G-GRADED SECONDARY REPRESENTATION OF THE G-GRADED MODULES

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Abstract

Let A be a commutative ring which is graded by a finitely generated abelian group G. In this paper we introduce the G-graded secondary representation of a G-graded module M of A.

1. Introduction

The results that are presented here are a sort of dual of the theory of graded primary decomposition of a group graded module over a graded commutative ring graded over finitely generated abelian group which was discussed in [2] and [5].

We shall begin by recalling briefly the salient feature of the theory, in a form convenient to the present discussion. Throughout, all rings will be commutative graded rings graded over finitely generated abelian group, and all modules will be unital.

Let A be a G-graded ring, M an G-graded A-module where G is finitely generated abelian group. It is clear that there exists a family $\{A_g\}_{g\epsilon G}$ of additive subgroups of A such that $A = \oplus_{g\epsilon G} Ag$ and $A_g A_h \subseteq A_{gh}$ for all g,h ϵ G and similarly for the G-graded A-module M there exists a given family $\{M_g\}_{g\epsilon G}$ of additive subgroups of M such that $M = \oplus_{g\epsilon G} Mg$ and $A_g M_h \subseteq M_{gh}$ for all g,h ϵ G. An element of a graded ring A is called homogeneous if it belongs to $\bigcup_{g\epsilon G} Ag$. Also, we write $h(A) = \bigcup_{g\epsilon G} Ag$. The summands Ag are called homogeneous components and elements of these summands are called homogeneous elements. If **a** ϵ A, then **a** can be written as $\sum_{g\epsilon G} \mathbf{a_g}$ where $\mathbf{a_g}$ is the component of **a** in Ag. In this case, Ae is a sub ring of A and 1_A ϵ Ae. Also, we write $h(M) = \bigcup_{g\epsilon G} Mg$.

A sub module N of M is G-graded if $N = \oplus_{g\epsilon G} Ng$, where $Ng = N \cap Mg$ for g ϵ G. In this case Ng is called the g-component of N for g ϵ G. More over M/N becomes a G-graded module with g-component $(M/N)_g = (Mg + N) / N$ for g ϵ G. Clearly 0 is G-graded sub module of M.

Also a graded ideal I of a G-graded ring A is an ideal verifying $I = \oplus_{g\epsilon G} (I \cap Ag) = \oplus_{g\epsilon G} Ig$. An ideal I of G-graded ring A is said to be G-prime ideal if $I \neq A$ and whenever ab ϵ I, we have either a ϵ I or b ϵ I, where a,b ϵ h(A) or equivalently a G-graded ideal I of A is G-prime if and only if for every two G-graded ideals J,K, $J K \subset I$ implies either $J \subset I$ or $K \subset I$.

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For each **a ϵ h(A),** Let $\lambda_{a,M}$ denote the endomorphism of M defined by multiplication by **a** i.e. $\lambda_a(x) = ax$ **for all x ϵ M.** Let $(\lambda_{a,M})^n = (\lambda_a)^n(x) = a^n x$ for some Natural number n and for all x ε M .Let $N^G(M)$ be that set of all a ε h(A) such that $\lambda_{a,M}$ is nilpotent. We prove that $N^G(M)$ is a G-graded ideal.

Proposition 1.1. If M is a G-graded module of the graded ring A, then the set

$$N^G(M) = \{ a \in h(A) : \lambda_{a,M} \text{ is nilpotent} \} \text{ a G-graded ideal.}$$

Proof.

$$\begin{aligned} N^G(M) &= \{ a \in h(A) : \lambda_{a,M} \text{ is nilpotent} \} \\ &= \{ a \in h(A) : (\lambda_{a,M})^n = 0 \text{ for some natural number } n \} \\ &= \{ a \in h(A) : (\lambda_a)^n(x) = 0 \text{ f or all } x \in M \} \\ &= \{ a \in h(A) : a^n x = 0 \} \end{aligned}$$

Clearly for any **a ϵ** $N^G(M)$ and for any homogeneous element **r ϵ** M we have **(-a) ϵ** $N^G(M)$ and **ar ϵ** $N^G(M)$ Now let **a, b ϵ** $N^G(M)$. There exists **x, y ϵ M** and **m, n** be Natural numbers such that $a^m x = 0$ **and** $b^n y = 0$.

Let $z = (x + y) \in M$.

Consider $(a+b)^{m+n}\, z = \sum_{i+j=m+n} a^i\, b^j\, z$

$$= \sum_{i+j=m+n} a^i\, b^j\, (x+y)$$

$$= \sum_{i+j=m+n} a^i\, b^j\, x \;+\; \sum_{i+j=m+n} a^i\, b^j\, y$$

$= 0 + 0$ (By Binomial Theorem $(a+b)^{m+n} = \sum_{i+j=m+n} a^i b^j$. Each term on the right hand side is as either $i \geq n$ or $j \geq m$.)

$= 0$

Therefore (a + b) ϵ $N^G(M)$ and hence $N^G(M)$ is a G-graded ideal of M.

The graded ideal $N^G(M)$ is called the G-graded nilradical of the A-module M. Also, in view of Definition 1.1 of [6], it can be observed as G-graded radical of the annihilator of M i.e. $N^G(M) = Gr(Ann(M))$.

Definition 1.2. A G-graded module M is said to be G-graded coprimary or G-coprimary if $M \neq 0$ and if, for each a ε h(A), the endomorphism $\lambda_{a,M}$ is either injective or nilpotent.

By the analog of proposition 2.4 of [1] we can conclude that $N^G(M)$ is a G-prime ideal say **p** and therefore **M** is said to be G-graded **p**- coprimary or G-**p**-coprimary. If M is any G-graded A-module of the G-graded ring A and **p** any G-prime ideal of A, a submodule Q of M is called G-graded p-primary or G-p-primary if the quotient M/Q is G-p-coprimary.

Now let M be a G-graded A-module, N a G-graded submodule of M. A G-graded primary decomposition of N in M is an expression of N as a finite intersection of G-p-primary submodules, say N = Q1 ∩ Q2 ∩ . . . ∩ Qn. The G-graded primary decomposition is minimal if (a) the G-prime ideals $p_i = N^G(M/Q_i) = Gr(Ann(M/Q_i))$ are all distinct and (b) none of the components in the intersection is redundant. Any G-graded primary decomposition can be refined to a minimal one. See [2] for detailed discussion. If N has a G-graded primary decomposition in M, we shall say that N is decomposable graded submodule of M, If in particular the zero submodule of M is decomposable, we shall say that M is G-good. A graded submodule N of M is decomposable if and only if the quotient G-graded module M/N is G-good. The aim of this paper is to dualize the theory developed in [2] and [5].

Definition 1.3. A G-graded A-module M is said to be G-secondary if $M \neq 0$ and if, for each $a \in h(A)$ the endomorphism $\boldsymbol{\lambda}_{\mathbf{a,M}}$ is either surjective or nilpotent.

Claim 1.4. If an A-module M is G-secondary, then $N^G(M) = Gr(Ann(M))$ is a G-prime ideal p.
Proof.
For a, b $\in$ h(A), let $ab \in Gr(Ann(M)) \Rightarrow (ab)^n M = 0$ for some Natural number $n > 0$.
If $b \notin Gr(Ann(M))$ $\boldsymbol{\lambda}_{\mathbf{b,M}}$ is surjective, that is $b^n M = M$. Then $a^n M = a^n(b^n M) = (ab)^n M = 0 \Rightarrow$ a is nilpotent. Thus, $a \in Gr(Ann(M))$ and $N^G(M)$ is G-prime.

Definition 1.5. Following Claim 1.4, M is said to be G-graded p-secondary or G-p-secondary.

Let M be a G-graded A-module. A G-graded secondary representation of M is an expression of M as a sum of G-graded secondary submodules, say $M = N_1 + N_2 + . . . + N_n$. This representation is said to be minimal if (a) the G-prime ideals $N^G(N_i)$ are all distinct and (b) none of the summands N_i is redundant. Any G-graded secondary representation of M can be refined to a minimal one. If M has a G-graded secondary representation, we shall say that M is G-representable. Let M be a G-representable G-graded A-module and let $M = N_1 + N_2 + . . . + N_n$ be a G-graded minimal secondary representation.
Our presentation and treatment of G-graded secondary representation and G-attached primes closely follows the one in MacDonald [3].

2. Graded Secondary representations

Let A be a G-graded commutative ring with identity where G is finitely generated abelian group. As stated in the introduction, a G-graded A-module M is said to be G-secondary if $M \neq 0$ and if, for each $a \in h(A)$, the endomorphism on M, $\boldsymbol{\lambda}_{\mathbf{a,M}}$ (i.e., multiplication by a in M) is either surjective or nilpotent. It is immediate that the graded nilradical of M is a prime ideal p and M is said to be G-graded p-secondary.

Proposition 2.1. Finite direct sums and non-zero quotients of G-graded p-secondary modules are G-graded p-secondary.
Proof.
It is sufficient to prove the result for two G-graded p-secondary modules. Let L and M be any two G-graded p-secondary modules with $p = Gr(Ann(L)) = Gr(Ann(M))$. Let $a \in h(A)$, assume $\boldsymbol{\lambda}_{\mathbf{a,M}}$ is not surjective. Then $a(L \oplus M) \neq (L \oplus M)$ which implies that either $aL \neq L$ or $aM \neq M$. Suppose $aL \neq L$ then $\exists\ k > 0$ such that $a^k L = 0$,which implies $a^k \in Ann(L)$ which in turn implies

$a \in p = Gr(Ann(L))$. But this means that $\exists\ l > 0$ such that $a^l \in Ann(M)$, that is, $a^l M = 0$. So, taking $n = \max(k, l)$ then $a^n (L \oplus M) = 0$. Thus, $(L \oplus M)$ is secondary. To show that $(L \oplus M)$ is p-secondary, let $ab \in Gr(Ann(L \oplus M))$, which means that $\exists\ n > 0$ such that $(ab)^n(L \oplus M) = 0$. But L, M were G-p-secondary, so either $a \in p = Gr(Ann(L))$ or $b \notin p$, that is a is nilpotent. From claim 1.4 we have $L \oplus M$ is G-p-secondary. Hence by induction finite direct sums of G-graded p-secondary modules are G-graded p-secondary.

Let M be a G-graded p-secondary module, so that $p = Gr(Ann(M))$.Let $\lambda : M \to M' = M/N$ be the natural projection from M to a non-zero quotient of M . Let $a \in h(A)$, with $\lambda_{a,M}$ is surjective, that is $aM = M$ which implies $aM' = M'$ as $aM + N = M$. Otherwise, $\exists\ k > 0$ such that $a^k M = 0 \Rightarrow a^k M' = 0 \Leftrightarrow a^k M + N = N$ as $a^k M = 0$. This also shows that $a^k \in Ann(M/N)$ $\Rightarrow x \in Gr(Ann(M/N))$. Then, as before $Gr(Ann(M')) = p$ which implies that M is G-p-secondary.

Proposition 2.2. The annihilator of a G-graded p-secondary module is G-graded p-primary ideal.
Proof.
Let M be a G-graded p-secondary module and let $Ann(M) = I$. Let $ab \in I$ and assume $b^n \notin I$ for all n. Then for $b \in h(A)$ either $bM = M$ or $\exists\ n > 0$ such that $b^n \in Ann(M)$ which we assumed otherwise, so $bM = M$. Thus, $ab \in I \Rightarrow abM = 0 \Rightarrow aM = 0 \Rightarrow a \in I$. Thus, I is primary. Since $p = Gr(Ann(M) = \sqrt{I} \Rightarrow$ I is G-p-primary.

References

[1] S. Behara and S. D. Kumar, Group graded associated ideals with flat base change of rings and short exact sequences, Proceedings-Mathematical Sciences, (2011)
[2] S. D. Kumar, S. Behara, Uniqueness of graded primary decomposition of graded modules graded over finitely generated abelian groups, Communications in Algebra, 39, (2011).
[3] I.G.Macdonald,Secondary representation of modules over a commutative ring, Sympos.Math. XI, (1973), 23-43.
[4] C. Năstăsescu, F. van Oystaeyen, Methods of graded rings, LNM 1836, Springer, 2004.
[5] M. Perling and S. D. Kumar, Primary decomposition over rings graded by finitely generated Abelian groups, J. Algebra, 318 (2007), 553–561.
[6] M. Refai, K. Al-Zoubi, On Graded Primary Ideals, Truk. J. Math, 28 (2004), 217-229.
[7] R.Y. Sharp, Asymptotic behavior of certain sets of attached prime ideals. J. London Math. Soc., 34 (1986), 212-218.