# A resolution to the Multiple IRR problem: Modified capital amortization schedule (MCAS) method that eliminates multiple IRR and leads to a real and unique IRR[[1]](#footnote-2)

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**Abstract**

The problems of reinvestment and multiple IRR remained unresolved for decades. This problem is associated only with some of the non-normal net cash flow (NNCF). A new method, based on capital amortization schedule (CAS) used to estimate IRR and NPV, transparently reveals reinvestment, if there is any. A modified CAS (MCAS) method is used to eliminate the reinvestment thereby resolving the problem of reinvestment and multiple IRR. The salient findings of the present analysis are:

1. The CAS method estimated IRR and NPV are consistent with that estimated by discounted cash flow (DCF) method.
2. The CAS method indicates if there is any reinvestment of intermediate income and that the reinvestment income under the benefit stream causes multiple IRRs and distorts NPVs too.
3. The MCAS method eliminates the reinvestment income associated with NNCF investments and leads to a unique and real IRR and consistent NPV.
4. The assumption of reinvestment at IRR or at hurdle rate in NPV is a false assertion in the cases of all normal NCF and NNCFs. However, such reinvestment is evident only with some NCFs and NNCFs with positive opening balance in one or more years under the CAS.
5. The MCAS method is appropriate for both normal NCF and NNCF as illustrated in this paper. CAS or DCF method is appropriate only for normal NCF investments.
6. The alternative methods like generalized IRR (GIRR) and the Average IRR (AIRR) are not appropriate estimates for NNCF and not NCF consistent.

In conclusion, the MCAS method resolves the problem of multiple IRR and leads to a unique IRR that is real and NCF-consistent. Neither the NPV nor the MIRR could resolve the problem of multiple IRR.

1. Introduction

Economic theory indicates that measurement of efficiency requires cost-benefit analysis (CBA). The CBA tool has been often criticised because of the multiple IRR and inconsistency between NPV and IRR ranking of investments. The problem of multiple IRR was first discussed in detail by Lorie and Savage (1955). Till date, this problem continues to dominate the debate on estimation methods for non-normal[[3]](#footnote-4) net cash flow (NNCF). NNCF, also known as non-conventional or unorthodox NCFs, stream changes signs (positive and negative) several times. Such sign changes generally lead to the problems of multiple IRRs or no IRR and or negative IRR. There are projects with NNCF but only some of them end-up with multiple IRR. Discounted cash-flow (DCF) method used in cost-benefit analysis (CBA) and capital investment analysis (CIA) could not solve the problem of multiple IRR. The identification of an appropriate method to estimate a real rate of return for NNCF stream has been a challenge for more than a half century. There are several approaches introduced to estimate the IRR and NPV using NNCF data (for example, modified IRR (MIRR), generalized IRR (GIRR) by Kulakov and Kastro (2015), average IRR (AIRR) by Magni (2010) and selective IRR (SIRR) by Weber (2017)). The debate continues without consensus on the preferred criterion. Applied analysts and finance executives continue to remain within their comfort zone i.e. IRR and or NPV (Brealey et al. 2006, Osborne, 2010 and Brigham and Houston, 2006).

The purpose of this analysis is to identify the factors that lead to multiple IRR and to introduce an appropriate method to estimate NNCFs investment to resolve the problem of multiple IRR. This process also involves a review of some of the newly emerged methods to estimate the NNCF. The review provides a better insight into the conceptual issues and estimation methods. This analysis is also expanded to countercheck whether the estimated return is consistent with financing practice, accounting concept and NCF stream. This paper presents answers to these questions.

The present analysis is different from other published works and is more transparent. The evaluation of the estimated results and testing their consistency with NCF is a novelty in this paper. The new approaches involve: a. formulation of a modified capital amortization schedule (MCAS) method which is a modification of the CAS method (see Arjunan, 2017, for CAS method) to estimate NNCF investments; b. identification of the problem that causes multiple IRR using the CAS method;  c. estimating and comparing the IRR and NPV of selected NNCF data by the DCF, CAS and MCAS methods; d. evaluating whether the estimated IRR, NPV and MIRR are supported by the intrinsic value of NNCF or NCF (NCF-consistent[[4]](#footnote-5)) using CAS and MCAS methods; and e. evaluating other some of the method recently introduced by  other authors to estimate NNCF investment and compare the results with MCAS estimates.

### 2. Literature:

Non-normal cash flows are common with projects that involves substantial capital cost in the middle of the project for capital replacements or at the end the project to decommission or cleanup costs or mitigation of any damages to the land and water resources (Ross et al., 2013; Brigham and Houston, 2006; Brealey et al., 2006; Simon, 2006). A review of the published works mostly reveals that MIRR (Brigham and Ehrhardt (2016), Keirulff (2008), Balyeat et. al. (2015) and Kelleher et al. (2004)) is recommended as one of the best criteria to overcome the problem of multiple IRR. Some of the relevant published works are discussed here to provide a better appreciation of the problems and conclusions drawn.

Arjunan (2017a, b, c and d) conducted detailed numerical analyses, specifically focusing on the problems discussed in section 1 above, and concluded: a. NNCF is a data problem that affects both NPV and IRR; b. Neither MIRR nor NPV rule could solve the problems associated with multiple IRR; c. the reinvestment of intermediate income is common with NNCF and some normal NCF; and c. IRR is the best available criteria for independent and mutually exclusive projects. These findings contradict most of the earlier published works including finance and economic texts.

Weber (2017) introduced a selective IRR (SIRR) criterion that he claimed to be equivalent to the NPV-rule. He argued that “an investor with a cost of capital of r = 10% would report the return of a project with the cash-flow stream (-5, 16, -12) as minus infinity (and therefore completely unacceptable), whereas an investor with a cost of capital of r = 25% would report the return of the same cash-flow stream as 100% which is very attractive indeed.” The cumulative undiscounted sum of  the above NCF is -1 (cumulative loss) and therefore 100% return is highly unrealistic and purely a mathematically generated return and not supported by the project benefit stream.

Ross et. al (2013) clearly stated: "We will take a stand on one issue that frequently comes up in this context. The value of a project does not depend on what the firm does with the cash flows generated by that project. A firm might use a project’s cash flows to fund other projects, to pay dividends, or to buy an executive jet. It doesn’t matter: How the cash flows are spent in the future does not affect their value today. As a result, there is generally no need to consider reinvestment of interim cash flows." Accordingly, the reinvestment income should not be included as part of the benefit stream. The inclusion of reinvestment income, which is not a benefit, leads to distorted IRR and NPV. This important statement is considered in the present analysis.

There are two interesting and thought-provoking papers on estimation of NNCF data by Magni (2010) and Kulakov and Kastro (2015). Their findings are discussed in detail in section 4. Magni (2010) used a NCF data (Table 1: -10, 30, -25) to illustrate the AIRR criterion. The data indicate that the project is not profitable (net loss of -5; capital loss of 50%, before discount). Magni reported an AIRR of -27.27% as the real rate that is not correct. Kulakov and Kastro (2015) discussed two estimation methods for NNCFs viz. a generalized internal rate of return (GIRR) for a project as an investment; and a generalized external rate of return (GERR) for a project as a loan. They discussed a sample NCF (see Table:1, -1600, 10000, -10000; Kulakov and Kastro, 2015) with a cumulative loss of -1600 (capital loss of 100%). That being the case, the rate of returns (GIRR or GERR) estimated looks unrealistic.

The problem of determining a rate of return for nonconventional projects cannot be solved within the bounds of the NPV method because the NPV method uses a single discount rate (Eschenbach and Nicholls 2012). Ben-Horin and Kroll (2012) studied the problem of multiple IRRs, using NNCFs. They concluded that the confusing multiple IRR solutions are only possible under unrealistic large fluctuations of the cash flows. A similar view is reported by Ng and Beruvides (2014).

Campani (2014) clarified that neither the MIRR nor the NPV could solve the problems associated with NNCF investment projects. He mathematically adjusted the NCF with an intention to eliminate intermediate negative cash flows and thereby avoiding multiple IRR. The data set used by him was NNCF, but an analysis reveals that without those adjustments made also there is no multiple IRR situation. All NNCF need not necessarily lead to multiple IRR. Hazen (2003) reported that with multiple IRR each rate could be interpreted as a rate of return on its own underlying investment stream. This argument is not convincing as discussed in section 4.

Mackie et. al. (2003) discussed the Adjusted IRR (AIRR) and indicated that the IRR will overstate the true rate of return, and the Adjusted IRR, with appropriate reinvestment rate, will give an estimate of the true value. This finding again depends on the doubtful reinvestment assumption. Gronchi (1986) reported that a unique IRR is feasible when the investment is pure lending rate and the investor should recover the capital invested with required return by the end. The investor should not borrow from the project at any time during its project life. This is an important work that partly identified the problem.

The Descartes’ rule of signs and Norstrom’s criterion are often quoted in discussions on the problem of multiple IRRs associated with NNCF. On Descartes’ rule, Osborne (2010) stated that “the number of changes of signs in the coefficients (cash flows) corresponds to the maximum number of real, positive roots. If the signs are reversed on all coefficients attached to odd powers, then the number of changes of sign in the coefficients (cash flows) corresponds to the maximum number of real, negative roots (less than minus 100%)”.  Nostrum criterion relates the multiple IRRs to the number of times the cumulative NCF changes signs. Butenko (2017) showed with an example (-1, 2, -1, 2, 4) that this criterion of Nostrom is not always true.

In summary, there is no method that could resolve the problem of multiple IRR. Many authors recommend MIRR or NPV approach to resolve the problem of multiple IRR. Other authors reject the MIRR due to several weaknesses. All other criterions recommended to deal with NNCF are not popular with practitioners. With NNCF data multiple IRR problems are common. Conceptually, NPV cannot be a better alternative because NPV will be zero as many times as there are IRRs. Multiple IRR is a NNCF data problem that could not be handled by the DCF and CAS methods. An appropriate method must be developed to resolve the problems of reinvestment and multiple IRR

### 3. Methodology

The primary focus of the present study is to identify and formulate a new method to estimate the rate of return (ROC and ROIC) for NNCF. During this process, various other methods recently proposed by other authors are also evaluated. The methodology is primarily based on the CAS method introduced by Arjunan (2017) that is more transparent than the DCF method. It clearly shows how much of each periodic future income or return from an investment is going towards interest payments (ROIC) and capital (principal) recovery (ROC). CAS analysis will indicate when the NCF is fully utilized (closing balance zero and NCF-consistency) to recover the capital invested (ROC) and a return on capital remains invested (ROIC). CAS estimation follows equation 1.

 (1)

Where OB (= -CF0, OB1 to OB t)) is the opening balance of capital each year, R is the interest rate (or return i.e. IRR), positive NCFs are the intermediate NCF per year, CB is the closing balance each year and ‘t’ is year 1 to n (terminal or final year of project life or maturity of investment). The CB of a year 1 to t is the OB of the immediate next year (2 to t). Equation 1 is transformed into Eq.2. The Eq. 2 reveals that the PV of the final CB is the NPV and the FV of the NPV is the CB and the ‘r’ that makes NPV = 0 is the IRR.

  (2)

The IRR can be estimated by using the “Goal Seek” command in excel (click data, then click What-if Analysis) with “Set cell” = final CB cell; “To value” = ‘0’; and “By changing cell” = Cell containing interest rate. This command will identify the rate that makes the final CB zero (that is IRR).

The key to understand and resolve the problems of multiple or negative or no IRR is to evaluate the NCF stream carefully to eliminate these problems (see Teichroew, Robichek, and Montalbano 1965a, 1965b, Bussey and Eschenbach 1992, Fleischer 1994, Eschenbach 1995, Park 1997). Many authors (Athanasopoulos 1978; Beaves 1988; Bernhard 1979, 1989; Lin 1976; Lohmann 1988; Solomon 1956) reported that the IRR method[[5]](#footnote-6) assumes reinvestment of positive interim cash ﬂows at the same rate (equivalent to IRR). Obviously, interest income on reinvested interim income if included as part of benefit stream, may cause the problem of multiple IRR. The proposed new method is developed to eliminate such interest income as that is not project generated income (see also Ross et al. 2013).

A modified CAS method (MCAS) is specifically designed to estimate the IRR and NPV for NNCF investments. The simple modification in Eq. 1 is through ‘r’. The ‘if function’ in the excel is used to apply interest or discount at ‘r’ (r\*1) if the OB is negative and if the OB is positive then the ‘r’ is zero (r\*0). Such a treatment is justified and consistent with accounting and finance practice as explained below:

1. Positive opening balance (OB) for any intermediate year leads to sign changes. This is normally the case with NNCF that leads to multiple IRR. In DCF and CAS analysis of the NNCF, the positive OB is an investment of the interim income (secondary investment) back into the project.
2. A positive OB (credit balance) indicates that the capital is already recovered (ROC) at a given interest rate (ROIC). That being the case, charging or debiting interest on the credit balance or charging and crediting interest income is not consistent with the accounting concept and financial practices. In simple words, no financial institution will charge or debit interest to a loan account when the account shows credit balance[[6]](#footnote-7).
3. A positive (credit) OB leads to positive interest income (rate equivalent to IRR) from the retained earnings or reinvested income. Interest income is a financial inflow and not the project generated income. Such an income from reinvestment is not an income flow from the investment as required under the DCF and CAS methods. Interest income should not be included as project benefit stream.
4. Any sinking fund created to pay-off any expected future capital expenditure or losses is a function of funds operation not a part of the internal rate of return of the investment project (some authors call it as external rate of return (see Arjunan 2017b, Kierulff, 2012, Cheremushkin, 2012, Crean 2005).
5. Finally, two basic principles followed are: a. the rate of return must be feasible with a given NCF or the NCF could support such rates (NCF-consistent); b. the mathematical relationship between NPV and IRR must be maintained.

Ille’s (2016) argued that the NPV estimate applies a reinvestment rate. The reinvestment assumption in the case of NNCF leads to a serious mistake in the calculations. Ille’s illustrated how this way of evaluation is wrong. This paper is important as the interest income on positive OBs are reinvestment income that leads to the multiple IRR problem (also see Arjunan 2017b).

Modification of DCF method is also possible with similar modification to the interest or discount rates. However, the CAS based method (or a modified CAS) is more transparent and easier to interpret. In the next section, the MCAS method is explained with a worked-out example so that analysts or economists can easily relate it to the real-life situation. CAS method is like the popular loan amortization schedule. In MCAS, the capital investment (ROC) and annual income (CF) is known and the unknown interest rate (ROIC) that makes the NCF ‘0’ (closing balance zero in MCAS) is the IRR (for full details please refer to Arjunan 2017a). Also, if discounted or interest applied at the hurdle rate (say 10%), the present value (PV) of the closing balance in the CAS is the NPV. The data used are available in public domain or hypothetical data sets are used.

### 4. Results and Discussion

Results of numerical analyses of NNCF data, both hypothetical and published works, are presented and discussed here in five parts viz.

1. Introducing the new method viz. a modified CAS method to estimate the NNCF data.
2. Identifying the factors that cause multiple IRR problem under DCF and CAS methods.
3. Evaluation of a NNCF data with a unique IRR and NPV under DCF / CAS methods vs IRR and NPV by MCAS method.
4. Resolving the problem of Multiple IRR by MCAS Method Versus MIRR; and
5. A Review of other methods to estimate NNCF viz. the GIRR and AIRR.

### 4.1 A Modified CAS Method (MCAS) to Estimate NNCF:

A new method, modified CAS (MCAS) method, is introduced to handle the problem associated with NNCF. As discussed earlier, the inclusion of reinvestment with some of the NNCF leads to multiple IRR and inconsistent NPV. The MCAS model is presented here and the discussions are in three parts:

1. A hypothetical NNCF data is used to estimate IRR and NPV under DCF and CAS methods (these estimates under both methods are the same) and MIRR. Using these estimates (IRR), MCAS are prepared to assess whether these estimates are consistent with the NNCF (NCF-consistent) and compared with the IRR and NPV estimated by the MCAS method.
2. Next NNCF data, used by Schmidt (2015) and Kyd (2017) are used to estimate IRR and NPV under DCF and CAS methods and MIRR. These estimates are then used in MCAS to find out whether they are NCF-consistent or the NCF (benefit stream) is adequate to support such estimates; and to evaluate whether the MIRR could solve the problem of multiple IRR as claimed.
3. Finally, NNCF data sets used by Magni (2010) and Kulakov and Kastro (2015) to demonstrate how the AIRR or GIRR are not the appropriate criterion to estimate the return for NNCF. The estimated criteria are used to develop MCAS to evaluate whether those criteria are superior to the unique IRR by MCAS.

**MCAS Method:** A hypothetical NNCF data with two sign changes is used to introduce the MCAS method. IRR and NPV under the DCF, CAS and MCAS methods are estimated and the results compared (See Table 1). Simultaneously, MIRRs under the DCF method and the closing balance under the CAS and MCAS methods are also estimated and furnished in the upper part of Table 1.

### Table: 1. NNCF Estimation using DCF, CAS and Modified CAS Methods

|  |  |  |  |
| --- | --- | --- | --- |
| **Years** | **NNCF** | **DCF & CAS Based estimates** | **MCAS Based Estimates** |
| 0 | -1000.0 | Criteria | Values | Criteria | Values\*\* | Criteria | Values |
| 1 | 800.0 | IRR – 1 | -5.5% | NPV at -5.5% | 0 (0) | IRR\*\*\* | 7.8% |
| 2 | 1000.0 | IRR – 2 | 45.2% | NPV at 45.2% | 0 (0) | NPV at 7.8% | 0 (0) |
| 3 | 1300.0 | MIRR (10, 10) \* | 11.9% | NPV at 11.9% | $106 ($166) | NPV at 11.9% | -$23 (-$36.0) |
| 4 | -2000.0 | MIRR (10,0) \* | 7.0% | NPV at 7.0% | $119 ($157) | NPV at 7.0% | $6.5 ($8.5) |
| SNCF | 100.0 |  |  |  |  | NPV at -5.5% | $111 ($163.0) |
|  |  |  |  | NPV at 7.8% | $147($216) | NPV at 45.2% | -$442 ($-647.0) |
|  |  |  |  | NPV at 10%  |  $164($241) | NPV at 10% | -$20.5 (-$30.0) |
| **CAS Method for the NNCF** |
| Years | 1 | 2 | 3 | 4 | Allocation of Benefits |
| **CAS at IRR = -5.5%** |
| Opening Balance | -1000 | -145.4 | 862.5 | 2115.4 | ROC | -1000.0 |
| Interest / Return at -5.5% | 54.6 | 7.9 | -47.1 | -115.4 | ROIC  | -100.0 |
| Income flows | 800.0 | 1000.0 | 1300.0 | -2000.0 | Income  | 1100.0 |
| Closing Balance | -145.4 | 862.5 | 2115.4 | 0.0 | Balance  | 0.0 |
| **CAS at IRR = 45.2%** |
| Opening Balance | -1000.0 | -652.0 | 53.3 | 1377.4 | ROC | -1000.0 |
| Interest / Return at 45.2% | -452.0 | -294.7 | 24.1 | 622.6 | ROIC | -100.0 |
| Income flows | 800.0 | 1000.0 | 1300.0 | -2000.0 | Income | 1100.0 |
| Closing Balance | -652.0 | 53.3 | 1377.4 | 0.0 | Balance | 0.0 |
| **CAS at IRR = 7.8% (IRR under MCAS)** |
| Opening Balance | -1000 | -278.2 | 700.0 | 2054.8 | ROC | -1000.0 |
| Interest / Return at 7.8% | -78.2 | -21.8 | 54.7 | 160.7 | ROIC | 115.5 |
| Income flows | 800.0 | 1000.0 | 1300.0 | -2000.0 | Income | 1100.0 |
| Closing Balance | -278.2 | 700.0 | 2054.8 | 215.5 | Balance | 215.5 |
| **Modified CAS (MCAS) Method for the NNCF**  |
| **MCAS at IRR = 7.8%** |
| Opening Balance | -1000.0 | -278.2 | 700.0 | 2000.0 | ROC | -1000.0 |
| Interest/ Return at 7.8% | -78.2 | -21.8 | 0.0 | 0.0 | ROIC | -100.0 |
| Income flows | 800.0 | 1000.0 | 1300.0 | -2000.0 | Income | 1100.0 |
| Closing Balance | -278.2 | 700.0 | 2000.0 | 0.0 | Balance | 0.0 |
| **MCAS at IRR = -5.5%** |
| Opening Balance | -1000 | -145.4 | 862.5 | 2162.5 | ROC | -1000.0 |
| Interest/ Return at -5.5% | 54.6 | 7.9 | 0.0 | 0.0 | ROIC | 62.5 |
| Income flows | 800.0 | 1000.0 | 1300.0 | -2000.0 | Income | 1100.0 |
| Closing Balance | -145.4 | 862.5 | 2162.5 | 162.5 | Balance | 162.5 |
| **MCAS at IRR = 45.2%** |
| Opening Balance | -1000 | -652.0 | 53.3 | 1353.3 | ROC | -1000.0 |
| Interest/ Return at 45.2% | -452.0 | -294.7 | 0.0 | 0.0 | ROIC | -746.7 |
| Income flows | 800.0 | 1000.0 | 1300.0 | -2000.0 | Income | 1100.0 |
| Closing Balance | -652.0 | 53.3 | 1353.3 | -646.7 | Balance | -646.7 |

\* The figures in parenthesis of MIRR (10,10) and MIRR (10,0) stands for financing (10%) and investment rates (10% or 0).

\*\* Figures in parenthesis is the closing balance (CB) in CAS or MCAS.

\*\*\* IRR is 7.9% by interpolation of the NPVs estimated under the MCAS using a MIRRs as the rates. Using MIRR does not have any significance other than to identify a positive and negative NPV to enable interpolation.

The estimated criteria under the DCF and CAS methods are used to prepare MCAS and vice versa. The CAS and MCAS for various rates already estimated are furnished in the lower part of Table 1.

The results are discussed here:

1. Under the DCF and CAS methods, the NNCF with two sign changes resulted in multiple IRR (IRRs -5.5% and 45.2%). MIRR estimate is unique (no multiple MIRR) but the MIRR is an underestimate as that fails to fully utilise the NNCF as reveals by the closing balance of $165.75 and $156.5 in CAS and are the unutilised NNCF NPV of $106 and $119 (NPV = PV closing balances) at MIRR (10,10) and (10,0) of 11.9% and 7.0%, respectively. MIRR is therefore not the appropriate criterion to eliminate the problem of multiple IRR.
2. Under the DCF and CAS methods, NPV at 10% is $164.4 and zero at IRR of -5.5% and 45.2%. Then the positive NPV of $164 at 10% is confusing and not consistent with the mathematical relationship between IRR and NPV. For instance, the NPV at 10% is positive but then it is zero at -5.5% and at 45.2%. The NPV is also spurious and not the appropriate criterion for NNCF data to overcome the problem of multiple IRRs (see Elle’s, 2016, Campani, 2014).
3. In Table 1, the sum of income or net benefit (of the NNCF) is given ($1100) and the rate of return must be within that NNCF value. The last column of Table 1 presents the allocation of the benefits towards the ROC (capital), ROIC (return) and balance (difference between the income and allocated benefits (ROC + ROIC). Some interesting observations from the CAS analysis:
* At IRR -5.5% and at 45.2%, the ROIC is only $100 that is equivalent to the SNCF of $100 (second column of Table 1);
* The sum of negative and positive interest (ROIC) accounted in the CAS at both IRRs is -$100 (at -5.5: $62.5 and -$162.5 and at 45.2%: -$746.7 and $646.7); The sum of ROIC of -$100 and the SNCF of $100, is zero (zero CB) and therefore the NPV is zero (PV of CB = 0);
* Under the CAS at 7.8% the ROIC is $115 (Sum of -$100 and $215). This spurious interest income of $115 along with the SNCF of $100 remain as unutilised closing balance of $215 and NPV of $147 (PV of $215).
1. Obviously, the positive OB in the CAS in year 3 and 4 leads to positive interest to be earned elsewhere or to be injected into the project to pay for the negative inflow in the final year. That interest income is not a benefit from the investment or project and causes the multiple IRR problem. This is often misinterpreted as problem of IRR but its neither a problem with IRR nor NPV. It is a problem associated with NNCF data and the inability of the estimation method (DCF) to handle such problem. The MCAS method overcomes such problem and a detailed discussion is forthcoming in the next section.
2. As discussed, the CASs prepared for the NNCF reveals that the OBs for years 3 and 4 are positive and therefore interest on the OBs (positive) are included as income flow which is not correct. In the CAS method, the interest income for the years three and four are $24.1 and $622.6, respectively.
3. Under the MCAS method, the positive interest income for year three and four are not considered as a benefit or income and therefore ignored. The interest is shown as zero for year 3 and 4, as could be seen in table 1.
4. The interest income is not an operating cash flow or benefit flow but an income flow from investment (or external to operating income). This treatment is the only difference between the CAS and MCAS methods.
5. Under the MCAS, the estimated IRR of 7.8% is unique and there is no multiple IRR. At 7.8% IRR, the NNCF is fully utilised and therefore the NPV and the closing balance are zero at 7.8% under the MCAS. Another important observation is that the ROIC under MCAS and CAS is -$100 (see last column in Table 1) at their respective IRRs (7.8% with MCAS and -5.5% and 45.2% with CAS methods). Although the interest is ignored under the MCAS, the net interest (ROIC) is $100 under both MCAS and CAS. This supports the treatment of interest income (ignored) on positive OBs under the MCAS method.
6. The NPVs at 10% are $164 and -$20.5 under the CAS and MCAS methods, respectively. The interest income on positive OB not only leads to multiple IRR but also inconsistent higher NPV. The current practice of preferring NPV to avoid the multiple IRR problem is also therefore inappropriate.
7. Neither the MIRR nor the IRR estimated using DCF or CAS method is supported by the NNCF under the MCAS. The MCAS estimated IRR of 7.8% is the only rate that fully utilises the NNCF and makes the closing balance and the NPV ‘zero’. NPV at 10% under the MCAS is -$20.5 and that is consistent for a NNCF with an IRR of 7.8%.
8. The NNCF is not adequate (as revealed by the closing balance of -$36.0 in MCAS) to support the MIRR (10,10) of 11.9%. MIRR (10,10) is an overestimate. The MIRR (10,0) is an underestimate as there is unutilised NNCF of $8.5 (as reveals by the closing balance in MCAS).
9. The MCAS method leads to a unique IRR of 7.8% and eliminated the problem of multiple IRR. When the MCAS estimated IRR of 7.8% is used in the CAS, there is a closing balance of $215.5 (see table). This indicates the 7.8% is an underestimate under CAS (also DCF) method. However, the closing balance of $215.5 results from the interest income (reinvestment income) accrued and included in year three and four (54.7 + 160.7 = 215.4). This interest income is unrealistic, and it leads to multiple IRRs or incorrect IRRs.
10. The IRR of 7.8% seems realistic and appropriate as explained below:
* At 7.8% the closing balance is zero in the MCAS that indicates the NNCF is fully utilised. An IRR of 7.8% is the maximum IRR that the NNCF could support;
* At 7.8% the NPV is zero which is consistent with the mathematical relationship between IRR and NPV; NPV at 10% is -$30.0 that indicates that the IRR must be less than 10%. MCAS estimated IRR of 7.8% is therefore realistic and consistent with NCF and NPV;
* NPV is negative at MIRR (10,10) of 11.9% and positive at MIRR (10,0) of 7% (investment rate ‘0’ used in MIRR (10,0) is reasonable as the intermediate interest income is not included under both the MCAS and MIRR (10,0);
* The IRR must be in between 11.9% and 7% because the NPV at these rates (MIRRs) under MCAS (results not furnished in Table 1) are negative and positive, respectively. By interpolating the NPV at 11.9% and 7.0%, the estimated IRR is 7.9% which is very close to 7.8% estimated by the MCAS method.
* For the investment of $1000, the SNCF is $100 (see column 2). That works out to 10% return before discounting and therefore after discounting the IRR of 7.9% is realistic.
* From an opportunity cost perspective, the interest income on positive OB needs careful analysis. The opportunity cost will be accounted for by investment elsewhere with market rate of interest. The project account need not act as a fund’s manager for the investor that too by assuming reinvestment at IRR or hurdle rate.

The above analysis demonstrates the usefulness of the MCAS method and solves the problem of multiple IRRs and leads to a unique IRR. The estimated IRR is consistent with the mathematical relationship between IRR and NPV and NCF.

**4.2 Factors causing Multiple IRR Problem:**

The possible number of IRRs is better explained using Descartes rule of signs. Eschenbachetal (2007) and several authors indicated that IRR can be estimated only when there are at least one or more sign changes in the NCF series. When there is no negative cash flow, there is no investment flow or sign changes and therefore no IRR. Six hypothetical cases of NNCF data with two or three sign changes are used to estimate the IRR, NPV and MIRR and presented in table 2.

* All the six NNCFs with two or three sign changes (see Table 2) do not end with multiple IRR. They all have unique IRRs. The IRR is either unique or zero or negative or no IRR.
* When the sum of NCF (SNCF) is zero (NNCF 1 and 2 with 2 or 3 changes of sign), the IRR is zero. Obviously, it is reasonable to expect an IRR of zero when the sum of NCF is zero;
* With negative SNCF (NNCF 3 and 4 with 2 or 3 changes of sign), the IRR is also negative or there is no IRR. This again is reasonable given that the sum of NCF is negative;
* NNCFs 5 and 6 are with two and three sign changes, respectively. With positive SNCFs, both cases 5 and 6 lead to unique IRRs that are positive. In these cases, IRR, NPV and MIRR indicate that those are acceptable. In all other cases, the IRR, NPV and MIRR do not support the investment.
* The estimated IRR and NPV by MCAS method (detailed discussions follow later) reveals some interesting findings:
1. In the cases of NNCFs 1 and 2 the IRRs are zero under CAS/DCF[[7]](#footnote-8) and MCAS methods but the estimated NPVs at 10% differ because MCAS method excludes the intermediate income;
2. NCCF 3 and 5 do not have reinvestment income and therefore the estimated IRRs and NPVs under CAS and MCAS methods are the same. This finding also indicates reinvestment assumption in MIRR is wrong.
3. There is no IRR in the case of NNCF 4 under CAS method but there is negative IRR of -7.4% and the NPV at 10% differs under MCAS method after elimination of reinvestment income.
4. As NNCF 6, IRR and NPV estimates by DCF and CAS methods, that includes reinvestment income, and MCAS method (reinvestment not included) diverge.

The results indicate that all NNCFs involve reinvestment income but need not necessarily end up with multiple IRR. The results are consistent with Descartes rule. In the cases of NNCF 3, the cumulative NCF has no change of sign but the IRR is -3.3%. In the case of NNCF 4, the NCF has two sign changes but the IRR is zero. Nostrom criteria are not always true. A similar case is reported by Butenko (2017).

### Table: 2. Hypothetical NNCFs with Two and Three Sign Changes but with Unique IRR (no multiple IRRs) with DCF and CAS Method

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Details | NNCF 1:with 2 Sign changes & SNCF = 0 | NNCF 2:with 3 sign Changes & SNCF = 0 | NNCF 3:with 2 Sign changes & SNCF < 0 | NNCF 4:with 3 sign Changes & SNCF < 0 | NNCF 5: with 2 Sign changes & SNCF > 0 | NNCF 6:with 3 sign Changes & SNCF > 0 |
| Year 0 | -10000 | -1000 | -10000 | -1000 | -10000 | -1000 |
| Year 1 | 5000 | 900 | 5000 | 900 | 5000 | 900 |
| Year 2 | -6000 | -300 | -7000 | -300 | -6000 | -300 |
| Year 3 | 7000 | 1000 | 7000 | 900 | 7000 | 1000 |
| Year 4 | 4000 | -600 | 4000 | -600 | 8000 | -300 |
| **Sum of NCF (SNCF)** | **0** | **0** | **-1000** | **-100** | **4000** | **300** |
| IRR (Guess at 0 to 500%) | 0.0% | 0.0% | -3.3% | No IRR | 11.1% | 18.1% |
| MIRR (10, 10) | 5.2% | 8.5% | 3.8% | 7.2% | 10.6% | 12.1% |
| MIRR (10, 0) | 1.7% | 3.5% | 0.3% | 2.1% | 7.5% | 6.9% |
| NPV at 10% | -$2,422.0 | -$88.2 | -$3,248.4 | -$163.4 | $310.1 | $116.7 |
| IRR by MCAS\* method | 0.0% | 0.0% | -3.3% | -7.4% | 11.1% | 16.1% |
| NPV at 10% by MCAS | -$3,546.0 | -$117.5 | -$3,248.4 | -$185.8 | $310.1 | $87.4 |

\* Worksheets for the above estimates are available with the author.

**4.3 A NNCF data with a unique IRR by DCF / CAS vs IRR by MCAS method:**

All NNCF data need not necessarily lead to multiple IRR problems. That does not mean the unique IRR and NPV estimated by DCF or CAS method is always real. A NNCF data that is available in public domain is used here to illustrate one such instance. The results are presented in Table 3. The NNCF data leads to a unique IRR of 23.0% and CAS indicates that the interest income of $56 is included as benefit or income which is a reinvestment income and should not be included. Under the MCAS method this interest income of $56 is not included and therefore the IRR is 21.1%. The MCAS method that excludes reinvestment income, is appropriate for both NNCF and NCF data.

Both the CAS and MCAS at IRR and at 10% (assumed hurdle rate) reveals an important point. Under CAS at 23.0% (IRR), in year four OB is $244. The interest income of $56 is a reinvestment income at 23.0% of the OB. Under CAS at 10%, the year four OB is $471.3 and an interest income at 10% ($47.1) is included under the benefit or income. In DCF or CAS methods, the interim income is reinvested at IRR or at 10% ONLY when the OB is a positive. A positive OB clearly indicates that the ROC and ROIC are fully recovered. Under the normal NCF and there is no negative flow afterwards. A positive CB is not possible with normal NCF. In the case of NNCF a positive OB is common. Under DCF and CAS estimates that include reinvested interest income the IRR is 23.0% and the NPV is $149. MCAS method eliminates the reinvestment both at IRR and 10%. After elimination, of the reinvested interest income, the IRR is 21.1% and the NPV is $117 under MCAS is lower than the CAS method. Both IRR and NPV suffers due to NNCF and the NPV rule becomes invalid under NNCF.

### Table: 3. NNCF with Unique IRR under DCF and CAS method vs IRR with MCAS Method

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| **Details** | **0** | **1** | **2** | **3** | **4** | **Sum** |  **Estimates**  |
| NNCF | -700 | 300.0 | 400.0 | 600.0 | -300.0 | 300.0 | Criteria | Values |
| **CAS Method Estimates** |
| Opening Capital |  | -700.0 | -560.8 | -289.5 | 244.0 | -700.0 | IRR | 23.0% |
| Interest at 23% |  | -161 | -129 | -66 | 56 | -300.0 | MIRR (10,10) | 14.3% |
| Income |  | 300.0 | 400.0 | 600.0 | -300.0 | 1000 | MIRR (10,0) | 9.5% |
| Closing Balance |  | -560.8 | -289.5 | 244.0 | 0.0 |  | NPV at 23.0% | 0 |
| **CAS at 10% (Hurdle rate)** |
| Opening Capital |  | -700.0 | -470.0 | -117.0 | 471.3 | -700.0 |  |  |
| Interest at 10% |  | -70.0 | -47.0 | -11.7 | 47.1 | -81.6 |  |  |
| Income |  | 300.0 | 400.0 | 600.0 | -300.0 | 1000 |  |  |
| Closing Balance |  | -470.0 | -117.0 | 471.3 | 218.4 |  | NPV at 10% | 149.2 |
| **MCAS Method Estimates**  |
| Opening Capital |  | -700.0 | -540.9 | -249.7 | 300.0 | -700.0 | IRR | 20.1% |
| Interest at 20.1% |  | -140.9 | -108.9 | -50.3 | 0.0 | -300.0 | MIRR (10,10) | 14.3% |
| Income |  | 300.0 | 400.0 | 600.0 | -300.0 | 1000 | MIRR (10,0) | 9.5% |
| Closing Balance |  | -540.9 | -249.7 | 300.0 | 0.0 |  | NPV at 21.1% | 0 |
| **MCAS at 10% (Hurdle Rate)** |
| Opening Capital |  | -700.0 | -470.0 | -117.0 | 471.3 | -700.0 |  |  |
| Interest at 10% |  | -70.0 | -47.0 | -11.7 | 0.0 | -128.7 |  |  |
| Income |  | 300.0 | 400.0 | 600.0 | -300.0 | 1000 |  |  |
| Closing Balance |  | -470.0 | -117.0 | 471.3 | 171.3 |  | NPV at 10% | 117.0 |

Source: http://www.accaglobal.com/an/en/student/exam-support-resources/professional-exams-study-resources/ p4/technical-articles/Modified-internal-rate-return.html

In summary, the above analysis illustrates that the reinvestment income is included only under the NNCF (see also Elles, 2016) and not under the normal NCF. The reinvestment affects the values of IRR and NPV and therefore the preference of NPV over IRR for NNCF investment is not appropriate. Also, the MIRR that assumes reinvestment rate for even normal NCF is inappropriate. The MIRR is unique but cannot solve the real problem of reinvestment and unrealistic IRR.

**4.4 Resolving the problem of Multiple IRR by MCAS Method Versus MIRR:**

As discussed, many authors suggested using MIRR to overcome the problem of multiple IRR. The NNCF data, used both by Kyd (2017) and Schmidt (2015), leads to multiple IRRs (0%, 100% and 200%). They concluded that all the three IRRs are acceptable, but this conclusion needs solid evidence. According to these authors, the problem of multiple IRR could be eliminated by MIRR. Kyd used a finance rate (FR) of 8% and a reinvestment rate (RR) of 3% and estimated a MIRR of 5.8%. For the same data, Schmidt used FR of 10% and RR of 5% and estimated a MIRR of 2.3% (the correct MIRR must be 7.7%). The same data is used to evaluate whether these estimates are consistent with NNCF and the analytical results are in Table 4.

1. For the given NNCF, there are multiple IRRs (three IRRs: 0%, 100% and 200%) under the CAS and DCF methods. For all the three IRRs, the NPV is zero and therefore the result is consistent with the mathematical relationships. However, the NPV at 10 % (assumed hurdle rate) is negative (-1284) and that being the case the IRR 100% and 200% are spurious or dubious.
2. With 0% or 100% or 200% IRRs, the ROIC is zero as per CAS that indicates such IRRs are spurious. This is obviously caused by the positive OBs in the CAS that leads to interest to be paid by the investor (8000 and 18000 at 100% and 200%, respectively) to achieve that false IRR of 100% and 200%. The multiple IRRs 100% and 200% is not supported by the NNCF and not NCF consistency.
3. The MCAS estimated IRR is 0% which is the real return and is consistent with NCF and NPV. The real return must be zero as indicated by the ROIC component for all other returns (IRRs and MIRRS) are not recovered and remain as closing balance. MCAS estimate does not include reinvestment and therefore the IRR is zero which is consistent with NNCF (Sum of NNCF = 0). MCAS eliminates the multiple IRR problem and leads to NCF consistent real IRR. The three different IRR solutions: 0%, 100%, and 200% may be the roots of the polynomial (NPV equation). Then the zero return is the real return and others are not real.
4. With NNCF that involves reinvestment, the NPV rule is invalid. For example, the NPV at 10% under CAS method is -1285 but under the MCAS method (that eliminates the reinvestment income) the NPV at 10% is -7100. The higher NPV under the DCF/CAS method is caused by the reinvestment income. These NPV results are not consistent with IRRs (at IRR, NPV = 0).
5. MCAS estimates reveal that the net benefit stream or income is not adequate to support each one of the MIRRs. These MIRRs are all overestimates due to reinvestment income. The ROIC at that MIRRs are just left unrecovered as negative closing balance between -4119 and -6763 (see last column of Table 3).
6. The claim by Kyd (2017) and Schmidt (2015) that the MIRR solves the problem of multiple IRR is not supported. MIRRs are unique returns but not the real return or feasible return and not NCF consistent. MIRR is again a spurious return (see also Arjunan 2017c). For example, at MIRR 5.8% the ROIC of $4119 is not recoverable from the income stream and remains as closing balance. MIRR estimate, by including a reinvestment rate, is not consistent with NCF.

### Table: 4. NNCF Estimation: MIRR vs IRR Estimated by DCF, CAS and MCAS Methods

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Years | 0 | 1 | 2 | 3 | SNCF |  |
| NNCF Data | -10000 | 60000 | -110000 | 60000 | 0 |  |
| **CAS and DCF based estimates** | **Allocation of benefit (NCF) in CAS** |
| Criteria | Values | NPV | Income (A) | ROC(B) | ROIC (C) | Balance (CB) = (B+C) – A |
| IRR (1) | 0.0% | 0 | 10000 | -10000 | 0 | 0 |
| IRR (2) | 100% | 0 | 10000 | -10000 | 0 | 0 |
| IRR (3) | 200% | 0 | 10000 | -10000 | 0 | 0 |
| NPV at 10% | 10% | -1284.7 | 10000 | -10000 | -1710.0 | -1710.0 |
| MIRR (10,10) | 9.5% | -1250 | 10000 | -10000 | -1642 | -1642 |
| MIRR (10,0) | 5.9% | -913 | 10000 | -10000 | -1085 | -1085 |
| MIRR (10,5) | 7.7% | -1096.4 | 10000 | -10000 | -1370.7 | -1370.7 |
| MIRR (8,3) | 5.8% | -915.4 | 10000 | -10000 | -1085.2 | -1085.2 |
| **MCAS Based Estimates** | **Allocation of benefit (NCF) in MCAS** |
| Criteria | Values | NPV | Income (A) | ROC(B) | ROIC (C) | Balance (CB) = (B+C) – A |
| IRR by MCAS | 0% | 0 | 10000 | -10000 | 0 | 0 |
| NPV at 10 % by MCAS | 10% | -7100.0 | 10000 | -10000 | -5334.3 | -5334.3 |
| **Evaluating the NCF Consistency of the Various Estimates Using MCAS** |
| MCAS at IRR (1) | 100% | -10000 | 10000 | -10000 | -80000 | -80000 |
| MCAS at IRR (2) | 200% | -6667 | 10000 | -10000 | -180000 | -180000 |
| MCAS at MIRR (10,10) | 9.5% | -5146 | 10000 | -10000 | -6763 | -6763 |
| MCAS at MIRR (10,0) | 5.9% | -3530 | 10000 | -10000 | -4197 | -4197 |
| MCAS at MIRR (10,5) | 7.7% | -4373.6 | 10000 | -10000 | -5467.6 | -5467.6 |
| MCAS at MIRR (8,3) | 5.8% | -3474.4 | 10000 | -10000 | -4118.9 | -4118.9 |

Data source: <https://www.propertymetrics.com/blog/2015/10/28/how-to-use-the-modified-internal-rate-of-return-mirr/> and <http://www.exceluser.com/formulas/irr.htm>

In summary, the real IRR is 0% estimated by the MCAS method. All other IRRs and MIRRs are not real returns. MIRR could never eliminate the real problem of multiple IRR. Next, the discussion will be about the AIRR and GIRR that are proposed based on strong mathematical analysis of the properties of the polynomials (NPV equation).

### 4.5 A Review of the GIRR and AIRR Methods to Estimate NNCF

Kulakov and Kastro (2015) and Magni (2010) studied the mathematical properties of the NNCF and presented useful mathematical insight into the NNCF analysis and the associated multiple roots. The former authors proposed a Generalised IRR (GIRR) and the later author proposed Average IRR (AIRR, also Aggregate Return on Investment - AROI) to estimate the NNCF investments. A critique of these two approaches are available in Arjunan (2017b). Multiple roots identified may contain valuable information (see Hazen, 2003 and Osborne 2011). That information will be relevant for that specific polynomial and not for the modified polynomials (as in the case of MCAS).

Kulakov and Kastro (2015) used in Table 1, the ‘oil pump installation project’ data studied by Solomon (1956). They concluded that 16.9% is the project’s rate of return and the GIRR. The same data is analysed and various estimates are prepared under DCF, CAS and MCAS methods and the results are furnished in Table 5 and summarized below:

1. The estimated IRRs are 25% and 400% by the CAS and DCF method. The multiple IRR problem associated with this NNCF is already reported by Solomon (1956).
2. The NPVs are zero at IRR 25% and 400% but not at GIRR of 16.9%. At GIRR of 16.9% the closing balance in the CAS is -497 (NPV = -363) and with MCAS the closing balance at 16.9% is -1870 (NPV = -1369). Under CAS and MCAS methods the closing balance indicates the NNCF is not enough to achieve a rate of return of 16.9% (GIRR). GIRR is therefore not the real rate of return. Under MCAS method the negative closing balance at 16.9% is due to the elimination of interest income on the positive OB. Neither the IRRs by DCF or CAS method nor the GIRR is real and the estimates are confusing with multiple IRR.
3. A unique IRR of -100% is estimated for this project by MCAS method. That means the entire capital of 1600 is lost (100% loss). The undiscounted SNCF is -1600 that also means the entire capital is lost (100% loss). The MCAS method indicates total capital loss (100%) (Net income is zero and therefore the ROC -1600 is unrecovered).
4. Kulakov[[8]](#footnote-9) and Kastro (2015, p.10) analysed the above NNCF and reported that "the highest interest rate resulting in a zero balance is equal to 16.9%. This rate is the project’s rate of return and it is the GIRR". However, under the MCAS at 16.9% neither the CB nor the NPV is zero and therefore not consistent with NPV and NCF. The MCAS estimated IRR of -100% is consistent with NPV and NNCF (both Zero at -100%) explains the capital loss of 100%. This finding rejects the argument of Magni (2010) that “the IRR is not capable of signaling the entire loss of investment”. Negative rate of return is normally common when the sum of NNCF is negative. The GIRR of 16.9% is not the correct IRR and therefore GIRR method is not appropriate.

Magni (2010) used a NNCF data set studied by Hazen (2003, p. 44) with an assumed market rate equal to 10% (see table 1, Magni 2010) and estimated an AIRR of -27.27% and claims that this rate is real rate. The estimated results are presented in Table 5 and summarized here:

### Table 5: An Evaluation of the AIRR and GIRR – An alternative to IRR

|  |
| --- |
| **CAS and DCF Estimates - Data Source: Kulakov & Kastro (2015)** |
| **Years** | **NNCF** | **DCF and CAS Estimates** |
| 0 | -1600 | Criteria | Values | Criteria | Value | Criteria | Value |
| 1 | 10000 | IRR 1 | 25% | NPV at 25% | 0 | MIRR (10,10) | 5.6% |
| 3 | -10000 | IRR 2 | 400% | NPV at 400% | 0 | MIRR (10,0) | 0.7% |
| SNCF | -1600 |  |  |  |  |  |  |
| **CAS at IRR (25%)** | **CAS at GIRR (16.9%)** |
| Opening Capital | -1600 | 8000.0 | Opening Capital | -1600 | 8129.6 |
| Interest at 25% | -400.0 | 2000.0 | Interest at 16.9% | -270.4 | 1373.9 |
| Income | 10000.0 | -10000.0 | Income | 10000.0 | -10000.0 |
| Closing Balance | 8000.0 | 0.0 | Closing Balance | 8129.6 | -496.5 |
| **MCAS at GIRR (16.9%)** | **MCAS at IRR (-100%)** |
| Opening Capital | -1600 | 8129.6 | Opening Capital | -1600 | 10000.0 |
| Interest at 16.9% | -270.4 | 0.0 | Interest at -100% | 1600.0 | 0.0 |
| Income | 10000.0 | -10000.0 | Income | 10000.0 | -10000.0 |
| Closing Balance | 8129.6 | -1870.4 | Closing Balance | 10000.0 | 0.0 |
| **CAS & DCF estimates - Data Source: Magni (2010)** |
| Years | NNCF | **DCF and CAS estimates** |
| 0 | -10 | Criteria | Values | Criteria | Value | Criteria | Value |
| 1 | 30 | IRR  | 0% | NPV | -5.00% | MIRR (10,10) | 3.7% |
| 3 | -25 | AIRR | 27.27% |  |  | MIRR (10,0) | -1.1% |
| SNCF | -5 |  |  |  |  |  |  |
| **CAS at IRR (0%)** |  | **CAS at AIRR (-27.27%)** |
| Opening Capital | -10 | 20.0 | Opening Capital | -10 | 22.7 |
| Interest - 0% | 0.0 | 0.0 | Interest - 27.27% | 2.7 | -6.2 |
| Income | 30.0 | -25.0 | Income | 30.0 | -25.0 |
| Closing Balance | 20.0 | -5.0 | Closing Balance | 22.7 | -8.5 |
| **MCAS at AIRR (-27.27%)** | **MCAS at IRR (-50%)** |
| Opening Capital | -10 | 22.7 | Opening Capital | -10 | 25.0 |
| Interest at -27.27% | 2.7 | 0.0 | Interest at -50% | 5.0 | 0.0 |
| Income | 30.0 | -25.0 | Income | 30.0 | -25.0 |
| Closing Balance | 22.7 | -2.3 | Closing Balance | 25.0 | 0.0 |

1. As reported by Magni (2010), there is no IRR under the DCF and CAS method. When an assumed rate of return of 0% is used the CAS indicates that there is a closing balance of -5. This is reasonable when the rate of return is zero the SNCF of -5 is the closing balance.
2. Magni (2010) identified an AIRR of -27.27%. When a CAS is prepared using this AIRR, the closing balance is -8.5. This is the sum of SNCF of -5 and the net interest of -3.5% (ROIC). The CB indicates that at -27.27%, the ROIC is not fully covered.
3. A MCAS prepared with AIRR of -27.27% ends up with a closing balance of -2.3 and not ‘0’. A real root must lead to zero and therefore the AIRR is not the right rate of return.
4. The MCAS method estimated IRR is -50% for this data set and the NPV is zero at -50%. The SNCF is -5 which is 50% of the investment (-10). The investment is $10, and the net income is $5 ($30-25) and therefore capital loss of $5 works out to 50% ((5/10) \*100). The IRR -50% (capital loss of 50%) looks realistic and consistent with NPV and NCF.

The lesson learnt from the GIRR and AIRR analyses is that there may be multiple roots for the polynomial (NPV equation) but the root which is consistent with the NCF is real. Neither the GIRR nor the AIRR is real and unreliable. Ultimately IRR, estimated by MCAS method, is the best outcome available to investment, project and cost-benefit analysis.

### 5. Summary and Conclusions:

The analysis in this paper addresses the important problem of multiple IRR that persisted for more than half a century. Several attempts were made to resolve this problem, and some end up with recommendation to use NPV or MIRR. Neither of them is the appropriate solution. There are several other indicators introduced in the recent past viz. GIRR, GERR, AIRR and others. None of these indicators is popular with practitioners. This paper introduces a modified CAS (MCAS) method that leads to a unique and real IRR and resolves the problem of multiple IRR. Detailed analyses are undertaken using some NNCF data, hypothetical or other published NNCF. Based on these analyses the following problems are identified:

1. The CAS developed using NNCF data reveals that there is positive opening balance (OB) in one or more years of cashflows with some of the NNCF. This problem is not transparent in DCF method, but the CAS method is more transparent and reveals it.
2. Interest on the positive OBs is credited to the cash flow and included as income or benefit stream. This is the case with some NNCF.
3. The positive OBs are in effect reinvestments and the interest on the positive OBs is an income on reinvestment. Inclusion of reinvestment income causes the problem of multiple IRR.
4. The problem of multiple IRR has nothing to do with IRR or NPV, but a problem associated with some NNCF data and the failure of DCF or CAS methods to handle those problems.
5. Six hypothetical NNCFs, with two or three changes of signs, are estimated using DCF and CAS methods. They all end up with unique IRR without the problem of multiple IRR.
6. These unique IRRs are zero or negative or positive corresponding to the zero or negative or positive SNCF, respectively. The results are consistent with Descartes rule but reveals that the Nostrom criteria are not always true.
7. A modified CAS (MCAS) method is developed by eliminating the reinvestment of intermediate income from the CAS method. MCAS is used to conduct various analysis.
8. The MCAS method resolves the problem of multiple IRR by eliminating reinvestment income. The estimated IRR under MCAS method is NCF and NPV consistent as the closing balance becomes zero, that means the NCF is fully utilised.
9. With NNCF, involving reinvestment, the NPV estimate varies under the scenario of with and without reinvestment. With NNCF, the higher NPV under the DCF/CAS method is caused by the reinvestment income. These NPV results are not consistent with IRRs (at IRR, NPV = 0). The NPV rule is therefore invalid with NNCF investment.
10. The MCAS estimate, using NNCF data, indicates that the MIRR by DCF/CAS method is not supported by the NNCF and therefore not NCF consistent. MCAS method also reveals that MIRR is an overestimate because of the reinvestment rate used in MIRR. MIRR is not an appropriate criterion.
11. The opportunity cost of intermediate income will be taken care of by investment elsewhere with market rate of interest. This finding supports the view expressed by Ross et. al (2013) that reinvestment income should not be included as part of the benefit stream. The project or the investment need not assume reinvestment at IRR or hurdle rate.
12. Neither the NPV nor the MIRR is the appropriate criterion to resolve the multiple IRR problem as claimed by some authors (see Schmidt, 2015; Syd 2017; Kierulff 2008). MIRR assumes reinvestment under normal NCF which is inappropriate. MCAS method is appropriate for both NCF and NNCF data. CAS method can be appropriate ONLY for the normal NCF.

The main conclusions are:

1. The problem of multiple IRR is caused by reinvestment income (at IRR or hurdle rate) included as benefit under some NNCF data (not all NCF and NNCF). The DCF method and CAS methods need modification to eliminate the reinvestment income under the NNCF.
2. With normal NCF and some of the NNCF also, there is reinvestment at IRR or at hurdle rate as wrongly asserted in many published works.
3. MCAS is an appropriate method to estimate the rate of return (IRR and NPV) for both normal NCF and NNCF. CAS method can be used only for normal NCF data.
4. The estimated IRR and NPV by MCAS method are consistent with NCF.
5. The AIRR and GIRR criteria recently introduced are neither real rate of returns nor NCF and NPV consistent, and therefore not appropriate.
6. Ultimately, IRR and NPV, estimated by MCAS, is the best criterion available to investment, project and cost-benefit analysis.

### Bibliography

1. Arjunan, K. C. (2017a), A New Method to Estimate NPV and IRR from the Capital Amortization Schedule and an Insight into Why NPV is not the Appropriate Criterion for Capital Investment Decision (January 15, 2017). Available at SSRN: <http://dx.doi.org/10.2139/ssrn.2899648>.
2. Arjunan, K.C. (2017b), IRR Performs Better than NPV: A Critical Analysis of Cases of Multiple IRR and Mutually Exclusive and Independent Investment Projects (February 8, 2017). Available at SSRN: <http://dx.doi.org/10.2139/ssrn.2913905>.
3. Arjunan, K.C. (2017c), Modified IRR (MIRR) Is a Spurious Criterion and Should Not Be Used in Cost-Benefit Analysis (CBA) and Investment Analysis (March 28, 2017). Available at SSRN: <http://dx.doi.org/10.2139/ssrn.2942456>.
4. Arjunan, K.C. and K. Kannapiran, (2017), The Controversial Reinvestment Assumption in IRR and NPV Estimates: New Evidence against Reinvestment Assumption (February 16, 2017). Economic Papers - Australia, Forthcoming. Available at SSRN: <http://dx.doi.org/10.2139/ssrn.2918744>.
5. Balyeat, B. And Julie Cagle, 2015, MIRR: The Means to an End? Reinforcing Optimal Investment Decisions Using the NPV Rule, Journal of Financial Education Vol. 41, No. 1 (SPRING 2015), pp. 90-102
6. Balyeat, Julie Cagle, and Phil Glasgo, (2013), Teaching MIRR to Improve Comprehension of Investment Performance Evaluation Techniques, Journal of Economics and Finance Education ∙ Volume 12∙ Number 1 ∙Summer 2013.
7. Brealey, R.A., Myers S.C. and Allen, P., 2006, Principles of Corporate Finance, 8th Edition, McGraw-Hill/Irwin, New York, 2006.
8. Brigham, E.F. and Houston, J.F. 2006. Fundamentals of Financial Management, Concise 5th Edition, Thomson South-Western, Ohio.
9. Brigham, E.F. and M.C. Ehrhardt, 2016. Financial Management: Theory & Practice, Edition 15, Cengage Learning, 2016, p.428.
10. Bussey, L.E. and Eschenbach, T.G. 1992. The Economic Analysis of Industrial Projects, Prentice–Hall, Englewood Cliffs, New Jersey.
11. Butenko, S. 2017, Rate of Return Analysis, The University of Toledo, College of Engineering, Available at: <http://www.eng.utoledo.edu/~nkissoff/lessons/Lesson7.html>,
12. C.S. PARK, 1997. Contemporary Engineering Economics, Addison–Wesley, Menlo Park, California.
13. Campani, C.H. 2014. On the Rate of Return and Valuation of Non-Conventional Projects, Business and Management Review Vol. 3(12) pp.
14. Eschenbach, T.G. 1995. Engineering Economy: Applying Theory to Practice, Irwin, Chicago.
15. Fleischer, G.A. 1994 Introduction to Engineering Economy, PWS Publishing, Boston.
16. Gronchi, S. 1986. On investment criteria based on the internal rate of return. Oxford Economic Papers, 1 (March), 174–180.
17. Hazen, G.B. 2003. Anewperspectiveonmultipleinternalratesofreturn.TheEngineeringEconomist, 48, 31–51.
18. Ille’s, M. 2016. The Real Reinvestment Rate Assumption as a Hidden Pitfall, ‘Club of Economics in Miskolc' TMP Vol. 12., No. 1., pp. 47-60.
19. Kulakov, N. Yu. and Kastro, A.B. 2015. Evaluation of Nonconventional Projects: GIRR and GERR vs. MIRR, *The Engineering Economist*, 00:1–14.

# Kyd, C. 2017, How Excel's MIRR Function Can Fix the IRR Function, Accessed on 8 July 2017, <http://www.exceluser.com/formulas/irr.htm>

1. Lohmann, J.R. 1988. “The IRR, NPV and the Fallacy of the Reinvestment Rate Assumptions,” The Engineering Economist, Vol. 33, No. 4, pp. 303– 330.
2. Lorie, J.H. and Savage, L.J. (1955) Three problems in rationing capital. The Journal of Business, 28(4), 229–239
3. Mackie, Peter; Nellthorp, John; Laird, James. 2005. Notes on the Economic Evaluation of Transport Projects: Fiscal Impacts. Transport Notes Series; No. TRN 8. World Bank, Washington, DC. Https://openknowledge.worldbank.org/handle/10986/11794 License: CC BY 3.0 Unported.”
4. Magni C.A. 2010. Average Internal Rate of Return and Investment Decisions: A New Perspective. *The Engineering Economist*, 55(2), 150-180”.
5. Moshe Ben-Horin & Yoram Kroll (2012): The Limited Relevance of the Multiple IRRs, The Engineering Economist, 7:2, 101-118.
6. Ng, E and Beruvides, M.G. 2014. Multiple internal rate of return revisited: frequency of occurrences. The Engineering Economist, 59, 1–13.
7. Norstrom, C.J. 1972. A Sufficient Condition for a Unique Nonnegative Internal Rate of Return, The Journal of Financial and Quantitative Analysis, Vol. 7, No. 3, pp. 1835-1839
8. Osborne, M. 2010, A resolution to the NPV-IRR debate? Quarterly Review of Economics and Finance, 50 (2). Pp. 234-239. ISSN 10629769.
9. Osborne, M. 2011, On the meaning of internal rates of return, SSRN Id: 1634819.
10. S. Ross, S., Westerfield, R. and Jordan, B. 2013. Fundamentals of Corporate Finance, New York, The McGraw-Hill, Irwin, p.290.
11. Schmidt, R. 2015. How to Use the Modified Internal Rate of Return (MIRR), propertymetrics, <https://www.propertymetrics.com/blog/2015/10/28/how-to-use-the-modified-internal-rate-of-return-mirr/>
12. Simon, B. 2006. Principles of Finance with Excel, Oxford University Press, New York.
13. Teichroew, D., Robichek, A.A. and Montalbano, M. 1965b. “Mathematical Analysis of Rates of Return under Certainty,” Management Science, Vol. 11, pp. 395–403.
14. Teichroew, D., Robichek, A.A. and Montalbano, M. 1965a. An Analysis of Criteria for Investment and Financing Decisions under Certainty, Management Science, Vol. 12, pp. 151–179.
15. Weber, T. A. 2014. On the (Non-)Equivalence of IRR and NPV, Journal of Mathematical Economics, Vol. 52, pp. 25--39, 2014.
1. Copyright © 2017 Kannapiran Chinna Arjunan. All rights reserved. Readers may make verbatim copies of this document for non-commercial purposes by any means, provided this copyright notice appears on all such copies. [↑](#footnote-ref-2)
2. The opinions expressed in this article are those of the author(s) based on private research, and they do not reflect in any way those of the institutions to which the author(s) is affiliated. [↑](#footnote-ref-3)
3. Normal Net cash flow (NCF) change signs only once. It starts with negative cash flows during the first year or first few years (investment years) followed by only positive NCFs for all the subsequent years. Non-normal NCF changes signs more than once with negative intermediate cashflows. [↑](#footnote-ref-4)
4. NCF-consistent means that the rate of return fully utilizes the NCF or the NCF is just enough to support that rate of return (NCF = 0 at that return). [↑](#footnote-ref-5)
5. There is no IRR or NPV method. Both IRR and NPV are estimated by DCF or CAS methods. IRR and NPV are, inter-related, estimates and if one is wrong the other must be wrong too. [↑](#footnote-ref-6)
6. Financial institutions do credit interest on credit balances in some form of loan accounts or alternatively transfer the credit balance to the customers’ other interest-bearing account. [↑](#footnote-ref-7)
7. Estimated IRR and NPV by DCF or CAS method perfectly match and therefore throughout this paper DCF or CAS or DCF/CAS refers to the same method. [↑](#footnote-ref-8)
8. N. Kulakov (personal communication by email dated 18 July 2017) communicated, after a review of this paper, that “for the project considered by Solomon: -1600, 10000, -10000, IRR1=25%, IRR2=400%, GIRR (0%)= -100%, GIRR (23%) = 16.9%, GIRR (25%)=25%, GIRR (27.7%)=35.6%. Accordingly, -100% is the GIRR that is “the highest interest rate resulting in a zero balance”. MCAS estimate exactly identified -100% as the unique IRR and not multiple rates. [↑](#footnote-ref-9)