On the Linearization of the Ricatti's equation via the Sundman Transformation

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Abstract

A Ricatti equation is any equation of the form $x' = a_2x^2 + a_1x + a_0$ where a_i (for $i = 0, 1, 2$) are functions of the independent variable. In this paper, we show that in general the Ricatti's equation is not linearizable using the Sundman transformation. We use a substitution to convert the equation into a linearizable second order non-linear form and then we determine completely the class of the Ricatti's equation that are linearizable using this transformation (under the particular substitution).

1 Introduction

Equations of the form $x' = a_2x^2 + a_1x + a_0$, where a_i (for $i = 0, 1, 2$) are functions of t, are known as the Ricattis equations. These are first-order non-linear ordinary differential equations that normally require a prior solution in order to obtain a second solution. If a prior solution is not given, there is no known method of direct application to obtain any solution for such equations. In this paper the method of linearization of second order non-linear equations via the Sundman Transformation is used. We go through a step by step process and see that the conditions for linearization inherently constrains the functions a_i.

2 The Sundman Transformation

For a second order non-linear ODE of the form

$$x'' + \lambda_2 \left(x'\right)^2 + \lambda_1 x' + \lambda_0 = 0 \tag{1}$$

to be linearized and solved using the Sundman Transformation the following conditions are sufficient to be satisfied.

For $\lambda_3 = 0$, if $\lambda_6 = 0$ then we require

$$\begin{aligned} \lambda_{2tt} &= -\lambda_{2t}\lambda_1 \\ \lambda_{6t} &= \frac{3\lambda_{0t}\lambda_6}{2\lambda_0} \\ \lambda_{6x} &= \frac{\lambda_{0x}\lambda_6 + 2\lambda_0^2\lambda_{2t}}{\lambda_0} \\ \lambda_{4x} &= -2\lambda_1\lambda_4 \end{aligned} \tag{2}$$

If $\lambda_6 \neq 0$ then we require

$$\begin{aligned} \lambda_{2tt} &= -\lambda_{2t}\lambda_1 \\ \lambda_{6t} &= \frac{3\lambda_{0t}\lambda_6}{2\lambda_0} \\ \lambda_{6x} &= \frac{\lambda_{0x}\lambda_6 + 2\lambda_0^2\lambda_{2t}}{\lambda_0} \\ \lambda_{4t} &= \frac{\lambda_4\lambda_6^2 - 24\lambda_3^2\lambda_{2t}^2 - 4\lambda_0\lambda_1\lambda_4\lambda_6}{2\lambda_0\lambda_6} \end{aligned} \tag{3}$$

where

$$\begin{aligned} \lambda_3 &= \lambda_{1x} - 2\lambda_{2t} \\ \lambda_6 &= \lambda_{0t} + 2\lambda_0\lambda_1 \\ \lambda_4 &= 2\lambda_{0xx} - 2\lambda_{1tx} + 2\lambda_0\lambda_{2x} - \lambda_1\lambda_{1x} + 2\lambda_{0x}\lambda_2 + 2\lambda_{2tt} \end{aligned} \tag{4}$$

For $\lambda_3 \neq 0$, if $\lambda_5 = 0$ then

$$\begin{aligned} \lambda_{0t} &= -2\lambda_0\lambda_1 \\ \lambda_{2t} &= \lambda_3 \end{aligned} \tag{5}$$

are sufficient.

If $\lambda_5 \neq 0$ then we require

$$\begin{aligned}
\lambda_{2ttt} = \lambda_3^{-1}\lambda_5\left(\lambda_5+\lambda_{3t}\right) - 2\lambda_1\lambda_5 + \left(\lambda_1^2-\lambda_{1t}\right)\left(\lambda_3+\lambda_{2t}\right) - \lambda_{3tt} \\
\lambda_{0t} = \frac{2\lambda_0\left(\lambda_5-\lambda_1\lambda_3\right)}{\lambda_3} \\
\lambda_{2ttx} = -2\lambda_{2t}\left(\lambda_3+\lambda_{2t}\right) - \lambda_{3tx} - \lambda_1\frac{d}{dx}\left(\lambda_3+\lambda_{2t}\right) + \lambda_3^{-1}\lambda_{3x}\lambda_5
\end{aligned} \tag{6}$$

$$\begin{aligned}
\lambda_5\lambda_3 \\
\left(6\lambda_{ox}\lambda_{2t} + 2\lambda_{2tx}\lambda_0 + 4\lambda_{2t}\lambda_0\lambda_2 + 2\lambda_{3x}\lambda_0 + 4\lambda_2\lambda_2\lambda_3 + \lambda_1\lambda_5\right) \\
-6\lambda_3^2\left(\lambda_{2t}^2\lambda_0 - \lambda_{0x}\lambda_5 + \lambda_0\lambda_3\lambda_{1x}\right) - \lambda_5^2\left(\lambda_4 - 2\lambda_5\right) = 0
\end{aligned}$$

where

$$\lambda_5 = \lambda_{2tt} + \lambda_1\left(\lambda_3+\lambda_{2t}\right) + \lambda_{3t} \tag{7}$$

3 Conditions for Linearizing the Ricatti's equation

The Ricatti's equation is any first order non-linear ODE of the form

$$x' = a_2x^2 + a_1x + a_0 \tag{8}$$

where a_i for $i = 0, 1, 2$ are functions of t (only).

We make the substitution

$$x = p'$$

so that

$$x' = p'' \text{ and } x^2 = \left(p'\right)^2$$

Then equation (8) is transformed into

$$p'' - a_2\left(p'\right)^2 - a_1p' - a_0 = 0 \tag{9}$$

which is the form of the second order non-linear ODE where

$$\begin{aligned}
\lambda_2 &= -a_2 \\
\lambda_1 &= -a_1 \\
\lambda_0 &= -a_0
\end{aligned} \tag{10}$$

We now investigate the conditions. We have

$$
\begin{aligned}
\lambda_3 &= \lambda_{1p} - 2\lambda_{2t} \\
&= -a_{1p} + 2a_{2t} \\
&= 2a_{2t} \neq 0
\end{aligned} \tag{11}
$$

We now check λ_5.

$$
\begin{aligned}
\lambda_5 &= \lambda_{2tt} + \lambda_1 \left(\lambda_3 + \lambda_{2t}\right) + \lambda_{3t} \\
&= -a_{2tt} - a_1 \left(2a_{2t} - a_{2t}\right) + 2a_{2tt} \\
&= \lambda_{2tt} - a_1 a_{2t} \neq 0
\end{aligned} \tag{12}
$$

Since $\lambda_5 \neq 0$ we require the conditions in equation (6) to be satisfied. We start by checking the third equation of equation (6) and see that

$$
\begin{aligned}
2a_{2t}^2 &= 0 \\
\implies a_{2t} &= 0 \\
\implies a_2 = b &= constant
\end{aligned} \tag{13}
$$

Checking the rest of the equations in equation (6) shows that they are satisfied. But the condition that $a_{2t} = 0$ puts a further constraint on λ_3 since this would imply

$$
\lambda_3 = 0
$$

Then we check λ_6 and see that

$$
\lambda_6 = 2a_0 a_1 - a_{0t} \tag{14}
$$

If $\lambda_6 = 2a_0 a_1 - a_{0t} \neq 0$ then we check the conditions in equation (3). The first, third and fourth equations in equation 3 are clearly satisfied. The second equation of equation (3) gives the requirement

$$
2a_0 a_1 a_{0t} + a_{0tt} - 3a_{ot}^2 - 4a_0^2 a_{1t} = 0 \tag{15}
$$

which is sufficient. If $\lambda_6 = 2a_0 a_1 - a_{0t} = 0$ this would imply $2a_0 a_1 = a_{0t}$. Substituting this into equation (15) gives the requirement $a_0 = 0$. All other conditions are satisfied.

In summary, The classes of Ricatti's equations solvable via the Sundman transformation are

$$
x' = bx^2 + a_1 x + a_0 \tag{16}
$$

provided equation (15) holds.

Also all equations of the form

$$x' = bx^2 + a_1 x \tag{17}$$

for any time dependent function a_1 are linearizable, where b is a real constant $\neq 0$ (since $b = 0$ reduces the equation to a linear one anyway). We recognize the class in equation (17) as a Bernoulli's equation with $n = 2$. We will later see how we use our method to solve such equations.

Now for the first class of equations $x' = bx^2 + a_1 x + a_0$ the set of partial differential equations to be solved [2] in order for such class to be linearized are given by

$$\begin{aligned} F_{pp} &= \frac{G_p F_p + F_p G \lambda_2}{G} \\ G_p &= 0 \\ G_t &= G\frac{\lambda_{0t}}{2\lambda_0} = G\frac{a_{0t}}{2a_0} \end{aligned} \tag{18}$$

The second equation of equation (18) can be solved to give

For the class in equation (17) where $\lambda_6 = 0$, $a_{0t} = 2a_0 a_1$. This reduces the third equation of equation (18) to

$$G_t = G a_1 \tag{19}$$

Solving equation (19) gives

$$G = e^{\int a_1 dt} \tag{20}$$

Since $G_x = 0$ the first equation of equation (18) reduces to

$$F_{pp} = -bF_p \tag{21}$$

which is the other equation to be solved.

Solving equation (21) gives

$$F = -\frac{A}{b} e^{-bp} + c \tag{22}$$

for arbitrary constants A and c.

The third equation of equation (18) can be solved (without further restriction) to give

$$G = B a_0 \tag{23}$$

for some arbitrary constant B.

In summary, for the class in equation (16), F and G are given by equations (22) and (23) respectively. For the class in equation (17), F and G are given by equations (22) and (20) respectively.

4 The Solution

Once our functions F and G are known we set out to find our parameters α, β and γ given by

$$\begin{aligned}\alpha &= \frac{-a_1 G + G_t}{G^2}\\ \beta &= \frac{\lambda_0\lambda_2 G + \lambda_{0x} G - \lambda_0 G_x}{G^3}\\ \gamma &= \frac{\beta F G^2 - \lambda_0 F_x}{G^2}\end{aligned} \tag{24}$$

For the class in equation (16),

$$\begin{aligned}\alpha &= \frac{a_{0t} - a_0 a_1}{B a_0^2}\\ \beta &= \frac{b}{B^2 a_0}\\ \gamma &= 0\end{aligned} \tag{25}$$

Our linearized equation then becomes

$$u'' + \left(\frac{a_{0t} - a_0 a_1}{B a_0^2}\right) u' + \left(\frac{b}{B^2 a_0}\right) u = 0 \tag{26}$$

for arbitrary constant B where $B \neq 0$ and $a_0 \neq 0$. If it so happens that a_0 and a_1 are such that the coefficients in equation (26) are constants, immediately solutions can be found using normal methods for solving linear ODEs with constant coefficients. If the coefficients are nonconstant, then there are known methods to solve such equations for which a guess solution would be required and then we could use reduction of order to solve. However, for a particular case where equation (26) is exact, a first integral can be completely obtained. Suppose we have an equation of the form

$$p_0 u'' + p_1 u' + p_2 u = 0 \tag{27}$$

where p_i for $i = 0, 1, 2$ are functions of the independent variable T only. We can rewrite equation (27) as

$$\left(p_0 u' - p_0' u\right)' + \left(p_1 u\right)' + \left(p_0'' - p_1' + p_2\right) u = 0 \tag{28}$$

Then equation (27) is called exact if $p_0'' - p_1' + p_2 = 0$. This reduces equation (28) to the first order equation

$$p_0 u' + \left(p_1 - p_0'\right) u = c \tag{29}$$

for some constant c whose solution is given by

$$u = ce^{-\int \frac{p_1 - p_0'}{p_0} dT} \int e^{\int \frac{p_1 - p_0'}{p_0} dT} dT \tag{30}$$

For the case under consideration (equation (26)), $p_0 = 1$ and $p_1 = \left(\frac{a_{0T} - a_0 a_1}{B a_0^2}\right)$ where we have written the $a_i's$ in terms of T using the relation $T = B \int a_0 dt$ from the transformation equation $dT = Gdt$ and equation (23). Then the condition of exactness in equation (28) (where we required $p_0'' - p_1' + p_2 = 0$) reduces to

$$p_2 - p_1' = 0 \implies p_1' = p_2 \tag{31}$$

Once equation (31) is satisfied an exact solution to equation (26) is given by

$$u = e^{-\int p_1 dT} \int e^{\int p_1 dT} dT \tag{32}$$

Using the transformation equations $u = F \implies u = -Ae^{-p}$ and $dT = Gdt$ [1, 2] where G is given by equation (23) and writing p_1 in terms of t gives the solution of equation (16) as

$$p' = x = -\frac{1}{b}\frac{d}{dt}\left(ln\left(-bAe^{-B\int p_1 a_0 dt} \int e^{B\int p_1 a_0 dt} dt\right)\right) \tag{33}$$

for constants A, B. Otherwise other methods requiring a prior solution may be used to solve equation (26).

For the class in equation (17)

$$\begin{aligned} \alpha &= 0 \\ \beta &= 0 \\ \gamma &= 0 \end{aligned} \tag{34}$$

our linearized equation becomes

$$u'' = 0 \tag{35}$$

4.1 Examples

1. Consider the equation

$$x' = x^2 - 1 \tag{36}$$

Clearly this falls into the first class in equation (16) as $b = 1$ and $a_0 \neq 0$. Here $a_2 = 1, a_1 = 0, a_0 = -1$. This reduces equation (26) to the form

$$u'' - \frac{1}{B^2}u = 0 \tag{37}$$

the general solution which is given by

$$u = c_1 e^{\left(-\frac{1}{B}\right)T} + c_2 e^{\left(\frac{1}{B}\right)T} \tag{38}$$

Now from the transformation equations $u = F \implies u = -Ae^{-p}$ and $dT = Gdt \implies dT = -Bdt \implies T = -Bt$ (where we keep ignoring the constant in F since it will vanish for all differentiations). Then

$$\begin{aligned} p = -ln\left(-\frac{u}{A}\right) &= -ln\left(-\frac{c_1e^t + c_2e^{-t}}{A}\right) \\ p' = x &= -\frac{c_1e^t - c_2e^{-t}}{c_1e^t + c_2e^{-t}} \end{aligned} \tag{39}$$

for A and $B \neq 0$. It can be easily verified that equation (39) is a solution to the differential equation (36).

2. Consider another differential equation

$$x' = x^2 + \frac{1}{t}x \tag{40}$$

We see that this is an equation of the class in equation (17). We have now shown that for all equations in the class in equation (17) the linearized form is the form in equation (35) whose solution is given by

$$u = cT \tag{41}$$

for some constant c. Again from the transformation equations $u = F \implies u = -Ae^{-p}$ and $dT = Gdt \implies dT = e^{\int \frac{1}{t}dt}dt \implies dT = Btdt \implies T = \frac{B}{2}t^2$ (where we again ignore the constant in F). Then

$$\begin{aligned} p = -ln\left(-\frac{u}{A}\right) &= -ln\left(-\frac{cB}{2A}t^2\right) \\ \implies p' = x &= -\frac{2}{t} \end{aligned} \tag{42}$$

Again the above equation can easily be verified to be a solution to the differential equation (40).

For the class in equation (17) we can write explicitly the solution as

43

$$
\begin{gathered}
x=-\frac{1}{b}\frac{d}{dt}\left(ln\left(-bA\int e^{\int a_1 dt}dt\right)\right)\\
or\\
x=-\frac{1}{b}\frac{d}{dt}\left(ln\left(\int e^{\int a_1 dt}dt\right)\right)
\end{gathered}
\tag{43}
$$

for some arbitrary constant A. We neglect the constants since they will cancel out upon differentiation.

5 Conclusion

We conclude that in general the Ricatti's equation is not solvable using the Sundman transformation. Only particular classes of such equations can be linearized (equations (16) and (17)) for which exact solutions can be directly extracted.

References

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