12o_{n-1.1}^2) cos^2 o_{n-1.1})i \quad (2.34).$$

Now look at (2.34); noticing (by property (2.1.0)) that $o_{n.1} = o_{n-1.1}$, then it becomes trivial to deduce that equality (2.34) clearly says that

$$\mu_{n-1.1} = (1 + 16o_{n.1}^2 - 20o_{n.1}^3 - o_{n.1}^4 + 4o_{n.1}^6 + (12o_{n.1}^3 - 12o_{n.1}^2) cos^2 o_{n.1})i \quad (2.35).$$

Clearly

$$\mu_{n-1.1} = \mu_{n.1} \quad (2.36),$$

by using equalities (2.14) and (2.35). It is immediate that equality (2.36) says that

$$\mu_{n-1.1} - \mu_{n.1} = 0 \quad (2.37).$$

Using equalities (2.22) and (2.29) and (2.33) and (2.37), then it becomes trivial to see that

$$\nu_{n-1.0} - \nu_{n.0} + \mu_{n-1.0} - \mu_{n.0} + \nu_{n-1.1} - \nu_{n.1} + \mu_{n-1.1} - \mu_{n.1} = \rho_{n.0} + \rho'_{n.0} \quad (2.38),$$

where

$$\rho_{n.0} = (1 + o_{n.1} - 4o_{n.1}^2 - 14o_{n.1}^3 - 15o_{n.1}^4 - 15o_{n.1}^5 + 15o_{n.1}^2 sin^2 o_{n.1}) \quad (2.39),$$

and

$$\rho'_{n.0} = (-1 - o_{n.1} - 5o_{n.1}^2 - 13o_{n.1}^3 - 12o_{n.1}^4 - 12o_{n.1}^5 + 12o_{n.1}^2 sin^2 o_{n.1})i \quad (2.40).$$

It is trivial that equality (2.38) clearly says that

$$\nu_{n-1.0} + \mu_{n-1.0} + \nu_{n-1.1} + \mu_{n-1.1} - (\nu_{n.0} + \mu_{n.0} + \nu_{n.1} + \mu_{n.1}) = \rho_{n.0} + \rho'_{n.0} \quad (2.41).$$

Now look at (2.41), then using equalities (2.10) and (2.15), it becomes trivial to deduce that equality (2.41) clearly says that

$$\phi_{n-1.2} - \phi_{n.2} = \rho_{n.0} + \rho'_{n.0} \qquad (2.42).$$

That being so, consider equality (2.42) , then, using expressions of the couple $(\rho_{n.0}, \rho'_{n.0})$ given by equalities (2.39) and (2.40), it becomes very easy to check (by elementary computation) that equality (2.42) is of the form

$$\phi_{n-1.2} - \phi_{n.2} = c + 5yo_{n.1} + (-c + 4yo_{n.1})i \qquad (2.43),$$

$$where\, c = 1 + o_{n.1} + o_{n.1}^2 + o_{n.1}^3 \text{ and } y = -o_{n.1} - 3o_{n.1}^2 - 3o_{n.1}^3 - 3o_{n.1}^4 + 3o_{n.1}sin^2 o_{n.1} \qquad (2.44).$$

Now look at $z_{n.1}$ introduced in Recalls 2.1, then using the two equalities of (2.44), it becomes trivial to check (by elementary computation) that $\frac{c(\ -o_{n.1}^5+5o_{n.1}^2-4o_{n.1}+(3o_{n.1}-3o_{n.1}^2)cos^2 o_{n.1}\ )}{o_{n.1}^3+o_{n.1}^2+o_{n.1}+1} + y(o_{n.1}-1) = -z_{n.1}$, and clearly

$$\frac{c(\ -o_{n.1}^5 + 5o_{n.1}^2 - 4o_{n.1} + (3o_{n.1} - 3o_{n.1}^2)cos^2 o_{n.1}\ )}{o_{n.1}^3 + o_{n.1}^2 + o_{n.1} + 1} + y(o_{n.1} - 1) \equiv 0 \ mod(z_{n.1}),$$

by using the previous equality. Using the previous congruence along with (2.43) and (2.44), then it becomes very easy to deduce that

$$there\ exists\ (c, y) \in \mathcal{Z} \times \mathcal{R}\ such\ that, \qquad (2.45)$$

$$\phi_{n-1.2} - \phi_{n.2} = c + 5yo_{n.1} + (-c + 4yo_{n.1})i, \qquad (2.46)$$

$$and\ \frac{c(\ -o_{n.1}^5 + 5o_{n.1}^2 - 4o_{n.1} + (3o_{n.1} - 3o_{n.1}^2)cos^2 o_{n.1}\ )}{o_{n.1}^3 + o_{n.1}^2 + o_{n.1} + 1} + y(o_{n.1} - 1) \equiv 0 \ mod(z_{n.1}). \qquad (2.47).$$

Clearly $\phi_{n-1.2} - \phi_{n.2}$ tackles $z_{n.1}$ $\Big($use (2.45) and (2.46) and (2.47) and the definition of tackle introduced in Definition 2.1 $\Big)$. Property (2.1.1) follows and Proposition 2.1 immediately follows. □

The previous simple Proposition made, we are now ready to give the analytic simple proof of the Fermat composites problem.

3. The short proof of the Fermat composites problem. In this Section, the definitions of $\mathcal{FCO}(n)$, o_n and $o_{n.1}$ (see Definitions 1.1), the definition of relative integers (see Recalls 2.1), the definition of tackle (see Definition 2.1), and the definition of equation $\phi_{n.2}$ (see Definitions 2.2), are fundamental and crucial.

Now the following Theorem immediately implies the Fermat composites problem.

Theorem 3.1. *Let n be an integer $\geq F_5$ and let $o_{n.1}$ (see Abstract and Definitions for the meaning of F_5, and see Definitions* 1.1 *for the meaning of $o_{n.1}$); look at equation $\phi_{n.2}$ introduced in Definitions* 2.2. **If** $o_{n.1} \leq 2n + 2$, *then*

$$\phi_{n.2}\ tackles\ z_{n.1},$$

where $z_{n.1}$ is introduced in Recalls 2.1 *and is of the form*

$$z_{n.1} = 4o_{n.1}^5 - 7o_{n.1}^2 + 3o_{n.1} + (3o_{n.1} - 3o_{n.1}^2)sin^2 o_{n.1} + (3o_{n.1}^2 - 3o_{n.1})cos^2 o_{n.1}.$$

We are going to prove simply Theorem 3.1. But before, let us remark.

Remark 3.1. *Let n be an integer $\geq F_5$ and let $o_{n.1}$. We have the following three trivial properties.*
*(*3.1.0.*)* **If** $o_{n.1} \geq 2n + 4$, *then Theorem* 3.1 *is satisfied by $o_{n.1}$.*
*(*3.1.1.*)* **If** $o_{n.1} = 2n + 2$, *then Theorem* 3.1 *is satisfied by $o_{n.1}$.*
*(*3.1.2.*)* **If** $n \leq 2 + F_5$, *then Theorem* 3.1 *is satisfied by $o_{n.1}$*

Proof. Property (3.1.0) is trivial. Property (3.1.1) is immediate $\Big($**indeed** let n be an integer $\geq F_5$; observing (by the hypotheses) that $o_{n.1} = 2n + 2$, then

$$\phi_{n.2}\, tackles\ z_{n.1} \qquad (3.1),$$

by using Example.**2** of Definition 2.1. *(3.1)* clearly says that Theorem 3.1 is satisfied by $o_{n.1}$. Property (3.1.1) follows$\Big)$.

Property (3.1.2.) is immediate $\Big($ **indeed**, observing (by using property (1.1.0) of Remark 1.1) that $o_{n.1} > F_5^{F_5}$, and remarking (by the hypotheses) that $n \leq 2 + F_5$, then, using the previous two inequalities, it becomes trivial to deduce that

$$o_{n.1} > F_5^{F_5} > 6 + 3F_5 > 2n + 4 \qquad (3.2);$$

so

$$o_{n.1} > 2n + 4 \qquad (3.3),$$

by using *(3.2)*. Clearly Theorem 3.1 is satisfies by $o_{n.1}$, by using inequality *(3.3)* and property (3.1.0)$\Big)$.□

Using Remark 3.1, let us Remark.

Remark 3.2. *Suppose that Theorem* 3.1 *is false; then there exists an integer* $n \geq F_5$ *such that* $o_{n.1}$ *does not satisfied Theorem* 3.1. (*Proof.* Immediate.□)

From Remark 3.2, let us define

Definitions 3.1 *(Fundamental).* **(i).** We say that n is a *counter-example to Theorem* 3.1, if $n \geq F_5$ and if $o_{n.1}$ does not satisfied Theorem 3.1 (If Theorem 3.1 is false, then such a n exists, by using Remark 3.2) .
(ii). We say that n is a *minimum counter-example to Theorem* 3.1, if n is a counter-example to Theorem 3.1 with n minimum (If Theorem 3.1 is false, then such a n exists, by using **(i)**).

The previous simple remarks and definitions made, we now prove simply Theorem 3.1.

Proof of Theorem 3.1. Otherwise (we reason by reduction to absurd), let n be a minimum counter-example to Theorem 3.1 (such a n exists, by using Remark 3.2 and Definitions 3.1). We observe the following.
*Observation.*3.1.*i. Look at* n *(recall* n *is a minimum counter-example to Theorem* 3.1*), and let* $o_{n.1}$*. Then* $n > 2 + F_5$ *and* $o_{n.1} \leq 2n + 2$.

Clearly $n > 2 + F_5$ $\Big($ Otherwise $n \leq 2 + F_5$ and clearly Theorem 3.1 is satisfied by $o_{n.1}$ (use the previous inequality and property (3.1.2) of Remark 3.1); a contradiction, since in particular $o_{n.1}$ does not satisfied Theorem 3.1$\Big)$; and clearly $o_{n.1} \leq 2n+2$ $\Big($ Otherwise $o_{n.1} > 2n+2$; noticing that $o_{n.1}$ and $2n+2$ are even ($o_{n.1}$ is even (use the definition of $o_{n.1}$) and $2n+2$ is trivially even), then it becomes trivial to deduce that the previous inequality implies that $o_{n.1} \geq 2n+2+2$; so $o_{n.1} \geq 2n+4$ and clearly Theorem 3.1 is satisfied by $o_{n.1}$ (use the previous inequality and property (3.1.0) of Remark 3.1); we have a contradiction since $o_{n.1}$ does not clearly satisfied Theorem 3.1. Observation.3.1.i follows.
*Observation.*3.1.*ii. Look at* n *(recall* n *is a minimum counter-example to Theorem* 3.1*), and let* $o_{n.1}$*. Then*

$$\phi_{n.2} \textit{ does not tackle } z_{n.1}.$$

Immediate, since in particular, n is a counter-example to Theorem 3.1.
*Observation.*3.1.*iii. Look at* n*, and let* $o_{n.1}$*. Then*

$$o_{n.1} \leq 2n \textit{ and } o_{n.1} = o_{n-1.1}.$$

Firstly, we are going to show that $o_{n.1} \leq 2n$. *Fact:* $o_{n.1} \leq 2n$. Otherwise,

$$o_{n.1} > 2n; \qquad (3.4);$$

remarking that $o_{n.1}$ and $2n$ are even $\Big($ $o_{n.1}$ is even (use the definition of $o_{n.1}$) and $2n$ is trivially even$\Big)$, then inequality *(3.4)* immediately implies that $o_{n.1} \geq 2n+2$. Note (by using Observation.3.1.i) that $o_{n.1} \leq 2n+2$. Now using the previous two inequalities, then it becomes trivial to see that $o_{n.1} = 2n+2$; so Theorem 3.1 is satisfied by $o_{n.1}$ $\Big($ use the previous equality and property (3.1.1) of Remark 3.1$\Big)$, and we have a contradiction, since $o_{n.1}$ does not clearly satisfied Theorem 3.1. So

$$o_{n.1} \leq 2n \qquad (3.5).$$

Now we show that $o_{n.1} = o_{n-1.1}$. Indeed, using inequality *(3.5)* and property (1.1.2) of Remark 1.1 ,then it becomes trivial to deduce that $o_{n.1} = o_{n-1.1}$. Observation.3.1.*iii* follows.

Observation.3.1.*iv*. *Look at* n. *Now let* $\phi_{n.2}$ *(see Definitions* 2.2*), and via* $\phi_{n.2}$, *consider* $\phi_{n-1.2}$ $\big($ *this consideration gets sense, since* $n > 2 + F_5$ *(use Observation*.3.1.*i), and so* $n - 1 > 1 + F_5 > F_5\big)$. *Then* $\phi_{n-1.2} - \phi_{n.2}$ *tackles* $z_{n.1}$.

Indeed, observing (by Observation.3.1.*iii*) that $o_{n.1} \leq 2n$ and noticing (by Observation.3.1.*i*) that $n > 2 + F_5$, then using the previous two inequalities, it becomes trivial to deduce that all the hypotheses of Proposition 2.1 are satisfied, therefore, all the conclusions of Proposition 2.1 are satisfied; in particular property (2.1.1) of Proposition 2.1 is satisfied; consequently $\phi_{n-1.2} - \phi_{n.2}$ tackles $z_{n.1}$. Observation.3.1.*iv* follows.

Observation.3.1.*v*. *Look at* n *(recall* n *is a minimum counter-example to Theorem* 3.1*). Now let* $\phi_{n.2}$ *(see Definitions* 2.2*), and via* $\phi_{n.2}$, *consider* $\phi_{n-1.2}$ $\big($ *this consideration gets sense, since* $n > 2 + F_5$ *(use Observation*.3.1.*i), and so* $n - 1 > 1 + F_5 > F_5\big)$. *Then* $\phi_{n-1.2}$ *tackles* $z_{n.1}$.

Indeed, look at n $\big($recall n is a minimum counter-example to Theorem 3.1$\big)$, and via n, consider $n-1$ $\big($this consideration gets sense, since $n > 2 + F_5$ (use Observation.3.1.*i*), and so $n - 1 > 1 + F_5 > F_5\big)$. Observing (by Observation.3.1.*iii*) that $o_{n.1} = o_{n-1.1}$ and $o_{n.1} \leq 2n$, then, by the minimality of n, it becomes trivial to deduce that $n-1$ is not a counter-example to Theorem 3.1 and $\phi_{n-1.2}$ tackles $z_{n-1.1}$; the previous clearly says that $\phi_{n-1.2}$ tackles $z_{n.1}$ (observe that $o_{n.1} = o_{n-1.1}$ (use Observation.3.1.*iii*)), and the previous equality immediately implies that $z_{n.1} = z_{n-1.1}$ (use the fact that $o_{n.1} = o_{n-1.1}$ and the definition of $z_{n.1}$ introduced in Recalls 2.1)). Observation.3.1.*v* follows.

Observation.3.1.*vi*. *Look at* n *and let let* $\phi_{n.2}$. *Then* $\phi_{n.2}$ *tackles* $z_{n.1}$.

Indeed, using Observation.3.1.*iv* and the definition of *tackle* (see Definition 2.1), then it becomes trivial to deduce that

$$\textit{there exists } (c, y) \in \mathcal{Z} \times \mathcal{R} \textit{ such that} \qquad (3.6)$$

$$\phi_{n-1.2} - \phi_{n.2} = c + 5yo_{n.1} + (-c + 4yo_{n.1})i \qquad (3.7),$$

$$\textit{and} \quad \frac{c(\ -o_{n.1}^5 + 5o_{n.1}^2 - 4o_{n.1} + (3o_{n.1} - 3o_{n.1}^2)cos^2 o_{n.1}\ )}{o_{n.1}^3 + o_{n.1}^2 + o_{n.1} + 1} + y(o_{n.1} - 1) \ \equiv\ 0 \ mod(z_{n.1}) \qquad (3.8).$$

That being so, using Observation.3.1.*v* and the definition of *tackle* (see Definition 2.1), then it becomes trivial to deduce that

$$\textit{there exists } (c', y') \in \mathcal{Z} \times \mathcal{R} \textit{ such that} \qquad (3.9)$$

$$\phi_{n-1.2} = c' + 5y'o_{n.1} + (-c' + 4y'o_{n.1})i \qquad (3.10),$$

$$\textit{and} \quad \frac{c'(\ -o_{n.1}^5 + 5o_{n.1}^2 - 4o_{n.1} + (3o_{n.1} - 3o_{n.1}^2)cos^2 o_{n.1}\ )}{o_{n.1}^3 + o_{n.1}^2 + o_{n.1} + 1} + y'(o_{n.1} - 1) \ \equiv\ 0 \ mod(z_{n.1}) \qquad (3.11).$$

Now using *(3.6)* and *(3.7)* and *(3.8)* and *(3.9)* and *(3.10)* and *(3.11)*, then it becomes trivial to deduce that

$$\phi_{n.2} = (c' - c) + 5(y' - y)o_{n.1} + (\ -(c' - c) + 4(y' - y)o_{n.1}\ )i \qquad (3.12),$$

$$\textit{where} \quad \frac{(c' - c)(\ -o_{n.1}^5 + 5o_{n.1}^2 - 4o_{n.1} + (3o_{n.1} - 3o_{n.1}^2)cos^2 o_{n.1}\ )}{o_{n.1}^3 + o_{n.1}^2 + o_{n.1} + 1} + (y' - y)(o_{n.1} - 1) \ \equiv\ 0\, mod(z_{n.1}) \qquad (3.13),$$

$$\textit{and where } (c' - c, y' - y) \in \mathcal{Z} \times \mathcal{R} \qquad (3.14).$$

(3.12) and *(3.13)* and *(3.14)* clearly say that

$$\textit{there exists } (c'', y'') \in \mathcal{Z} \times \mathcal{R} \textit{ such that} \qquad (3.15)$$

$$\phi_{n.2} = c'' + 5y''o_{n.1} + (-c'' + 4y''o_{n.1})i \qquad (3.16),$$

$$where\ \frac{c''(\ -o_{n.1}^5+5o_{n.1}^2-4o_{n.1}+(3o_{n.1}-3o_{n.1}^2)cos^2o_{n.1}\ )}{o_{n.1}^3+o_{n.1}^2+o_{n.1}+1}+y''(o_{n.1}-1)\ \equiv\ 0\ mod(z_{n.1})\quad (3.17),$$

$$and\ where\ c''=c'-c\,and\ y''=y'-y \quad (3.18).$$

Clearly $\phi_{n.2}$ tackles $z_{n.1}$ (use *(3.15)* and *(3.16)* and *(3.17)* and *(3.18)*). Observation.3.1.vi follows.

These simple observations made, then it becomes trivial to see that Observation.3.1.vi clearly contradicts Observation.3.1.ii. Theorem 3.1 follows.□

Now the Fermat composites problem directly results from the following Theorem.

Theorem 3.2 (the using of Example.**3** of Section.**2**). *For every integer* $n \geq F_5$*, we have* $o_{n.1} > 2n$.
Proof. Otherwise $\Big($ we reason by reduction to absurd$\Big)$, let n be a minimun counter-example and let $o_{n.1}$; then

$$o_{n.1} \leq 2n \quad (3.19),$$

and we observe the following.
*Observation.*3.2.1. $n > 2 + F_5$.

Otherwise $n \leq 2 + F_5$; now observing (by using property (1.1.0) of Remark 1.1) that $o_{n.1} > F_5^{F_5}$ and using the previous two inequalities, then it becomes trivial to deduce that $o_{n.1} > F_5^{F_5} > 2n + 4$; so $o_{n.1} > 2n + 4$ and the previous inequality contradicts inequality *(3.19).* Observation.3.2.1 follows.
*Observation.*3.2.2. $o_{n.1} = o_{n-1.1}$.

Indeed, remarking (by *(3.19)*) that $o_{n.1} \leq 2n$, then, using the previous inequality and property (1.1.2) of Remark 1.1, it becomes trivial to deduce that $o_{n.1} = o_{n-1.1}$ Observation.3.2.2 follows.
*Observation.*3.2.3. $o_{n.1} = 2n$.

Indeed, look at n, and via n, consider $n - 1$ $\Big($ this consideration gets sense, since $n > 2 + F_5$ (by Observation.3.2.1), and therefore $n - 1 > 1 + F_5 > F_5\Big)$. Then, by the minimality of n, $n - 1$ is not a counter-example to Theorem 3.2; consequently $o_{n-1.1} > 2(n - 1)$ and the previous inequality clearly says that

$$o_{n-1.1} > 2n - 2 \quad (3.20).$$

Note that

$$o_{n.1} = o_{n-1.1} \quad (3.21),$$

by Observation.3.2.2. Now using *(3.20)* and *(3.21)*, then it becomes trivial to deduce that

$$o_{n.1} > 2n - 2 \quad (3.22).$$

Noticing that $o_{n.1}$ and $2n-2$ are even ($o_{n.1}$ is even (use the definition of $o_{n.1}$) and $2n-2$ is trivially even), then it becomes trivial to deduce that inequality *(3.22)* implies that $o_{n.1} \geq 2n - 2 + 2$; the previous inequality clearly says that

$$o_{n.1} \geq 2n \quad (3.23).$$

Clearly $o_{n.1} = 2n$ (use inequalities *(3.23)* and *(3.19)*). Observation.3.2.3 follows.
*Observation.*3.2.4*(the using of Example.***3** *of Section.***2***). Look at* $o_{n.1}$ *and consider* $\phi_{n.2}$ *(see Definitions* 2.2*). Then* $\phi_{n.2}$ *does not tackle* $z_{n.1}$*, where* $z_{n.1}$ *is introduced in Recalls* 2.1.

Indeed, observing (by Observation.3.2.3) that $o_{n.1} = 2n$ and using Example.**3** of Definition 2.1, then it becomes trivial to deduce that $\phi_{n.2}$ does not tackle $z_{n.1}$. Observation.3.2.4 follows.

These simple observations made, look at $o_{n.1}$ and consider $\phi_{n.2}$ (see Definitions 2.2); observing that $o_{n.1} = 2n$ (by Observation.3.2.3) and remarking that $n > 2 + F_5$ (by Observation.3.2.1), then using the previous, it becomes immediate that all the hypotheses of Theorem 3.1 are satisfied, therefore, the conclusion of Theorem 3.1 is satisfied; consequently

$$\phi_{n.2} \text{ tackles } z_{n.1} \qquad (3.24).$$

(3.24) clearly contradicts Observation.3.2.4. Theorem 3.2 follows.□

Theorem 3.2 immediately implies the Fermat composites problem.

Theorem 3.3 *(The Proof of the Fermat composites problem). There are infinitely many Fermat composites.* [*Proof.* Observe [by using Theorem 3.2] that

$$\text{For every integer } n \geq F_5 \text{ we have } o_{n.1} > 2n \qquad (3.25);$$

consequently, there are infinitely many Fermat composites, by using *(3.25)* and Proposition 1.1. □]

Epilogue. *Our simple article clearly shows that divisibility helps to characterize Fermat composites as we did in [7] and [8] and [9] and [10], and elementary arithmetic congruences coupled with trivial arithmetic calculus help to give a simple analytic proof of problem posed by the Fermat composites.*

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