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Extinction Growth Model

Daniel Ochieng Achola [1 2]

Abstract

Magin and Cock in their roan antelopes recovery plan considered the effect of poaching in their model. Okseandal and Lungu developed a growth model in a crowded environment by introducing randomness in their differential equation via additional noise term. Inbreeding in small population have substantial impact in population growth rate. In this paper, we develop a mathematical model that incorporates genetic defect in estimating the growth rate of roan antelopes in Ruma Park, Kenya.

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1 Introduction

Verhust in his classical logistics growth model

$$\frac{dP_t}{dt} = fP_t = \lambda P_t\left(1 - \frac{P_t}{M}\right) \tag{1}$$

with 0 and M the equilibrium levels of the equation and letting P_0 be the initial value corresponding to the equilibrium stable solution

$$P_t = \frac{MP_0e^{-\lambda t}}{(M-P_0)+P_0e^{\lambda(t-t_0)}} \quad : P_0 \neq M \;\text{ and }\; (M-P_0)+P_0e^{\lambda(t_*-t_0)} \neq 0 \tag{2}$$

[1]Kabarak University. e-mail: dachola@kabarak.ac.ke
[2]

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whenever t_* is a point of jump discontinuity with

$$e^{\lambda(t_*-t_0)} = \left(\frac{P_0 - M}{P_0}\right)$$

Since $P_0 > M$ and $P_0 \neq 0$ the RHS > 0 therefore a positive logarithm (raising to the log).

$$\lambda(t_* - t_0) = \ln\left(\frac{P_0 - M}{P_0}\right) > 0$$

$$t_* - t_0 = \frac{1}{\lambda}\ln\left(\frac{P_0 - M}{P_0}\right) > 0 \quad (3)$$

When $t_0 = 0$ then

$$t_* = \frac{1}{\lambda}\ln\left(\frac{P_0 - M}{P_0}\right) > 0$$

We solve for t. According to Greisen [3] analysis of Voltera is insightful but has no intra-specific competition i.e natural resources has no diminishing returns. Several variations of Verhust logistics growth models have been modified for resource management. A case in point, Shaffer who modeled fish population

$$\frac{dP_t}{dt} = \lambda\left(1 - \frac{P_t}{M}\right) - EP_t \quad (4)$$

Where E is a positive constant that measures total effort made to harvest given species of fish. Genetic drift is the cumulative and non-adaptive fluctuation in allele frequencies resulting from random sampling of genes in each generation that can impede or accelerate wildlife population [7]. Inbreeding is not strictly a component of genetic drift but correlated with it has been documented to cause loss of fitness and reduces the ability of the population to adapt to future changes in the environment [1, 12].

Gilpin [4] described these synergistic destabilizing effects of stochastic process on small wildlife population as extinction vortices. Most population growth processes are inherently stochastic yet much theoretical analysis involves deterministic models with the assumption that biological systems consist of large collection of individuals in the same ecological interaction. This assumption implies that dynamics of measure (mean) is sufficient description and ignores the influence of variance [16].

Oksendal & Lungu [9] proposed a stochastic logistic model in estimating population growth at any time. We have worked along this line and derived a mathematical model that estimates population growth of roan antelopes by incorporating genetic defect that was not considered by Magin & Kock [5].

2 Preliminaries

Randomness is an intrinsic property of biological observation which makes deterministic models incomplete.

$$P_t = \lambda P_t\left(1 - \frac{P_t}{M}\right)dt \tag{5}$$

where

λ is the intrinsic growth rate,

P_t is the population at any time t and

M is the carrying capacity.

However, for Pivato if some intrinsic randomness in the system which makes perfect prediction of the future impossible but strong trends or correlation exists, the mathematical structure used to model this phenomenon is stochastic process [10]. Stochastic process consist of space, time and probability measure

Definition 2.1. *If Ω is a given set, then a σ algebra F on Ω is a family F of subset of Ω with the following properties.*

(i)$\Phi \in F$

(ii)$f \in F \Rightarrow f^c \in F$

where $f^c = \Omega/F$ is the compliment of F in Ω

(iii)

$$A_1, A_2, \cdots \in F \Rightarrow \bigcup_{i=1}^{\infty} A_i \in F$$

The pair (Ω, F) is called a measurable space.

A probability measure P on a measurable space (Ω, F) is a function $P : F \to [0, 1]$ such that

(a) $P(\Phi) = 0, P(\Omega) = 1$

(b) If $A_1, A_2, \cdots \in F$ and

$$\left(A_i\right)_{i=1}^{\infty} \text{ is a disjoint i.e.} \left(A_i \bigcap A_j = \Phi \quad : i \neq j\right)$$

then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

The triple (Ω, F, P) is called a probability space. It is called a complete probability space if F contains all the subsets of G of Ω with P outer measure zero.

$$P^*(G) = \inf\{P(F) \quad : f \in F, G \subset F\} = 0$$

Given any family μ of subsets of Ω there is a smallest σ algebra H_μ containing μ namely

$$H_\mu = \bigcap\{H : H\ \sigma \quad \text{algebra of } \ \Omega, \mu \subset H\}$$

Let (Ω, F, P) denote a complete given probability space, then a random variable X is F measurable function $X : \Omega \to \mathbb{R}^n$ Every random variable induces probability measure μ_x on $\mathbb{R}^n$ defined by $\mu_x(B) = P(x^{-1}(B))$, μ_x is the distribution of $\mathbb{X}$. If

$$\int_\Omega |X(\omega)| dP(\omega) < \infty$$

then

$$E[X] = \int_\Omega X(\omega) dP(\omega) = \int_{R^n} x d\mu_x$$

where x is called the expectation of $X(w.r.t.P)$

Definition 2.2. *A stochastic process is a parameterized collection of random variables $\{X_t\}_{t\in T}$ and defined on probability space (Ω, F, P) and assuming values in $\mathbb{R}^n$. The parameter space T is usually half line $[0, \infty)$ but may belong to $[a, b]$ the non-negative integers and even subsets of R^n for $n \geq 1$ such that for each $t \in T$ fixed we have a random variable $W \to X_t(\omega) : \omega \in \Omega$ and on fixing $\omega \in \Omega, t \in T$ which is called the path of X_t*

For clarity $X_t \equiv X(t)$.

A stochastic process $X = \{X(t), t \in T\}$ is a collection of random variables. For each T in the index set $T, X(t)$ is a random variable with t interpreted as time and X_t the state of the process at a time t [14]. If we let X be some set, time for some other set and we let W be some σ-algebra on X the W measurable stochastic process on the state space X over time T is a probability measure W [10]

Stochastic processes are sequences of events governed by probabilistic laws. These systems occupy one state at a given time and could make transition

probabilities from one state to another. The set X of possible status may be finite or infinite depending on application. X consist of discrete elements X_i for $i = 0, 1, 2 \ldots$ with element X_i being possible states of the systems at any time t.
The probability $P_{i,j}(t)$ of the system making transition from the state i to j in the interval time t is the conditional probability defined as

$$P_{i,j}(t) = Pr\{X_{t_{0+1}}/X_{t_0} = X_i\} \tag{6}$$

where X_{t_0} is the state of the state of the system at the time t_0. The index set T is a countable set and X discrete time stochastic process or continuous time stochastic if it forms a continuum.

Definition 2.3. *A discrete time stochastic process is the probability measure on $(X^+, \bigotimes_n \in T)$,[31]. Discrete time stochastic processes are ranked in increasing order of complexity.*
This hierarchy follows either Bernoulli or Markov processes. Discrete time processes can be demonstrated by random walks with probability p of a particle moving to the right and probability $[(p-1) = q]$ of particle moving to the left. Let $P_{i,j}$ be the transition probability then

$$P_{i,j+1} = P = 1 - P_{i,j-1} : i = \pm 1, \pm 2, \pm 3, \ldots$$

suppose for abitrary time i, x in a random variable X_i takes $p = 1, q = -1$ and X_i are independent and identically distributed (iid) with identity function.

$$\rho\delta(x-1) - (1-q)d(x+1)$$

$$E[X] = 2p - 1$$

and

$$Var[X] = 4p(1-p)$$

If the n^{th} partial sum of the random variable

$$Y_n = X_1 + X_2 +, \ldots X_n = \sum_{i=1}^{n} X_i$$

Then the sequence for the random variable

$$\{Y_1, Y_2, \ldots Y_n\}$$

is the random walk with the probability distribution

$$E[Y_n] = n(2p - 1)$$

and

$$Var[Y_n] = 4npq$$

at stage n. *If we let* μ *and* σ^2 *be the mean and variance respectively then*

$$E[Y_n] = n\mu$$

and

$$Var[Y_n] = n\sigma^2$$

Definition 2.4. *Let* X *be some set and time* t *be some open set and closed interval in* $\mathbb{R}$ *representing an interval of time and* W *be some* σ *algebra, then* W *be some measurable continuous time stochastic process on state space* X *over time interval* T *is the probability measure* W *Continuous time stochastic process* $\{X(t), t \geq 0\}$ *has independent increments if* $\forall t_0 < t_1 < t_n$ *the random variables*

$$X(t_1) - X(t_0), X(t_2) - X(t_1), \dots X(t_n - X(t_{n-1}))$$

are independent.
They may make stationary increments if $X(t+s) - X(t)$ *has distribution values* $\forall t$ *i.e. the distribution only depends on* s. *This implies that for* n *time points the random variables set*

$$\{X(t_1), X(t_2), \dots X(t_n)\}$$

and

$$\{X(t_1 + s), X(t_2 + s), \dots X(t_n + s)\}$$

has the same joint probability distribution thus

$$E[E(t)] = E[X(t + s)]$$

Markov process is a continuous time $X = \{X(t), t \geq 0\}$ *with the*

$$Pr\{X(t) \leq x | X(\mu), \mu \in [0, s] = Pr[X(t)] \leqslant x | X(s)\}$$

Markov processes are stochastic processes for which all its future knowledge is summarized in current value. Examples of these processes are Brownian motion, stable processes, Poisson processes and even Levy processes. We can therefore ascertain that stochastic processes are variable with both the expected variable term (drift term) and random term (diffusion term).
The drift-coefficient term, models dominant actions while diffusion-coefficients represents randomness along the dominant curve. Roan antelope population growth varies in random number and represents stochastic process.

3 Brownian Motion and Stochastic Differential Equations

An irregular movement of pollen grains suspended in water as was observed by a botanist Robert Brown in 1828 has a wide range of application. Nobert Wiener came up with a concise and rigorous mathematical definition of Brownian motion, sometimes called Weiner Process.

Definition 3.1. *A Brownian motion or Wiener process is a stochastic process* $\xi(t) \geq 0$ *satisfying*

(i)$\xi(0) = 0$

(ii) For any $0 \leq t_o < t_1, \ldots < t_n$ *the random variables*

$\xi(t_k), -x(t_k)(1 \leq k \leq n)$ *are independent*

(iii) If $0 \leq s \leq t, x(t) - x(s)$ *is normally distributed with*

$$E(P(t) - P(s)) = (t - s)\mu E(\xi(t) - \xi(s)^2) = (t - s)\sigma^2,$$

where μ *and* σ *are constants,* $\sigma \neq 0$

If $\xi(t)$ is a Brownian motion,then μ is the drift and σ^2 is the variance. Brownian motion can be a Weiner process $dW = \varepsilon\sqrt{dt}$: ε is a random drawn from standard normalized if $\mu = 0$ and $\sigma^2 = 1$ any continuous time process with stationary independent increments and can be proved to be Brownian motion. Brownian motions are used in models that resemble random movements of particles. A (μ, σ) Brownian motion $\xi = \{\xi(t), t \geq 0\}$ can be expressed as a Weiner process i.e

$$\xi(t) = \mu t + \sigma W_t$$

and a normal variable with mean of zero and a variance of one. The values of dW for any two intervals are independent such that small infinite change can be written as $\Delta W_t = \xi\sqrt{\Delta t}$ adding up each of those intervals, we obtain

$$W_t - W(0) = \lim_{t\to 0}\left\{\sum_{i=1}^{n} \varepsilon_i \sqrt{\Delta t}\right\}$$

One dimensional Weiner process has $\xi(t)$ determined by the stochastic differential equation(SDE) of the form of

$$d\xi(t) = \mu dt + \sigma dW_t \quad : P(0) = P_t, \tag{7}$$

where μ (drift rate) and σ standard deviation.

Thus $d\xi(t)$ is the sum of the deterministic term dt and the stochastic term $(dW_t)$and in the short term interval $[t_{i-1}, t]$ and the increase may be given by

$$\xi_i(t) - \xi_{i-1}(t) = \mu \int_{i-1}^{i} dt + \sigma \int_{i-1}^{i} dW_t \tag{8}$$

With a general solution of the form

$$\xi(t) = \xi_{i-1}(t_{i-1}) + \mu(t_i - t_{i-1}) + \sigma(W(t_i) - W(t_{i-1})) \tag{9}$$

and in particular if the interval is [0,1] the equation (8) becomes

$$\xi(t) = \xi_0 + \mu \int_0^1 dt + \sigma \int_0^1 dW_t \tag{10}$$

whose solution is

$$\xi_t = \xi_0 + \mu t + \sigma W_t \tag{11}$$

with $\xi(0) = 0$ and $\lambda W(0) = 0$

A generalized Weiner process with non-constant coefficient

$$d\xi = \mu(\xi, t)dt + \sigma(\xi, t)dW_t \tag{12}$$

where $\mu(\xi, t)$ and $\sigma(\xi, t)$ are functions of variable ξ and time t is called Ito's process if it solves the equation

$$\xi(t) = \xi_0 + \int_0^t \mu(\xi(t), t)dt + \int_0^t \sigma(\xi(t), t)dW_t : t \geq 0 \tag{13}$$

where ξ_0 is the initial value, $\mu(\xi(t),t)$ is the drift term and $\sigma(\xi(t),t)$ is the diffusion term. A special type of Ito's with linear coefficient is the geometric Brownian motion (gBm) and has the stochastic differential equation of the form

$$d\xi(t) = \mu\xi(t)dt + \sigma\xi(t)dW_t \quad : \mu > 0, \sigma > 0, \tag{14}$$

where μ is the mean growth rate and σ is the rate of diffusion. Equation (14) can be expressed as a growth function

$$\frac{d\xi(t)}{\xi(t)} = \mu dt + \sigma dW_t, \quad \xi(0) = \xi_t \tag{15}$$

over infinitely short time interval $(t, t + \Delta t)$.
Solutions to equation (15) can not be obtained from standard Reinman Calculus formula for total derivative. If we let $f(x,t)$ be a continuous function with $(x,t) \in \mathbb{R} \times [0,\infty)$ together with its derivatives f_t, f_x, f_{xx} then the process $f(\xi(t),t)$ has the SDE (16). Ito achieved a rigorous treatment for integrating such Weiner like differential equation,thus Ito calculus,[13].

The solution to equation (15) is the stochastic differential equation

$$\begin{aligned} df(\xi(t),t) &= [f_t(\xi(t),t) + f_x(\xi(t),t)\mu(t) + \frac{1}{2}(\xi(t),t)b^2(t)]dt + \\ &\quad f_x(\xi(t),t)\sigma(t)dW_t \end{aligned} \tag{16}$$

This is called Ito's formula. It is noticeable that if $W(t)$ were continuously differentiable in t then by Reinman calculus the term $\frac{1}{2}f_{xx}b^2dt$ would not appear.

Proof. See Friedman [2] □

Theorem 3.2. *Let* $d\xi_i(x) = \mu_i(t)dt + \sigma_i(t)d\xi \quad : 1 \le i \le m$ *and let* $f(x_i, \dots x_m, t)$ *be a continuous function in* (x,t) *where* $x = (x_i, \dots x_m) \in R^m, t \ge 0$ *together with its first* t *derivative and second* x *derivative then* $f(\xi_i(t), \dots \xi_m, t)$ *stochastic differential given by [2]*

$$\begin{aligned} df(X(t),t) &= \Bigg[f_t(X(t),t) + \sum_{i=1}^{m} f_{x_i}(X(t),t)\mu_i(t) + \\ &\quad \frac{1}{2}\sum_{i,j=1}^{m} f_{x_i x_j}(X(t),t)\sigma_i(t)\sigma_j(t)\Bigg]dt + \\ &\quad \sum_{i=1}^{m} f_{x_i}(X(t),t)\sigma_i(t)dW_t \end{aligned} \tag{17}$$

where $X(t) = (\xi_i(t), \dots \xi_m(t))$
Equation (17) is the Ito's formula. From theorem (6), equation (17) the geometric Brownian motion (gBm) is given by

$$d\xi(t) = \mu\xi(t)dt + \sigma\xi(t)dW_t, \tag{18}$$

where $\mu\xi(t)dt$ is the drift and $\sigma\xi(t)dW_t$ is the diffusion term $dW_t = \varepsilon\sqrt{dt}$
Dividing both sides of equation (18) by $\xi(t)$, we obtain

$$\frac{d\xi(t)}{\xi(t)} = \mu dt + \sigma dW_t \tag{19}$$

and in order to get the strong solution of equation (19) we let $f(\xi(t), t)$ be a function of ξ and t twice differentiable in ξ and once in t such that

$$f(\xi(t), t) = \ln \xi(t)$$

Note

$$\frac{d\xi(t)}{\xi(t)} = \mu dt + \sigma dW_t$$

suggests the nature of $f(\xi(t), t)$ differentiating $f(\xi(t), t)$ twice with respect to ξ and once with respect to t gives

$$\frac{d(\xi(t), t)}{d\xi} = \frac{1}{\xi} \quad \frac{\partial^2(\xi(t), t)}{d\xi^2} = \frac{-1}{\xi^2} \quad \frac{\partial(\xi(t), t)}{\partial t} = 0$$

and by equation (16) we have integral in the form

$$df(\xi(t), t) = d(\ln \xi(t)) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\varepsilon\sqrt{t} \tag{20}$$

Equation (20) follows a generalized Weiner process with the drift rate $\left(\mu - \frac{\sigma^2}{2}\right)$ and diffusion coefficient σ,which are constants. The distribution of this process is given by

$$df(\xi(t), t) \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)dt, \sigma\sqrt{dt}\right)$$

or

$$\partial\left(\ln \xi(t) \sim N\left(\mu - \frac{\sigma^2}{2}\right)dt, \sigma\sqrt{dt}\right)$$

whose solution over the interval (t_{i-1}, t_i) is given by

$$\ln \xi(t) = \ln \xi(t_{i-1}) + \left(\mu - \frac{\sigma^2}{2}\right)(t_{i-1}, t_i) + \sigma\xi_i(\sqrt{t_{i-1}, t_i}) \tag{21}$$

Moreover, on putting like terms together, we obtain

$$\ln\left(\frac{\xi(t_i)}{\xi(t_{i-1})}\right) = \left(\mu - \frac{\sigma^2}{2}\right)(t_{i-1}, t_i) + \sigma\varepsilon(\sqrt{(}t_{i-1}, t_i) \tag{22}$$

And in considering the interval $(0, 1)$ then equation (20) becomes

$$\ln \xi(t) = \ln \xi_0 + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\varepsilon\sqrt{t} \quad \xi(0) = \xi_0 > 0 \tag{23}$$

Thus $\ln \xi(t)$ is normally distributed for any time t with the mean given by $\ln \xi_0 + \left(\mu - \frac{\sigma^2}{2}\right)$ and variance by $\sigma^2 t$ and the change in logarithm of the population size in the interval $(0, 1)$ results in

$$\ln \xi(t) - \ln \xi_0 = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\varepsilon\sqrt{t} \tag{24}$$

with the corresponding distribution given by

$$\ln \xi(t) - \ln \xi_0 \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma\sqrt{t}\right)$$

From equation (24) the strong solution becomes

$$\xi(t) = \xi_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\varepsilon\sqrt{t}\right] \tag{25}$$

which has the log-normal distribution given by

$$\xi(t) \sim \text{ log-normal}\left(\xi_0 \exp \mu t, \xi_0\sqrt{\exp(2\mu t), \exp(\sigma^2 t)^{-1}}\right)$$

such that if $\sigma = 0$ then equation (24) becomes

$$\xi(t) = \xi_0 \exp(\mu t)$$

Thus $\xi(t)$ has the exponential growth with the expectation $\xi_0 \exp(\mu t)$ and variance zero.

4 Extinction Growth Model Equation

In addition to competition for resources and predation. We consider the genetic defect on the population growth rate for the roan antelopes. From the Verhulst logistic growth rate equation (5) and adding genetic growth component to the logistic growth model we have

$$dP_t = \lambda P_t\left(1 - \frac{P_t}{M}\right)dt - \Psi(P_t), \tag{26}$$

where

λ is the growth ratio

P_t is the population at time t

M is the carrying capacity

$\Psi(P_t)$ is the function of P_t representing genetic defect

Letting $\Psi(P_t) = \gamma$ a constant then equation (26) becomes representing genetic defect

$$\frac{dP_t}{dt} = \lambda P_t\left(1 - \frac{P_t}{M}\right) - \gamma \tag{27}$$

And equating equation (27) to zero we obtain

$$\lambda P_t^2 - \lambda M P_t + \gamma M = 0 \tag{28}$$

whose solution is given by

$$P_t = \frac{\lambda M \pm \sqrt{(\lambda^2 M^2 - 4\lambda\gamma M)}}{2\lambda} \quad : P(0) = P_0 \tag{29}$$

The nature of solution of equation (29) depends on the genetic defect γ such that

$\gamma > \frac{\lambda M}{4}$ there is no real valued function implying genetic defect rate leads to extinction,

$\gamma = \frac{\lambda M}{4}$ has unique solution thus absolute growth rate in the absence of genetic defect and

$\gamma < \frac{\lambda M}{4}$ has positive growth rate with genetic defect.

Suppose we have a genetic defect at the rate proportional to P_t and if we let $\Psi = \gamma P_t dt$ then equation (26) becomes

$$\frac{dP_t}{dt} = \lambda P_t\left(1 - \frac{P_t}{M}\right) - \gamma P_t \tag{30}$$

Integrating equation (30) and solving for P_t we obtain the solution

$$P_t = \frac{(\lambda M - \gamma)P_0}{[\lambda(M - P_0) - \gamma]e^{-(\lambda M - \gamma)t} + \lambda P_0} \quad : P(0) = P_0$$

As $t \to \infty, P_t \to P_0$ and $t \to \infty, P_t \to \frac{(\lambda-\gamma)M}{\lambda}$ with the following steady states

$$P_t = 0, \quad P_t = \frac{(\lambda - \gamma)M}{\lambda}$$

Stochastic models are probabilistic in structure. This helps in solving the effects of uncertainty in ecological models. Hence, analysis of systems with white noise gives better results. If we consider population growth process

$$\frac{1}{P_t}\frac{dP(t)}{dt} = \lambda(M - P_t)$$

adding noise to the continuous growth process above, we obtain

$$\frac{1}{P_t(M - P_t)}\frac{dP_t}{dt} = \lambda dt + \text{noise} \tag{31}$$

If noise$= \sigma dW_t = \sigma\varepsilon\sqrt{dt}, \varepsilon \sim N(0,1).$, equation (31) can be written as

$$\frac{1}{P_t}\frac{dP_t}{(M - P_t)} = \lambda dt + \sigma dW_t, \quad M \neq P_t \tag{32}$$

On making dP_t the subject of the formula, we obtain the logistic stochastic differential equation

$$dP_t = \lambda P_t(M - P_t)dt + \sigma P_t(M - P_t)dW_t \tag{33}$$

with the distribution

$$\left[dP_t \sim N(\lambda P_t)dt, \sigma P_t(M - P_t)\sqrt{dt}\right]$$

On using the variable

$$Y(t) = \log\left(\frac{P(t)}{|M - P(t)|}\right) \quad M \neq P_t$$

and simplifying equation (32) we obtain

$$dY = \lambda M dt + \sigma M dW_t \tag{34}$$

Equation (34) is the generalised Weiner process with $\lambda M dt$ as drift and $\sigma M dt$ as variance. Equation (34) has the explicit solution

$$Y(t) = Y(0) + \lambda M(t - t_0) + \sigma M W_t, \quad W_0 = 0 \tag{35}$$

If we let

$$Y(t) = \log\left(\frac{P_t}{M - P(t)}\right) \text{ and } Y(0) = \left(\frac{P(0)}{M - P(0)}\right)$$

Equation (34) becomes

$$\log\left(\frac{P(t)}{M - P(t)}\right) = \log\left(\frac{P(0)}{M - P(0)}\right) + \lambda M(t - t_0) + \sigma M W_t \tag{36}$$

and making P_t the subject of the formula we have the Verhulst Logistic Brownian motion

$$P_t = \frac{MP_0}{(M - P_0)e^{-\{\lambda M(t-t_0)+\sigma M W_t\}} + P_0} \quad : P(0) = P_0 \tag{37}$$

Considering roans resources whose population P_t varies randomly due to natural factors (e.g predation, diseases) according to autonomous diffusion process

$$dP_t = \lambda P_t(M - P_t)dt + \sigma P_t(M - P_t)dW_t \tag{38}$$

Equation (38) is an Ito process called logistic geometric Brownian motion, and can be solved by use of Ito's lemma. Let $F(P_t, t)$ be function of P_t and t be twice differentiable in P_t and once in t , we have

$$dF(P_t, t) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial P_t}dP_t + \frac{1}{2}\frac{\partial^2 F}{\partial P_t}dP_t^2$$

But

$$dP_t = \lambda P_t(M - P_t)dt + \sigma P_t(M - P_t)dW_t$$

Hence $dP_t^2 = \sigma^2 P_t^2 (M - P_t)^2 dt$ and by Ito's calculus we obtain

$$\begin{aligned} dF(P_t, t) \quad = \quad & \frac{\partial F}{dt}dt + \frac{\partial F}{dP_t}\lambda P_t(M - P_t)dt + \\ & \frac{\partial F}{dP_t}\sigma P_t(M - P_t)dW_t + \\ & \frac{1}{2}\frac{\partial F}{dP_t^2}\sigma^2 P_t^2 (M - P_t)^2 dt \end{aligned} \tag{39}$$

We can rewrite equation (39) in the form

$$\begin{aligned} dF(P_t,t) &= \left\{ \frac{\partial F}{dt} + \frac{\partial F}{dP_t}\lambda P_t(M-P_t) + \right. \\ & \left. \frac{1}{2}\frac{\partial F}{dP_t^2}\sigma^2 P_t^2 (M-P_t)^2 \right\} dt + \\ & \frac{\partial F}{dP_t}\sigma P_t(M-P_t)dW_t \end{aligned} \tag{40}$$

If we use the variable $F = \ln\left(\frac{P_t}{M-P_t}\right)$ then

$$\frac{\partial F}{dt} = 0, \quad \frac{\partial F}{\partial P_t} = \frac{M}{P_t(M-P_t)}, \quad \frac{\partial F}{\partial P_t^2} = \frac{2M(P_t - M)}{P_t^2(M-P_t^2)}$$

Substituting this in equation (39), we obtain

$$dF(P_t,t) = \left\{ \lambda M - \frac{1}{2}\sigma^2(M^2 - 2MP_t) \right\} dt + \sigma M dW_t \tag{41}$$

Equation (41) is similar to to the Brownian motion in equation (33). Its solution is got by integration.Thus

$$dF(P_t,t) \sim \lambda M - \frac{1}{2}\sigma^2(M - 2MP_t)dt, \sigma M dW_t$$

It can be solved by Ito calculus.When $\sigma = 0$ then equation (41) is a deterministic differential equation given by

$$dF(P_t,t) = \lambda M dt = \left(\frac{M}{P_t(M-P_t)}\right)dP_t$$

and making dP_t the subject of the subject of the formula, we obtain

$$dP_t = \lambda P_t(M-P_t)dt$$

If we let

$$F(P_t,t) = \ln\left(\frac{P_t}{(M-P_t)}\right) \text{ then } dF(P_t,t) = \left(\frac{M}{P_t(M-P_t)}\right)dP_t$$

and rewriting equation (38)

$$dP_t = \lambda P_t(M-P_t)dt + \sigma P_t(M-P_t)dW_t$$

we obtain

$$\frac{dP_t}{P_t(M-P_t)} = \lambda dt + \sigma dW_t \tag{42}$$

But

$$dF(P_t,t)P_t(M-P_t) = MdP_t$$

hence

$$dP_t = \frac{dF(P_t,t)P_t(M-P_t)}{M}$$

and when substituted in equation (42) we obtain

$$\frac{dF(P_t,t)P_t(M-P_t)}{MP_t(M-P_t)} = \lambda dt + \sigma M dW_t$$

$$dF(P_t,t) = \lambda M dt + \sigma dW_t \tag{43}$$

This is a generalised Weiner process with $\lambda M dt$ as the drift and $\sigma M dt$ as the variance. It has the explicit solution

$$F(P_t,t) = F(P_0,0) + \lambda M_t + \sigma M dW_t$$

which is equivalent to

$$\ln\left(\frac{P_t}{M-P_t}\right) = \ln\left(\frac{P_0}{M-P_0}\right) + \lambda M_t + \sigma M W_t$$

Solving for P_t we obtain

$$P_t = \frac{MP_0}{(M-P_0)e^{-\lambda M_t - \sigma M W_t} + P_0} \quad : P(0) = P_0 \tag{44}$$

When $\sigma = 0$ in equation (44) we obtain the deterministic logistic differential equation given by

$$P_t = \frac{MP_0}{(M-P_0)e^{-\lambda M_t} + P_0} \qquad as \quad t \to \infty,\ e^{-\lambda M_t - \sigma M W_t} \to 0$$

To take care of fluctuations in the roan antelopes population growth rate due to genetic defect at the rate proportional to $P_t(M-P_t)$ so as to ensure positive population growth rate, we add genetic defect in equation (38) to obtain

$$dP_t = (\lambda - \gamma)P_t(M-P_t)dt + \sigma P_t(M-P_t)dW_t \tag{45}$$

where

P_t roan antelopes population at time t,

λ roan antelope growth ratio,

γ genetic defect,

M carrying capacity,

σ diffusion rate and

W_t random variable.

Suppose $F(P_t, t) = F$ is twice differentiable function in P_t and once in t, then by Ito's lemma

$$dF(P_t, t) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial P_t}dP_t + \frac{1}{2}\frac{\partial^2 F}{\partial P_t^2}dP_t^2$$

which is equivalent to

$$\begin{aligned} dF(P_t, t) &= \left\{\frac{\partial F}{\partial t}dt + (\lambda - \gamma)P_t(M - P_t)\frac{\partial F}{\partial P_t} + \right. \\ &\left. \frac{1}{2}\sigma^2 P_t^2(M - P_t^2)\frac{\partial^2 F}{\partial P_t}\right\}dt + \\ &\sigma P_t(M - P_t)\frac{\partial F}{\partial P_t}dW_t \end{aligned} \tag{46}$$

Using the variable

$$F(P_t, t) = \ln\left(\frac{P_t}{M - P_t}\right) \tag{47}$$

where,

$$\frac{\partial F}{\partial t} = 0, \quad \frac{\partial F}{\partial P_t} = \frac{M}{P_t(M - P_t)}, \quad \frac{\partial^2 F}{\partial P_t^2} = \frac{2M(P_t - M)}{P_t^2(M - P_t)^2}$$

We substitute the above results in equation (46) to obtain

$$\begin{aligned} dP_t &= \frac{M}{P_t(M - Pt)}\Big[(\lambda - \gamma)P_t(M - P_t)dt + \sigma P_t(M - P_t)dW_t\Big] + \\ &\frac{1}{2}\left(\frac{2P_t - M^2}{P_t^2(M - P_t^2)}\right)\sigma P_t^2(M - P_t)^2 dt \end{aligned}$$

$$\Rightarrow dP_t = M\left\{(\lambda - \gamma) + \frac{1}{2}\sigma^2(2P_t - M)\right\}dt + \sigma M dW_t \tag{48}$$

with
$$dP_t \sim N\left\{M(\lambda-\gamma)+\frac{1}{2}\sigma^2(2P_t-M)dt,\ \ \sigma M\sqrt{dt}\right\}$$
On rewriting equation (45) as
$$\frac{dP_t}{P_t(M-P_t)} = (\lambda-\gamma)dt+\sigma dW_t \tag{49}$$
and using the variable in equation (46) we can rewrite equation (48) as
$$\partial F(P_t,t) = (\lambda-\gamma)Mdt+\sigma dW_t \tag{50}$$
Integrating equation (50) with respect to t, we obtain
$$F(P_t,t) = F(P_0,0)+(\lambda-\gamma)M_t+\sigma MW_t, \tag{51}$$
which on substitution with the variable in equation (47),yields
$$\ln\left(\frac{P_t}{M-P_t}\right) = \ln\left(\frac{P_0}{M-P_0}\right)+(\lambda-\gamma)M_t+\sigma MW_t \tag{52}$$
Solving equation (52), we obtain
$$P_t = \frac{MP_0}{(M-P_0)e^{-(\lambda-\gamma)M_t-\sigma MW_t}+P_0} \quad : P(0)=P_0 \tag{53}$$
From equation (53) when $\lambda=\gamma$,we have
$$P_t = \frac{MP_0}{(M-P_0)e^{-\sigma MW_t}+P_0} \quad : P(0)=P_0 \tag{54}$$
Equation (54) is a function of random variable W_t only. This implies that the population may approach extinction.

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