**The New General Integral Transform Decomposition method for solving nonlinear Volterra integral equation based on Numerical formula**

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**Abstract**

In this paper, we propose a new method, namely New General Integral Decomposition method (GIDM) for solving nonlinear Volterra integral equations. This method is a combination of the new general integral transform method and decomposition method. The nonlinear terms can be easily handled by the use of Adomian polynomials. The technique is described and illustrated with some examples. The results reveal that the proposed method is very efficient, simple and can be applied to other nonlinear problems.

**Keywords :**New Genenral integral decomposition method, Nonlinear integral equations, Adomian polynomials, Noise terms phenomena.

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# 1 Introduction

For solving nonlinear functional equation , Adomian decomposition method was introduced by George Adomian in 1980[3]. This technique provides an infinite series solution of the equation and nonlinear term is decomposed into an infinite series Adomian poynomials. Several linear and nonlinear ordinary ,partial and stochastic differential equations are solved easily by Adomian decomposition method . In this work , New general integral transform technique in combination with Adomian decomposition method is presented and modified This article considers the effectiveness of the new general integral decomposition method (NGDM )in solving nonlinear equations. In 2020 Jafari H.[1] introduced a new integral transform, named the the new general integral transform. In the present paper we focus solve nonlinear volterra integral equations. Nonlinear volterra integral equation arise in many scientific fields such as the popukation dynamics, spread of epidemics and semiconductor devices. In this paper we have followed the combined New general integral transform and Adomian decomposition method but while decomposing the nonlinear terms using decomposing the nonlinear term using Adomian polynomials, we have substituted the term $u_i$ with Newton Raphson formula. As we know that Newton Raphson formula is used for fibding the better approximate solution of real valued function.

## 2 Preliminaries

**Definition 2.1 The New General Integral Transform**
*Let $f(t)$ be a integrable function defined for t≥0 ,p(s)≠0 and q(s) are positive real functions , we define the general integral transform T(s) of $f(t)$ by the formula*

$$T[f(t):s] = T(s) = p(s)\int_{o}^{\infty} f(t)e^{-q(s)t}dt \tag{1}$$

*provided the integral exists for some q(s).*

**Definition 2.2 Nonlinear Volterra integral equation of the second kind**
*Consider the following nonlinear Volterra integral equation with difference kernel i.e.$k(x,t) = k(x-t)$ defined as*

$$u(x) = f(x) + \int_{0}^{x} k(x-t)F(u(t))dt \tag{2}$$

*where $f(x)$ is known real valued function and $F(u(x))$ is the nonlinear function of $u(x)$.*

Apply New General integral transform on both sides of (2) . After that using the linear property and convolution theorem of New General integral transform , we have

$$T[u(x)] = T[f(x)] + T[k(x-t)]T[F(u(t))] \tag{3}$$

The Methodology consists of approximating the solution of (2) as an infinite series given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{4}$$

However , the nonlinear term $F(u(x))$is decomposed as

$$F(u(x)) = \sum_{n=0}^{\infty} A_n(x) \tag{5}$$

where $A_n's$ are modified Adomian polynomials which are based on newton raphson formula given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}F\left(\sum_{i=0}^{n}\lambda^i\left(u_i - \frac{F(u_i)}{F'(u_i)}\right)\right) \tag{6}$$

substituting equation (4) and (5) into (3), we get

$$T\left[\sum_{n=0}^{\infty}u_n(x)\right] = T[f(x)] + T[k(x-t)]T\left[\sum_{n=0}^{\infty}A_n(x)\right] \tag{7}$$

using the linearity property of New general integral transform , we get

$$\left[\sum_{n=0}^{\infty}T[u_n(x)]\right] = T[f(x)] + T[k(x-t)]\left[\sum_{n=0}^{\infty}T[A_n(x)]\right] \tag{8}$$

To determine the terms $u_0(x), u_1(x), u_2(x), ...$ of infinite series, comparing both sides of (8), we have the following iterative scheme

$$T[u_0(x)] = T[f(x)] \tag{9}$$

In general , the relation is given by

$$T[u_{(n+1)}(x)] = T[k(x-t)]T[A_n(x)] \tag{10}$$

applying the inverse general transform to (9) and (10) we get

$$u_0(x) = T^{-1}[T[f(x)]] \tag{11}$$

$$[u_{(n+1)}(x)] = T^{-1}[T[k(x-t)]T[A_n(x)]] \tag{12}$$

Adapting the value of $u_0(x)$ into (6) gives the value of $A_0$ and then using the general iterative relation (12, we get values of $u_1(x)$,$u_(2)$,$u_3(x)$,...and so on , which finally gives solution (4) to the given Volterra integral equation.
The effectiveness of this technique for solving Volterra integral equations is shown by following numerical examples. here we have also find the maximum absolute error estimation to show the adequancy of technique given as
$e_j = Max|u_e x - u_a pp|$
where $e_j$ denotes the maximum absolute error at some $x_j$ in the given interval.

**Example 2.1** *Consider the following Volterra integral equation [13,25]*

$$u(x) = x + \int_0^x u^2(t)dt \tag{13}$$

*which has the exact solution as $u(x) = tan(x)$.*

**Solution.**Taking General integral transform on both side of equation (13) ans using the linearity property of general integral transform , we have

$$T[u(x)] = T[x] + T\left[\int_0^x u^2(t)dt\right] \tag{14}$$

that is

$$T[u(x)] = \frac{p(s)}{q(s)^2} + \frac{1}{q(s)}T[u^2(t)] \tag{15}$$

Using above technique , we have

$$T\left[\sum_{n=0}^{\infty} u_n(x)\right] = \frac{p(s)}{q(s)^2} + \frac{1}{q(s)}T\left[\sum_{n=0}^{\infty} A_n(x)\right] \tag{16}$$

where the nonlinear term $F(u(x)) = u^2(x)$ is decomposed using the formula given by (6). Certain terms of modified Adomian Polynomials are as follows:

$A_0 = (\frac{1}{2})^2 {u_0}^2$

$A_1 = (\frac{1}{2})^2 u_0 u_1$

$A_2 = (\frac{1}{2})^2 (2u_0u_2 + {u_1}^2)$

$A_3 = (\frac{1}{2})^2 (2u_0u_3 + 2u_1u_2)$

Comparing both side of equation (16) gives,

$$T[u_0(x)] = \frac{p(s)}{q(s)^2} \tag{17}$$

In general

$$T\left[u_{(n+1)}(x)\right] = \frac{1}{q(s)}T\left[A_{(n)}(x)\right] \tag{18}$$

Applying inverse General integral transform on both side of (17),gives

$$u_0(x) = x, \tag{19}$$

use genereal relation, we have

$$u_1(x) = \frac{x^3}{12}, \tag{20}$$

Continuing in this manner, we get

$u_2(x) = \frac{x^5}{420}$,

$u_3(x) = \frac{11x^7}{20160}$,

$u_4(x) = \frac{x^9}{13440}$,

Subsequently, the approximate solution becomes

$u(x) = x + \frac{x^3}{12} + \frac{x^5}{420} + \frac{11x^7}{20160} + \frac{x^9}{13440} + ...$

The exact solution and the one obtained by our technique corresponding to distinct values of x are presented in table 1 and demonstrated through figure 1. The absolute error laid out in the table admit that the solutions are very much closed to the exact solution and the maximum absolute error is 0.0002

| x | Exact Solution | Approximate Solution | Absolute Error |
|---|---|---|---|
| 0 | 0 | 0 | 0.000 |
| 0.01 | 0.010000333 | 0.01000008333 | 2.4967E-07 |
| 0.02 | 0.020002667 | 0.02000066668 | 2.0002E-06 |
| 0.03 | 0.030009003 | 0.03000226479 | 6.7382E-06 |
| 0.04 | 0.040021347 | 0.04000533376 | 1.6013E-05 |
| 0.05 | 0.050041708 | 0.05001041797 | 3.129003E-05 |
| 0.06 | 0.060072104 | 0.06001800324 | 5.410076E-05 |
| 0.07 | 0.070114558 | 0.07002859034 | 8.5698966E-05 |
| 0.08 | 0.080171105 | 0.08004268033 | 1.2842467E-04 |
| 0.09 | 0.090243790 | 0.09006077463 | 1.8301537E-04 |
| 0.1 | 0.100334672 | 0.1000833751 | 2.530921E-04 |

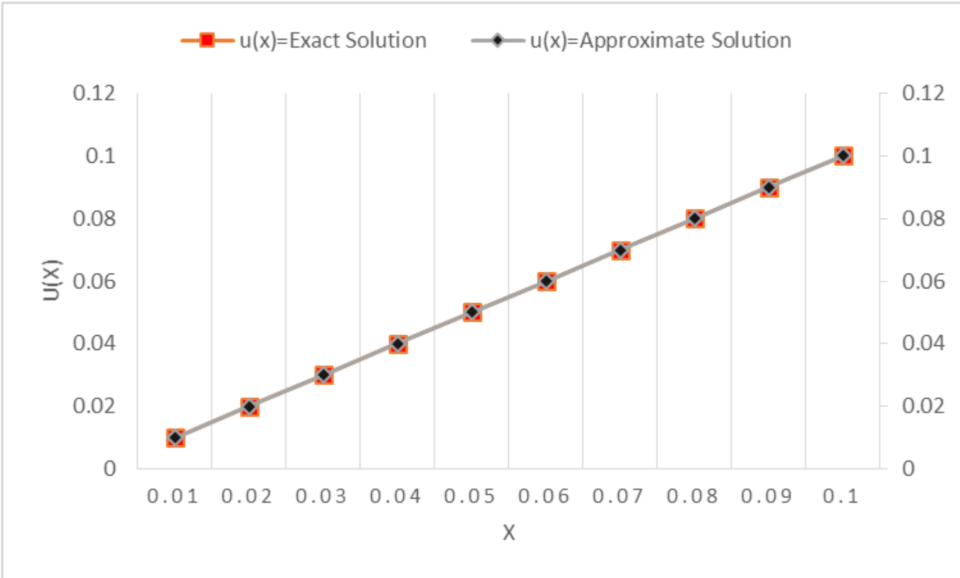


Fig. 1 – Comparison of Exact Solution and Approximate solution.

**Example 2.2** *Consider the following Volterra integral equation [21]*

$$u(x) = 2x - \frac{x^4}{12} + 0.25\int_0^x (x-t)u^2(t)dt \tag{21}$$

*which has the exact solution as* $u(x) = 2x$.

**Solution.** Taking General integral transform on both side of equation (21) ans using the linearity property of general integral transform , we have

$$T[u(x)] = T[2x - \frac{x^4}{12}] + .0.25T\left[\int_0^x (x-t)u^2(t)dt\right] \tag{22}$$

that is Using above technique , we have

$$T\left[\sum_{n=0}^{\infty} u_n(x)\right] = T[2x - \frac{x^4}{12}] + \frac{1}{p(s)}T[x]T\left[\sum_{n=0}^{\infty} A_n(x)\right] \tag{23}$$

Comparing both side of equation (23) gives,

$$T[u_0(x)] = T\left[2x - \frac{x^4}{12}\right] \tag{24}$$

In general

$$T\left[u_{(n+1)}(x)\right] = \frac{1}{p(s)} T[x] T\left[\sum_{n=0}^{\infty} A_n(x)\right] \tag{25}$$

Applying inverse General integral transform on both side of we gives

$$u_0(x) = 2x - \frac{x^4}{12}, \tag{26}$$

use genereal relation, we have

$$u_1(x) = \frac{x^4}{48} - \frac{x^7}{2028} + \frac{x^10}{207360}, \tag{27}$$

Continuing in this manner, we get

$u_2(x) = -\frac{x^6}{4777574400} + \frac{37x^13}{905748480} - \frac{11x^10}{2903040} + \frac{x^7}{8064}$

Subsequently, the approximate solution becomes

$u(x) = 2x - \frac{x^4}{16} - \frac{x^7}{2688} + \frac{x^{10}}{967680} + \frac{37x^{13}}{905748480} - \frac{11x^{16}}{4777574400} + ...$

The numerical results shown in table

| x | Exact Solution | Approximate Solution | Absolute Error |
|---|---|---|---|
| 0 | 0 | 0 | 0.000 |
| 0.05 | 0.1 | 0.09999609 | 3.9063E-07 |
| 0.1 | 0.2 | 0.19999375 | 6.2500E-06 |
| 0.15 | 0.3 | 0.299968359 | 3.1641E-05 |
| 0.2 | 0.4 | 0.3999899995 | 1.0000E-04 |
| 0.25 | 0.5 | 0.499755837 | 2.4416E-04 |
| 0.3 | 0.6 | 0.599493669 | 5.0633E-04 |
| 0.35 | 0.7 | 0.69906187 | 9.813E-04 |
| 0.40 | 0.8 | 0.798399391 | 1.6006E-04 |
| 0.45 | 0.9 | 0.89743572 | 2.5643E-03 |
| 0.5 | 1 | 0.996090845 | 3.9092E-03 |

Maximum absolute error is 0.03

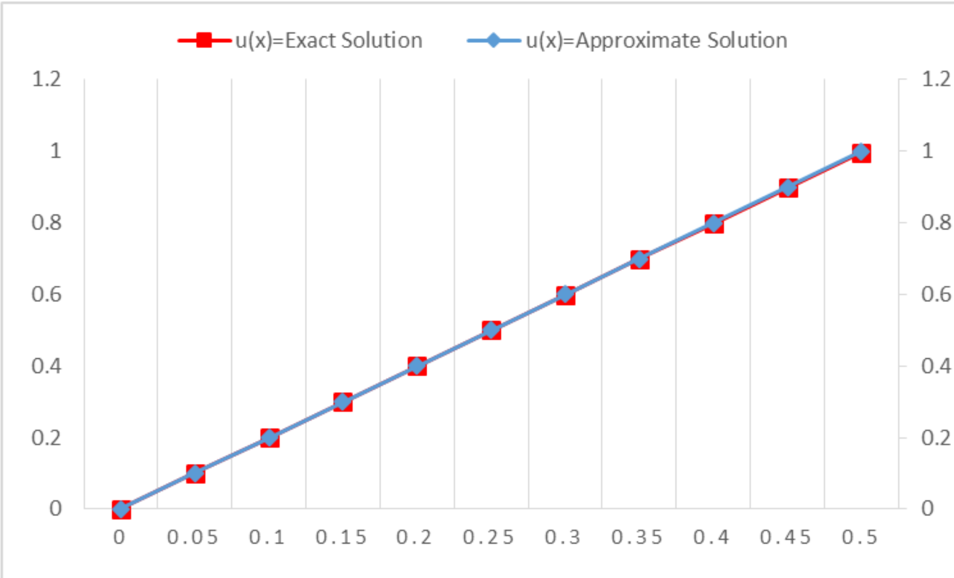


Fig. 2 – Comparison of Exact Solution and Approximate solution.

# 3 Conclusion

In this paper we used combination od Adomian Polynomials method and general integral transform for solution of nonlinear integral equation as demonstrated through solution tables and their graphs , it is observed that the approximate solution obtained by this technique are more accurate and error is minimum. The technique used in the paper is easy to implement and provides more accurate solution for nonlinear integral equation.

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