Reducing Modified Formulas For Sinh-Gordon Equation to the Painleve Equations

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Abstract:
By using the mixed variable transformation $\varphi = lny(X); X = ax + bt + \gamma$ we derived some modified formulas for sinh-Gordon equation reducible to the Painleve equations and we found the exact solutions for them and for other equations derived from them.

Keyword: Sinh-Gordon Equation; Painleve equations.

1. Introduction

Investigation of exact solutions of nonlinear evolution equations [7] has been a hot topic of research for several decades. The powerful methods used for this purpose are for example, Backlund transformation[4], inverse scattering technique[1],Hirota's direct method [8], tanh method [11],series method [3,10], Jacobian elliptic function expansion method and its extension [9, 12, 13], and the algebraic method[2]. For importance of Painleve equations (P-type equations) [14,15], some researchers tried using transformations to transform the partial differential equations or new modified equations to P-type equations, In [5], new modified forms for Sine-Gordon equation $\phi_{xx} - \phi_{tt} = \sin\phi$ reduced to the p-type equations by using mixed variables transformation (MVT):

$$\phi = 4\tan^{-1} u(X) \;;\; X = \frac{x - ct}{\sqrt{1 - c^2}}$$

The general form of ordinary differential equation with no movable critical points of order two is[6]:

$$\frac{d^2 w}{dx^2} = L(x,w)\left(\frac{dw}{dx}\right)^2 + M(x,w)\frac{dw}{dx} + N(x,w) \tag{1.1}$$

where *L, M and N* are fractional functions in *w*.
There are two basic conditions to determine *L, M and N*:
a. $L(x,w) = 0$ or one of the following forms:

$$L(x,w) = \frac{m+1}{m(w-a_1)} + \frac{m-1}{m(w-a_2)} \qquad ,m \geq 1$$

$$L(x,w) = \frac{1}{2}\sum_{n=1}^{4}\frac{1}{w-a_n}$$

$$L(x,w) = \frac{2}{3}\sum_{n=1}^{3}\frac{1}{w-a_n} \tag{1.2}$$

$$L(x,w)=\frac{3}{4}\left(\frac{1}{w-a_1}+\frac{1}{w-a_2}\right)+\frac{1}{2}\left(\frac{1}{w-a_3}\right)$$

$$L(x,w)=\frac{1}{6}\sum_{n=1}^{3}\frac{n+2}{w-a_n}$$

where a_n are arbitrary function for x

b.M and N must be as the following forms:

$$M(x,w)=\frac{m(x,w)}{l(x,w)} \text{ and } N(x,w)=\frac{n(x,w)}{l(x,w)} \tag{1.3}$$

Where $l(x,w)$ with l degree in w is a least common denominator(LCD) for partial fractions in $L(x,w)$ and $m(x,w)$, $n(x,w)$ are polynomials in w with degree greater than or equal $l+1$ and $l+3$ respectively.

2. Main Result

Consider the nonlinear partial differential equation:

$$\varphi_{tt}-\varphi_{xx}=\sinh\varphi \tag{2.1}$$

Let the modified formula of the equation (2.1) as:

$$\varphi_{tt}-\varphi_{xx}-\sinh\varphi=H(\varphi,\varphi_x) \tag{2.2}$$

Let we take the mixed variables transformation:

$$\varphi=\ln y(X); X=ax+bt+\gamma \tag{2.3}$$

Where a, b and γ are constants and $a^2\neq b^2$.

By inserting (2.3) in (2.2) we have:

$$y_{XX}=\frac{1}{y}y_X^2+\frac{1}{2(b^2-a^2)}(y^2-1)+\frac{y}{(b^2-a^2)}G(y,y_X) \tag{2.4}$$

Where $G(y,y_x)$ is the result function from inserting (2.3) in $H(\varphi,\varphi_x)$.

Let $G(y,y_X)=G_1(y)y_X^2+G_2(y)y_X+G_3(y)$ (2.5)

By inserting (2.5) in (2.4) we have:

$$y_{XX}=\left(\frac{1}{y}+\frac{y}{(b^2-a^2)}G_1(y)\right)y_X^2+\frac{y}{(b^2-a^2)}G_2(y)y_X+\frac{y^2-1}{2(b^2-a^2)}+\frac{y}{(b^2-a^2)}G_3(y) \tag{2.6}$$

From compare equation (2.6) with equation (1.1) we have:

$$L(X,y)=\frac{1}{y}+\frac{y}{(b^2-a^2)}G_1(y)$$

$$M(X,y)=\frac{y}{(b^2-a^2)}G_2(y)$$

$$N(X,y)=\frac{y^2-1}{2(b^2-a^2)}+\frac{y}{(b^2-a^2)}G_3(y)$$

Case1:

If $L(X,y)=0$, then $M(X,y)$ and $N(X,y)$ are polynomials of degree 1 and 3 respectively, that is mean:

$$\frac{1}{y}+\frac{y}{(b^2-a^2)}G_1(y)=0$$

$$\frac{y}{(b^2-a^2)}G_2(y)=Ay+B$$

$$\frac{y^2-1}{2(b^2-a^2)}+\frac{y}{(b^2-a^2)}G_3(y)=Cy^3+Dy^2+Ey+F$$

Where $A,B,C,D,E,F$are arbitrary functions of X only and A,Ctakes the following special values.

$a)\ A=0, C=0$ $\quad$ $b)\ A=-2, C=0$

$c)\ A=-3, C=-1$ $\quad$ $d))\ A=-1, C=1$

$e) A=0, C=2$

Substituting $G_1(y), G_2(y), G_3(y)$in (2.5), we have:

$$G(y,y_X)=\frac{(a^2-b^2)}{y^2}y_X^2+\frac{(b^2-a^2)}{y}(Ay+B)+\frac{(b^2-a^2)(Cy^3+Dy^2+Ey+F)}{y}-\frac{y^2-1}{2y} \quad (2.7)$$

Since from (2.3) we have $y=e^{\varphi}$ then :

$$H(\varphi,\varphi_x)=\frac{(a^2-b^2)}{a^2}\varphi_x^2+(A_1e^{\varphi}+B_1)\varphi_x+C_1e^{2\varphi}+D_1e^{\varphi}+F_1e^{-\varphi}+E_1-sinh\varphi \quad (2.8)$$

Where A_1,B_1,C_1,D_1,E_1,F_1 are the new functions.

(i) If $A_1=B_1=C_1=D_1=E_1=F_1=0$, then equation (2.8) will be:

$$H(\varphi,\varphi_x)=\frac{(a^2-b^2)}{a^2}\varphi_x^2-sinh\varphi$$

That is mean under the transformation (2.3) the partial differential equation obtained is:

$$\varphi_{tt}-\varphi_{xx}=\frac{(a^2-b^2)}{a^2}\varphi_x^2 \quad (2.9)$$

And this will be reducible to the following P-type equation:

$$y_{XX}=0 \quad (2.10)$$

The general solution for (2.10) in terms of $x\ and\ t$ is:

$$y(x,t)=a_1x+b_1t+\theta$$

Exact solution for (2.9) is:

$$\varphi(x,t)=\ln(a_1x+b_1t+\theta)$$

From the transformation (2.3), we obtain:

$$\varphi_x=\frac{a}{y}y_X$$

And since the first integral of (2.10) is $y_X=c$, where c is arbitrary constant this leads to

$$\varphi_x=\frac{ac}{e^{\varphi}} \quad (2.11)$$

Substituting (2.11) in equation (2.9) we obtain the partial differential equation:

$$\varphi_{tt}-\varphi_{xx}=(a_1^2-b_1^2)e^{-2\varphi} \quad (2.12)$$

Where a_1,b_1 are new constants.

An exact solution for (2.12) can be obtained if we integrate (2.11) as:

$$\int e^{\varphi}\,d\varphi=\int ac\,dx$$

Therefore:

$$\varphi(x,t)=\ln(a_1x+k)$$

Where k is constant.

(ii) If $A_1=B_1=D_1=E_1=F_1=0\ and\ C_1=6$ then:

$$H(\varphi,\varphi_x)=\frac{(a^2-b^2)}{a^2}\varphi_x^2+C_1e^{2\varphi}-sinh\varphi$$

That is mean under the transformation (2.3) the partial differential equation obtained is:

$$\varphi_{tt} - \varphi_{xx} = \frac{(a^2-b^2)}{a^2}\varphi_x^2 + 6e^{2\varphi} \quad (2.13)$$

And this will be reducible to the following P-type equation:

$$y_{XX} = 6y^2 \quad (2.14)$$

From the transformation (2.3), we obtain:

$$\varphi_x = \frac{a}{y} y_X$$

And since the first integral of (2.14) is $y_X = \sqrt{4y^3 + k}$, where k is arbitrary constant , this leads to

$$\varphi_x = \frac{a}{e^{\varphi}}\sqrt{4e^{3\varphi} + k} \quad (2.15)$$

Substituting (2.15) in equation (2.13) we obtain the partial differential equation:

$$\varphi_{tt} - \varphi_{xx} = 4(a^2 - b^2)e^{\varphi} + k(a^2 - b^2)e^{-2\varphi} + 6e^{2\varphi} \quad (2.16)$$

An exact solution for (2.16) can be obtained as:

If $k = 0$ in (2.16),then:

$$\int e^{\frac{-\varphi}{2}} d\varphi = 2a \int dx$$

$$\varphi(x,t) = 2\ln\left(\frac{1}{ax + w(t)}\right)^2$$

Where $w(t)$ is arbitrary function.

Case2:

$L(X,y) \neq 0$

If we take the P-type equation

$$y_{XX} = \frac{1}{y} y_X^2 \quad (2.17)$$

And compare it with the equation (2.4) we obtain:

$$G(y, y_X) = \frac{1-y^2}{2y}$$

Therefore:

$$H(\varphi, \varphi_x) = -\sinh\varphi$$

That is mean under the transformation (2.3) the partial differential equation obtained is:

$$\varphi_{tt} - \varphi_{xx} = 0 \quad (2.18)$$

An exact solution for (2.18) can be obtained as:

From equation (2.17), we have:

$$y = c_1 e^{c_2 x}$$

Therefore:

$$\varphi(x,t) = k_1 x + k_2 t + \mu$$

Where k_1, k_2 *and* μ are constants.

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