Local structure-preserving algorithms for the RLW equation ☆

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Abstract

In this paper, two local structure-preserving algorithms for solving the regularized long wave (RLW) equation are constructed. By using the implicit midpoint method and the AVF method for the temporal and spatial discretization respectively, we present a local momentum-preserving algorithm for the RLW equation. Furthermore, by alternating the above two methods for the discretization in different direction, we get a local energy-preserving algorithm. The first algorithm is proved to be local momentum preserving and the second is local energy preserving. With periodic boundary conditions, the algorithms admit global momentum and energy conservation law. Propagation of single, double and triple solitary waves, the conservation properties of mass, energy and momentum of the RLW equation are presented to demonstrate the conservation properties of the two schemes.

Keywords: RLW equation; Local structure-preserving; Average Vector Field (AVF) method; Momentum; Energy.

1. Introduction

The regularized long wave (RLW) equation

$$u_t + u_x - \sigma u_{xxt} + \frac{\nu}{2}(u^2)_x = 0, \quad a < x < b, 0 < t < T, \tag{1.1}$$

subject to the initial and boundary condition

$$\begin{aligned} &u(a,t) = u(b,t), \quad 0 < t < T, \\ &u(x,0) = u_0(x), \quad a < x < b. \end{aligned} \tag{1.2}$$

was first put forward by Peregrine [1] to describe a model for long waves on the surface of water in a channel, and later by Benjamin et al. [2]. P. J. Olver [3] presented that the RLW equation posses only three independent conservation laws:

Mass conservation law

$$\mathcal{S} = \int_a^b u dx = C_1. \tag{1.3}$$

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Global momentum conservation law (GMCL)

$$\mathcal{M} = \frac{1}{3}\int_a^b u^2 + \sigma(u_x)^2 dx = C_2. \tag{1.4}$$

Global energy conservation law (GECL)

$$\mathcal{E} = \frac{1}{2}\int_a^b (\nu u^3 + 3u^2) dx = C_3. \tag{1.5}$$

This equation has drawn much attention for decades and has motivated a series of studies in physics and mathematics. Various numerical methods have been used to investigate the solutions for the RLW equation [4–7]. In particularly the finite difference method in [8], finite element methods such as the Galerkin method in [9], collocation methods in [10] and so on.

As is well known, a basic principle to design proper numerical methods is constructing structure preserving or conservative numerical methods that preserve intrinsic properties of the original system as well as possible. The numerical method which can preserve at least some structural properties of the system is called structure-preserving numerical method or geometric integrator.

In recent years much work have been devoted to finding unified frameworks for the structure-preserving methods. For example, the discrete variational derivative (DVD) method in [11], the average vector field (AVF) method in [12] and the discrete partial derivative (DPD) method in [13]. Theoretical analysis and practical computations shown that these methods have superior performance in long time simulations.

Lots of conservation PDEs have the properties of multi-symplectic structure, energy, mass and momentum conservations [14–17]. To inherit the multi-symplectic structure, many authors have paid attention to multi-symplectic scheme in recent years. The authors discussed the multi-symplectic scheme for the RLW equation, such as the multi-symplectic Preissman scheme in [18], the multi-symplectic Rung-kutta scheme in [19], the multi-symplectic Euler-box scheme in [20] and so on. All of these methods focus on the p-reservation of some kinds of discrete multi-symplecticity. Except the multi-symplectic conservation law, the RLW equation also preserve the energy and momentum conservation law. In [19] the authors proposed a multi-symplectic Euler-box scheme which can preserve the mass, energy and moment conservation law.

The conservation of energy and momentum are crucial properties for the Hamiltonian system. It should be noted that the structure preserving property has its own constrain. The reason is that the conservation law is defined on the global times level and it inevitably depends on the boundary conditions. In other words, if the boundary conditions are not suitable, the energy-preserving algorithms can not be applied to the problem. Therefore, whether and how to construct local energy preserving algorithms are interesting. In reference [21] Wang et al. proposed the concept of the local structure-preserving algorithms for PDEs and then constructed some algorithms preserved the multi-symplectic conservation law, local energy and momentum conservation laws for the Klein-Gordon-Schrödinger (KGS) equation. The outstanding advantage of the local structure-preserving algorithms is that they conserve the local structures of PDEs in any local time-space region. For example local energy-preserving algorithms preserve the discrete global energy under suitable boundary conditions such as periodic or homogenous boundary conditions. Thus in the case of multi-symplectic PDEs they cover the traditional global energy-preserving algorithms. In [22] Cai et al. and in [23] Chen et al. constructed some local structure-preserving algorithms for special multi-symplectic PDEs. In [24], Gong et al. developed a general approach to constructing local structure-preserving algorithms by using the AVF methods. In this paper we want to construct new local multi-symplectic structure-preserving algorithms for the RLW equation.

The AVF method is first written down in [25] and identified as energy-preserving and as a B-series method in [26]. For the differential equation

$$\dot{y} = f(y), \quad y \in \mathbb{R}^n,$$

the AVF method is the map $y \mapsto y'$ defined by

$$\frac{y' - y}{\tau} = \int_0^1 f((1-\xi)y + \xi y')dy,$$

where τ is the time step. The AVF method is a B-series method, is affine-covariant, self-adjoint and of order 2. When f is Hamiltonian with respect to a constant symplectic structure, i.e. $f = \Omega^{-1}\nabla H$ with Ω a nonsingular skew-symmetric matrix, the AVF method preserves the energy Ω. By using the AVF method, Ref. [12] gave a class of systematic energy-preserving methods based on symplectic formulation of Hamiltonian PDEs. Basic idea of the method is to first discrete PDEs with uniform spatial-discretization resulting in a large system of Hamiltonian ODEs. The resulting ODEs are then integrated by the AVF method. Currently the second-order AVF method

$$\frac{y_{n+1} - y_n}{\tau} = \int_0^1 f((1-\xi)y_n + \xi y_{n+1})dy,$$

attracts many attentions, see Ref. [15-18] and the references therein.

In this paper, we will focus on the local preserving algorithm for the RLW equation, including a local momentum-preserving scheme and a local energy preserving scheme. First we propose a local momentum-preserving scheme with implicit midpoint scheme in time and the AVF method in space for the RLW equation. Then we construct a local energy-preserving scheme with the AVF method in time and implicit midpoint scheme in space.

The remainder of this paper is organized as follows: In section 2, we introduce some operator definitions and its properties. In section 3, the local momentum-preserving algorithm is developed for the RLW equation. In section 4, the local energy-preserving algorithm is proposed for the RLW equation. We also prove some conservation properties of the methods. The numerical experiments are presented in section 5 to illustrate our theoretical predictions.

2. Preliminaries

2.1. Operator definitions and properties

In this section, we give some operator definitions and its properties. We first introduce some notations: the spatial domain $I = [a, b]$ and $L = b - a$, $x_j = a + hj$, $j = 0, 1, 2, \cdots N-1$, where $h = (b-a)/N$ is the spatial length. $t_k = k\tau$, $k = 0, 1, 2, \cdots$, where τ is temporal step span.

Some operators are also defined. Let u_j^k be the approximation of $u(x,t)$ at the node (x_j, t_k). We define the finite difference operators

$$\delta_t f_j^k = \frac{1}{\tau}(f_j^{k+1} - f_j^k), \quad \delta_x f_j^k = \frac{1}{h}(f_{j+1}^k - f_j^k)$$

and averaging operators

$$A_t f_j^k = \frac{1}{2}(f_j^{k+1} + f_j^k), \quad A_x f_j^k = \frac{1}{2}(f_{j+1}^k + f_j^k).$$

Besides, the following properties of discrete Leibnitz rules are also useful for us to proving the local energy conservation law in our next work

$$\delta_x(f\cdot g)_j^k = (\alpha f_{j+1}^k + (1-\alpha)f_j^k)\cdot\delta_x g_j^k + \delta_x f_j^k\cdot((1-\alpha)g_{j+1}^k + \alpha g_j^k), \quad \forall 0 \le \alpha \le 1.$$

Especially, we have

$$\delta_x(f\cdot g)_j^k = A_x f_j^k \cdot \delta_x g_j^k + \delta_x f_j^k \cdot A_x g_j^k,\ \alpha = \frac{1}{2}. \tag{2.1}$$

Similarly, we can obtain a series of analogous discrete Leibnitz rules in the time direction

$$\delta_t(f\cdot g)_j^k = (\alpha f_j^{k+1} + (1-\alpha) f_j^k)\cdot \delta_t g_j^k + \delta_t f_j^k \cdot ((1-\alpha) g_j^{k+1} + \alpha g_j^k), \forall 0 \le \alpha \le 1$$

and

$$\delta_t(f\cdot g)_j^k = A_t f_j^k \cdot \delta_t g_j^k + \delta_t f_j^k \cdot A_t g_j^k,\ \alpha = \frac{1}{2}.$$

2.2. *Multisymplectic structure of the RLW equation*

First we introduce the conjugate variables $u = \varphi_x, v = u_x, \omega = u_t$ and $p = \frac{1}{2}\varphi_t + \frac{1}{2}\varphi_x + \frac{\nu}{2}\varphi_x^2 - \sigma\varphi_{xxt}$, let $Z = (\varphi, u, v, \omega, p)^T$, equation (1.1) can be reformulated into the following PDEs

$$\begin{aligned}
&-\frac{1}{2}u_t - \frac{1}{2}u_x - p_x = 0,\\
&\frac{1}{2}\varphi_t - \frac{\sigma}{2}v_t + \frac{1}{2}\varphi_x - \frac{\sigma}{2}\omega_x = p - \frac{\nu}{2}u^2,\\
&\frac{\sigma}{2}u_t = \frac{\sigma}{2}\omega,\\
&\frac{\sigma}{2}u_x = \frac{\sigma}{2}v,\\
&\varphi_x = u.
\end{aligned} \tag{2.2}$$

Thus it can be written as a multi-symplectic system

$$MZ_t + KZ_x = \nabla_Z S(Z)$$

where

$$S(Z) = up - \frac{\nu}{6}u^3 + \frac{\sigma}{2}v\omega,$$

$$M = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 & 0\\ \frac{1}{2} & 0 & -\frac{\sigma}{2} & 0 & 0\\ 0 & \frac{\sigma}{2} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, K = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 & -1\\ \frac{1}{2} & 0 & 0 & -\frac{\sigma}{2} & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & \frac{\sigma}{2} & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proposition 2.1. *The system (2.2) possesses a local momentum conservation law (LMCL)*

$$\partial_t(up + \frac{\sigma}{2}\omega v - \frac{\nu}{6}u^3 + \frac{1}{2}u_x\varphi + p_x\varphi + \frac{\sigma}{2}\omega_x u) + \partial_x(-\frac{1}{2}u_t\varphi - p_t\varphi - \frac{\sigma}{2}\omega_t u) = 0. \tag{2.3}$$

Proof. Multiplying (2.2) by φ_t, u_t, v_t, ω_t and p_t respectively gives

$$\begin{aligned}
&-\frac{1}{2}u_t\varphi_t - \frac{1}{2}u_x\varphi_t - p_x\varphi_t = 0,\\
&\frac{1}{2}\varphi_t u_t - \frac{\sigma}{2}v_t u_t + \frac{1}{2}\varphi_x u_t - \frac{\sigma}{2}\omega_x u_t = pu_t - \frac{\nu}{2}u^2 u_t,\\
&\frac{\sigma}{2}u_t v_t = \frac{\sigma}{2}\omega v_t,\\
&\frac{\sigma}{2}u_x\omega_t = \frac{\sigma}{2}v\omega_t,\\
&\varphi_x p_t = up_t.
\end{aligned} \tag{2.4}$$

Adding the above equations (2.4) together, we have

$$-\frac{1}{2}u_x\varphi_t - p_x\varphi_t + \frac{1}{2}\varphi_x u_t - \frac{\sigma}{2}\omega_x u_t + \frac{\sigma}{2}u_x\omega_t + \varphi_x p_t = pu_t - \frac{\nu}{2}u^2u_t + \frac{\delta}{2}\omega v_t + \frac{\delta}{2}v\omega_t + up_t.$$

Noting that

$$\begin{aligned}
&-\frac{1}{2}u_x\varphi_t + \frac{1}{2}\varphi_x u_t = \partial_t(-\frac{1}{2}u_x\varphi + \frac{1}{2}u_t\varphi),\\
&-p_x\varphi_t + \varphi_x p_t = -\partial_t(p_x\varphi) + \partial_x(p_t\varphi),\\
&-\frac{\sigma}{2}\omega_x u_t + \frac{\sigma}{2}u_x\omega_t = \partial_t(-\frac{\sigma}{2}\omega_x u) + \partial_x(\frac{\sigma}{2}\omega_t u),\\
&pu_t - \frac{\nu}{2}u^2u_t + \frac{\delta}{2}\omega v_t + \frac{\delta}{2}v\omega_t + up_t = \partial_t(up + \frac{\sigma}{2}\omega v - \frac{\nu}{6}u^3),
\end{aligned}$$

we obtain (2.3).

□

If the boundary conditions are suitable, such as periodic boundary conditions (1.2), integration LMCL (2.3) over domain $I = [a, b]$, we obtain the GMCL (1.4).

Proposition 2.2. *The system (2.2) possesses a local energy conservation law (LECL)*

$$\partial_t(-\frac{1}{2}u_x\varphi - \frac{\sigma}{2}uv_x) + \partial_x(up + \frac{\sigma}{2}\omega v - \frac{\nu}{6}u^3 + \frac{1}{2}u_t\varphi + \frac{\sigma}{2}uv_t) = 0. \tag{2.5}$$

Proof. Multiplying (2.2) by φ_x, u_x, v_x, ω_x and p_x respectively gives

$$\begin{aligned}
&-\frac{1}{2}u_t\varphi_x - \frac{1}{2}u_x\varphi_x - p_x\varphi_x = 0,\\
&\frac{1}{2}\varphi_t u_x - \frac{\sigma}{2}v_t u_x + \frac{1}{2}\varphi_x u_x - \frac{\sigma}{2}\omega_x u_x = pu_x - \frac{\nu}{2}u^2u_x,\\
&\frac{\sigma}{2}u_t v_x = \frac{\sigma}{2}\omega v_x,\\
&\frac{\sigma}{2}u_x\omega_x = \frac{\sigma}{2}v\omega_x,\\
&\varphi_x p_x = up_x.
\end{aligned} \tag{2.6}$$

Adding the above equations (2.6) together, we have

$$\begin{aligned}
-p_x\varphi_x - \frac{\sigma}{2}\omega_x u_x + \frac{\sigma}{2}u_x\omega_x + \varphi_x p_x &= pu_x - \frac{1}{2}u^2u_x + \frac{\sigma}{2}\omega v_x + \frac{\sigma}{2}v\omega_x + up_x\\
&= \partial_x(up + \frac{\sigma}{2}\omega v - \frac{\nu}{6}u^3)
\end{aligned}$$

According the commutative law and discrete Leibnitz rules, we can obtain local energy conservation law (2.5). □

With the initial and periodic boundary conditions (1.2), we obtain the global energy conservation law (GECL) (1.5).

3. Local structure-preserving schemes

In this section, we propose two new local sructure-preserving schemes for the RLW equation.

3.1. *Local momentum-preserving algorithm*

Now we apply the implicit midpoint scheme in space and the AVF method in time to construct the local momentum-preserving algorithm for RLW equation.

LMP scheme: First we use the implicit midpoint scheme in space and obtain semi-discrete system

$$
\begin{aligned}
&-\frac{1}{2}\partial_t A_x u_j-\frac{1}{2}\delta_x u_j-\delta_x p_j=0,\\
&\frac{1}{2}\partial_t A_x\varphi_j-\frac{\sigma}{2}\partial_t A_x v_j+\frac{1}{2}\delta_x\varphi_j-\frac{\sigma}{2}\delta_x\omega_j=A_x p_j-\frac{\nu}{2}(A_x u_j)^2,\\
&\frac{\sigma}{2}\partial_t A_x u_j=\frac{\sigma}{2}A_x\omega_j,\\
&\frac{\sigma}{2}\delta_x u_j=\frac{\sigma}{2}A_x v_j,\\
&\delta_x\varphi_j=A_x u_j.
\end{aligned}
$$

By using the AVF method in time one can get that

$$
\begin{aligned}
&-\frac{1}{2}\delta_t A_x u_j^k-\frac{1}{2}\delta_x A_t u_j^k-\delta_x A_t p_j^k=0,\\
&\frac{1}{2}\delta_t A_x\varphi_j^k-\frac{\sigma}{2}\delta_t A_x v_j^k+\frac{1}{2}\delta_x A_t\varphi_j^k-\frac{\sigma}{2}\delta_x A_t\omega_j^k=\int_0^1((1-\xi)A_x p_j^k+\xi A_x p_j^{k+1})-\frac{\nu}{2}((1-\xi)A_x u_j^k+\xi A_x u_j^{k+1})^2 d\xi,\\
&\frac{\sigma}{2}\delta_t A_x u_j^k=\frac{\sigma}{2}\int_0^1((1-\xi)A_x\omega_j^k+\xi A_x\omega_j^{k+1})d\xi,\\
&\frac{\sigma}{2}\delta_x A_t u_j^k=\frac{\sigma}{2}\int_0^1((1-\xi)A_x v_j^k+\xi A_x v_j^{k+1})d\xi,\\
&\delta_x A_t\varphi_j^k=\int_0^1((1-\xi)A_x u_j^k+\xi A_x u_j^{k+1})d\xi.
\end{aligned}
$$

The full-discrete system can be written as

$$
\begin{aligned}
&-\frac{1}{2}\delta_t A_x u_j^k-\frac{1}{2}\delta_x A_t u_j^k-\delta_x A_t p_j^k=0,\\
&\frac{1}{2}\delta_t A_x\varphi_j^k-\frac{\sigma}{2}\delta_t A_x v_j^k+\frac{1}{2}\delta_x A_t\varphi_j^k-\frac{\sigma}{2}\delta_x A_t\omega_j^k=A_t A_x p_j^k-\frac{\nu}{6}[(A_x u_j^k)^2+(A_x u_j^{k+1})^2+(A_x u_j^k)(A_x u_j^{k+1})],\\
&\frac{\sigma}{2}\delta_t A_x u_j^k=\frac{\sigma}{2}A_t A_x\omega_j^k,\\
&\frac{\sigma}{2}\delta_x A_t u_j^k=\frac{\sigma}{2}A_t A_x v_j^k,\\
&\delta_x A_t\varphi_j^k=A_t A_x u_j^k.
\end{aligned}
\tag{3.1}
$$

Eliminating the auxiliary variables φ, v, ω and p yields the system

$$
\delta_t A_x^3 A_t u_{j-2}^k+\delta_x A_x^2 A_t^2 u_{j-2}^k-\sigma\delta_x^2\delta_t A_x A_t u_{j-2}^k+\frac{\nu}{6}\delta_x A_x A_t[(A_x u_{j-2}^k)^2+(A_x u_{j-2}^{k+1})^2+(A_x u_{j-2}^k)(A_x u_{j-2}^{k+1})]=0.
$$

Omitting the average operator A_x and A_t we can get a two time level scheme

$$
A_x^2\delta_t u_{j-1}^k+\delta_t A_x A_t u_{j-1}^k-\sigma\delta_x^2\delta_t u_{j-1}^k+\frac{\nu}{6}\delta_x[(A_x u_{j-1}^k)^2+(A_x u_{j-1}^{k+1})^2+(A_x u_{j-1}^k)(A_x u_{j-1}^{k+1})]=0. \tag{3.2}
$$

Theorem 3.1. *The scheme (3.1) or (3.2) preserve the discrete local momentum conservation law*

$$\begin{aligned}\mathcal{M}(x_j,t_k) =&\delta_t[A_x p_j^k\cdot A_x u_j^k+\frac{\sigma}{2}A_x\omega_j^k\cdot A_x v_j^k-\frac{\nu}{6}(A_x u_j^k)^3+\frac{1}{2}\delta_x u_j^k\cdot A_x\varphi_j^k+\delta_x p_j^k\cdot A_x\varphi_j^k+\frac{\sigma}{2}\delta_x\omega_j^k\cdot A_x u_j^k]\\&+\delta_x(-A_t\varphi_j^k\cdot\delta_t p_j^k-\frac{1}{2}\delta_t u_j^k\cdot A_t\varphi_j^k-\frac{\sigma}{2}A_t u_j^k\cdot\delta_t\omega_j^k)=0.\end{aligned}\tag{3.3}$$

Proof. Multiplying (3.1) by $\delta_t A_x\varphi_j^k$, $\delta_t A_x u_j^k$, $\delta_t A_x v_j^k$, $\delta_t A_x\omega_j^k$ and $\delta_t A_x p_j^k$ respectively, we have

$$\begin{aligned}&-\frac{1}{2}\delta_t A_x u_j^k\cdot\delta_t A_x\varphi_j^k-\frac{1}{2}\delta_x A_t u_j^k\cdot\delta_t A_x\varphi_j^k-\delta_x A_t p_j^k\cdot\delta_t A_x\varphi_j^k=0,\\&\frac{1}{2}\delta_t A_x\varphi_j^k\cdot\delta_t A_x u_j^k-\frac{\sigma}{2}\delta_t A_x v_j^k\cdot\delta_t A_x u_j^k+\frac{1}{2}\delta_x A_t\varphi_j^k\cdot\delta_t A_x u_j^k-\frac{\sigma}{2}\delta_x A_t\omega_j^k\cdot\delta_t A_x u_j^k\\&=A_t A_x p_j^k\cdot\delta_t A_x u_j^k-\frac{\nu}{6}[(A_x u_j^k)^2+(A_x u_j^{k+1})^2+(A_x u_j^k)(A_x u_j^{k+1})]\cdot\delta_t A_x u_j^k,\\&\frac{\sigma}{2}\delta_t A_x u_j^k\cdot\delta_t A_x v_j^k=\frac{\sigma}{2}A_t A_x\omega_j^k\cdot\delta_t A_x v_j^k,\\&\frac{\sigma}{2}\delta_x A_t u_j^k\cdot\delta_t A_x\omega_j^k=\frac{\sigma}{2}A_t A_x v_j^k\cdot\delta_t A_x\omega_j^k,\\&\delta_x A_t\varphi_j^k\cdot\delta_t A_x p_j^k=A_t A_x u_j^k\cdot\delta_t A_x p_j^k.\end{aligned}\tag{3.4}$$

Summing the above equations (3.4)together, we obtain

$$\begin{aligned}&-\frac{1}{2}\delta_x A_t u_j^k\cdot\delta_t A_x\varphi_j^k-\delta_x A_t p_j^k\cdot\delta_t A_x\varphi_j^k+\frac{1}{2}\delta_x A_t\varphi_j^k\cdot\delta_t A_x u_j^k-\frac{\sigma}{2}\delta_x A_t\omega_j^k\cdot\delta_t A_x u_j^k+\frac{\sigma}{2}\delta_x A_t u_j^k\cdot\delta_t A_x\omega_j^k\\&+\delta_x A_t\varphi_j^k\cdot\delta_t A_x p_j^k=A_t A_x p_j^k\cdot\delta_t A_x u_j^k-\frac{1}{6}[(A_x u_j^k)^2+(A_x u_j^{k+1})^2+(A_x u_j^k)(A_x u_j^{k+1})]\cdot\delta_t A_x u_j^k\\&+\frac{\sigma}{2}A_t A_x\omega_j^k\cdot\delta_t A_x v_j^k+\frac{\sigma}{2}A_t A_x v_j^k\cdot\delta_t A_x\omega_j^k+A_t A_x u_j^k\cdot\delta_t A_x p_j^k\end{aligned}\tag{3.5}$$

By using the discrete Leibniz rule and commutative law we can deduce

$$\begin{aligned}&-\frac{1}{2}\delta_x A_t u_j^k\cdot\delta_t A_x\varphi_j^k+\frac{1}{2}\delta_x A_t\varphi_j^k\cdot\delta_t A_x u_j^k=\delta_t(-\frac{1}{2}\delta_x u_j^k\cdot A_x\varphi_j^k)+\delta_x(\frac{1}{2}\delta_t u_j^k\cdot A_t\varphi_j^k),\\&-\delta_x A_t p_j^k\cdot\delta_t A_x\varphi_j^k+\delta_x A_t\varphi_j^k\cdot\delta_t A_x p_j^k=\delta_t(-\delta_x p_j^k\cdot A_x\varphi_j^k)+\delta_x(\delta_t p_j^k\cdot A_t\varphi_j^k),\\&-\frac{\sigma}{2}\delta_x A_t\omega_j^k\cdot\delta_t A_x u_j^k+\frac{\sigma}{2}\delta_x A_t u_j^k\cdot\delta_t A_x\omega_j^k=\delta_t(-\frac{\sigma}{2}\delta_x\omega_j^k\cdot A_x u_j^k)+\delta_x(\frac{\sigma}{2}\delta_t\omega_j^k\cdot A_t u_j^k),\\&A_t A_x p_j^k\cdot\delta_t A_x u_j^k+A_t A_x u_j^k\cdot\delta_t A_x p_j^k=\delta_t(A_x p_j^k\cdot A_x u_j^k),\\&\frac{\nu}{6}[(A_x u_j^k)^2+(A_x u_j^{k+1})^2+(A_x u_j^k)(A_x u_j^{k+1})]\cdot\delta_t A_x u_j^k=\frac{\nu}{6}\delta_t(A_x u_j^k)^3,\\&\frac{\sigma}{2}A_t A_x\omega_j^k\cdot\delta_t A_x v_j^k+\frac{\sigma}{2}A_t A_x v_j^k\cdot\delta_t A_x\omega_j^k=\delta_t(\frac{\sigma}{2}A_x\omega_j^k\cdot A_x v_j^k),\end{aligned}$$

So we can reduce the equation (3.5) to discrete local momentum conservation law (3.3). □

Remark 3.2. *In the above derivations, we use the Leibnite rules (2.1). The discrete LMCL (3.3) is independent of the boundary conditions and preserved in any local time-space region. It is consist with the continuous LMCL (2.3).*

Summing the discrete momentum conservation law (3.3) over index j reads

$$\delta_t \sum_{j=1}^{N}[A_x p_j^k \cdot A_x u_j^k + \frac{\sigma}{2}A_x\omega_j^k \cdot A_x v_j^k - \frac{\nu}{6}(A_x u_j^k)^3 + \frac{1}{2}\delta_x u_j^k \cdot A_x\varphi_j^k + \delta_x p_j^k \cdot A_x\varphi_j^k + \frac{\sigma}{2}\delta_x\omega_j^k \cdot A_x u_j^k]$$
$$+ \delta_x \sum_{j=1}^{N}(-\frac{1}{2}\delta_t u_j^k \cdot A_t\varphi_j^k - A_t\varphi_j^k \cdot \delta_t p_j^k - \frac{\sigma}{2}A_t u_j^k \cdot \delta_t\omega_j^k) = 0.$$

With the initial and periodic boundary conditions (1.2), we obtain the following global momentum conservation law.

Corollary 3.3. *For the initial and periodic boundary conditions (1.2), the LMP scheme preserves the discrete GMCL*

$$\mathcal{M}^{k+1} = \mathcal{M}^k = \cdots = \mathcal{M}^1 = \mathcal{M}^0, \tag{3.6}$$

where $\mathcal{M}^k = h\sum_{j=1}^{N}[A_x p_j^k \cdot A_x u_j^k + \frac{\sigma}{2}A_x\omega_j^k \cdot A_x v_j^k - \frac{1}{6}(A_x u_j^k)^3 + \frac{1}{2}\delta_x u_j^k \cdot A_x\varphi_j^k + \delta_x p_j^k \cdot A_x\varphi_j^k + \frac{\sigma}{2}\delta_x\omega_j^k \cdot A_x u_j^k]$.

*3.2. **Local energy-preserving algorithm***

Now we apply the implicit midpoint scheme in time and the AVF method in space to construct the local energy-preserving algorithm for RLW equation.

LEP scheme: First we use the implicit midpoint scheme in time and obtain semi-discrete system

$$\begin{aligned}
&-\frac{1}{2}\delta_t u^k - \frac{1}{2}\partial_x A_t u^k - \partial_x A_t p^k = 0,\\
&\frac{1}{2}\delta_t\varphi^k - \frac{\sigma}{2}\delta_t v^k + \frac{1}{2}\partial_x A_t\varphi^k - \frac{\sigma}{2}\partial_x A_t\omega^k = A_t p^k - \frac{\nu}{2}(A_t u^k)^2,\\
&\frac{\sigma}{2}\delta_t u^k = \frac{\sigma}{2}A_t\omega^k,\\
&\frac{\sigma}{2}\partial_x A_t u^k = \frac{\sigma}{2}A_t v^k,\\
&\partial_x A_t\varphi^k = A_t u^k.
\end{aligned}$$

Then we use the AVF method in space

$$\begin{aligned}
&-\frac{1}{2}\delta_t A_x u_j^k - \frac{1}{2}\delta_x A_t u_j^k - \delta_x A_t p_j^k = 0,\\
&\frac{1}{2}\delta_t A_x\varphi_j^k - \frac{\sigma}{2}\delta_t A_x v_j^k + \frac{1}{2}\delta_x A_t\varphi_j^k - \frac{\sigma}{2}\delta_x A_t\omega_j^k = \int_0^1((1-\xi)A_t p_j^k + \xi A_t p_{j+1}^k) - \frac{\nu}{2}((1-\xi)A_t u_j^k + \xi A_t u_{j+1}^k)^2 d\xi,\\
&\frac{\sigma}{2}\delta_t A_x u_j^k = \frac{\sigma}{2}\int_0^1((1-\xi)A_t\omega_j^k + \xi A_t\omega_{j+1}^k)d\xi,\\
&\frac{\sigma}{2}\delta_x A_t u_j^k = \frac{\sigma}{2}\int_0^1((1-\xi)A_t v_j^k + \xi A_t v_{j+1}^k)d\xi,\\
&\delta_x A_t\varphi_j^k = \int_0^1((1-\xi)A_t u_j^k + \xi A_t u_{j+1}^k)d\xi.
\end{aligned}$$

The full-discrete system can be written as

$$
\begin{aligned}
&-\frac{1}{2}\delta_t A_x u_j^k - \frac{1}{2}\delta_x A_t u_j^k - \delta_x A_t p_j^k = 0,\\
&\frac{1}{2}\delta_t A_x \varphi_j^k - \frac{\sigma}{2}\delta_t A_x v_j^k + \frac{1}{2}\delta_x A_t \varphi_j^k - \frac{\sigma}{2}\delta_x A_t \omega_j^k = A_x A_t p_j^k - \frac{\nu}{6}[(A_t u_j^k)^2 + (A_t u_{j+1}^k)^2 + (A_t u_j^k)(A_t u_{j+1}^k)],\\
&\frac{\sigma}{2}\delta_t A_x u_j^k = \frac{\sigma}{2}A_t A_x \omega_j^k,\\
&\frac{\sigma}{2}\delta_x A_t u_j^k = \frac{\sigma}{2}A_t A_x v_j^k,\\
&\delta_x A_t \varphi_j^k = A_t A_x u_j^k.
\end{aligned}
\tag{3.7}
$$

Eliminating the auxiliary variables φ, v, ω and p yields the system

$$
\delta_t A_x^3 A_t u_{j-2}^k + \delta_x A_x^2 A_t^2 u_{j-2}^k - \sigma\delta_x^2\delta_t A_x A_t u_{j-2}^k + \frac{\nu}{6}\delta_x A_x A_t[(A_t u_{j-2}^k)^2 + (A_t u_{j-1}^k)^2 + (A_t u_{j-1}^k)(A_t u_{j-2}^k)] = 0.
$$

Omitting the average operator A_x and A_t we can get a two time level scheme

$$
A_x^2\delta_t u_{j-1}^k + \delta_x A_x A_t u_{j-1}^k - \sigma\delta_x^2\delta_t u_{j-1}^k + \frac{\nu}{6}\delta_x[(A_t u_{j-1}^k)^2 + (A_t u_j^k)^2 + (A_t u_{j-1}^k)(A_t u_j^k)] = 0.
\tag{3.8}
$$

Theorem 3.4. *The scheme (3.7) or (3.8) preserve the discrete local energy conservation law*

$$
\begin{aligned}
\mathcal{E}(x_j, t_k) =&\delta_t(-\frac{1}{2}\delta_x u_j^k \cdot A_x\varphi_j^k - \frac{\sigma}{2}A_x u_j^k \cdot \delta_x v_j^k)\\
&+ \delta_x[A_t p_j^k \cdot A_t u_j^k + \frac{\sigma}{2}A_t\omega_j^k \cdot A_t v_j^k - \frac{\nu}{6}(A_t u_j^k)^3 + \frac{1}{2}\delta_t u_j^k \cdot A_t\varphi_j^k + \frac{\sigma}{2}\delta_t v_j^k \cdot A_t u_j^k] = 0
\end{aligned}
\tag{3.9}
$$

Proof. Multiplying (3.7) by $\delta_x A_t\varphi_j^k$, $\delta_x A_t u_j^k$, $\delta_x A_t v_j^k$, $\delta_x A_t\omega_j^k$ and $\delta_x A_t p_j^k$ respectively, we have

$$
\begin{aligned}
&-\frac{1}{2}\delta_t A_x u_j^k \cdot \delta_x A_t\varphi_j^k - \frac{1}{2}\delta_x A_t u_j^k \cdot \delta_x A_t\varphi_j^k - \delta_x A_t p_j^k \cdot \delta_x A_t\varphi_j^k = 0,\\
&\frac{1}{2}\delta_t A_x\varphi_j^k \cdot \delta_x A_t u_j^k - \frac{\sigma}{2}\delta_t A_x v_j^k \cdot \delta_x A_t u_j^k + \frac{1}{2}\delta_x A_t\varphi_j^k \cdot \delta_x A_t u_j^k - \frac{\sigma}{2}\delta_x A_t\omega_j^k \cdot \delta_x A_t u_j^k\\
&\ = A_t A_x p_j^k \cdot \delta_x A_t u_j^k - \frac{\nu}{6}[(A_t u_j^k)^2 + (A_t u_{j+1}^k)^2 + (A_t u_j^k)(A_t u_{j+1}^k)] \cdot \delta_x A_t u_j^k,\\
&\frac{\sigma}{2}\delta_t A_x u_j^k \cdot \delta_x A_t v_j^k = \frac{\sigma}{2}A_t A_x\omega_j^k \cdot \delta_x A_t v_j^k,\\
&\frac{\sigma}{2}\delta_x A_t u_j^k \cdot \delta_x A_t\omega_j^k = \frac{\sigma}{2}A_t A_x v_j^k \cdot \delta_x A_t\omega_j^k,\\
&\delta_x A_t\varphi_j^k \cdot \delta_x A_t p_j^k = A_t A_x u_j^k \cdot \delta_x A_t p_j^k.
\end{aligned}
\tag{3.10}
$$

Summing the above equations (3.10)

$$
\begin{aligned}
&-\frac{1}{2}\delta_t A_x u_j^k \cdot \delta_x A_t\varphi_j^k + \frac{1}{2}\delta_t A_x\varphi_j^k \cdot \delta_x A_t u_j^k - \frac{\sigma}{2}\delta_t A_x v_j^k \cdot \delta_x A_t u_j^k + \frac{\sigma}{2}\delta_x A_t v_j^k \cdot \delta_t A_x u_j^k\\
&= A_t A_x p_j^k \cdot \delta_x A_t u_j^k + A_t A_x u_j^k \cdot \delta_x A_t p_j^k - \frac{\nu}{6}[(A_t u_j^k)^2 + (A_t u_{j+1}^k)^2 + (A_t u_j^k)(A_t u_{j+1}^k)] \cdot \delta_x A_t u_j^k\\
&+ \frac{\sigma}{2}A_t A_x\omega_j^k \cdot \delta_x A_t v_j^k + \frac{\sigma}{2}A_t A_x v_j^k \cdot \delta_x A_t\omega_j^k.
\end{aligned}
\tag{3.11}
$$

By using the discrete Leibniz rule and commutative law we can deduce

$$
-\frac{1}{2}\delta_t A_x u_j^k \cdot \delta_x A_t\varphi_j^k + \frac{1}{2}\delta_t A_x\varphi_j^k \cdot \delta_x A_t u_j^k = \delta_x(-\frac{1}{2}\delta_t u_j^k \cdot A_t\varphi_j^k) + \delta_t(\frac{1}{2}\delta_x u_j^k \cdot A_x\varphi_j^k),
$$

$$-\frac{\sigma}{2}\delta_t A_x v_j^k \cdot \delta_x A_t u_j^k + \frac{\sigma}{2}\delta_x A_t v_j^k \cdot \delta_t A_x u_j^k = \delta_x(-\frac{\sigma}{2}\delta_t v_j^k \cdot A_t u_j^k) + \delta_t(\frac{\sigma}{2}A_x u_j^k \cdot \delta_x v_j^k),$$
$$A_t A_x p_j^k \cdot \delta_x A_t u_j^k + A_t A_x u_j^k \cdot \delta_x A_t p_j^k = \delta_x(A_t p_j^k \cdot A_t u_j^k),$$
$$\frac{\nu}{6}[(A_t u_j^k)^2 + (A_t u_{j+1}^k)^2 + (A_t u_j^k)(A_t u_{j+1}^k)] \cdot \delta_x A_t u_j^k = \frac{\nu}{6}\delta_x(A_t u_j^k)^3,$$
$$\frac{\sigma}{2}A_t A_x \omega_j^k \cdot \delta_x A_t v_j^k + \frac{\sigma}{2}A_t A_x v_j^k \cdot \delta_x A_t \omega_j^k = \delta_x(\frac{\sigma}{2}A_t \omega_j^k \cdot A_t v_j^k),$$

So we can reduce the equation (3.11) to discrete local energy conservation law (3.9). □

Remark 3.5. *The discrete LECL (3.9) is independent of boundary conditions and is consist with the continuous LECL (2.5).*

Summing the discrete energy conservation law (3.9) over index j reads

$$\delta_x \sum_{j=1}^{N}[A_t p_j^k \cdot A_t u_j^k + \frac{\sigma}{2}A_t \omega_j^k \cdot A_t v_j^k - \frac{\nu}{6}(A_t u_j^k)^3 + \frac{1}{2}\delta_t u_j^k \cdot A_t \varphi_j^k + \frac{\sigma}{2}\delta_t v_j^k \cdot A_t u_j^k]$$
$$+ \delta_t \sum_{j=1}^{N}(-\frac{1}{2}\delta_x u_j^k \cdot A_x \varphi_j^k - \frac{\sigma}{2}A_x u_j^k \cdot \delta_x v_j^k) = 0.$$

With the initial and periodic boundary conditions (1.2), we can obtain the following discrete global energy conservation law.

Corollary 3.6. *For the initial and periodic boundary conditions (1.2), the LEP scheme preserves the discrete GECL*

$$\mathcal{E}^{k+1} = \mathcal{E}^k = \cdots = \mathcal{E}^1 = \mathcal{E}^0, \tag{3.12}$$

where $\mathcal{E}^k = h\sum_{j=1}^{N}(-\frac{1}{2}\delta_x u_j^k \cdot A_x \varphi_j^k - \frac{\sigma}{2}A_x u_j^k \cdot \delta_x v_j^k)$.

4. Numerical examples

In this section, we conduct some numerical experiments to verify the theoretical results of the LMP and LEP schemes. The performance of the proposed method will be exhibited in following aspects:

(i) to show the accuracy of single soliton solutions;

(ii) to exhibit the performance in preserving the symplectic structure.

The RLW equation have the following soliton solution [1]

$$u(x,t) = 3c\,sech^2(k(x - x_0 - \varepsilon t)),$$

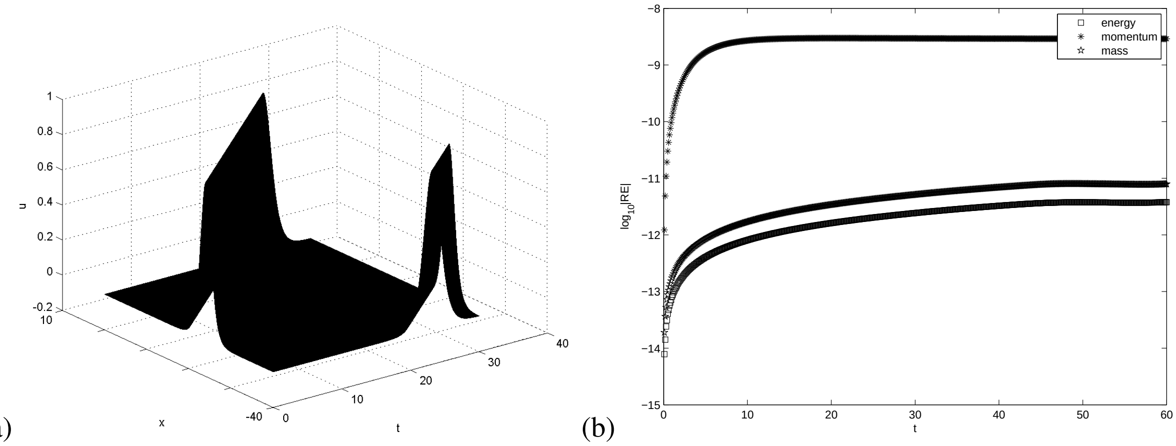
where $k = \frac{1}{2}\sqrt{\frac{\nu c}{\sigma(1+\nu c)}}$, $\delta = 1 + \nu c$ and $3c$ is amplitude and ε is velocity.

The relative mass error, relative momentum error and relative energy error $t = t_k$ are defined as

$$RI_1^k = \frac{|I_1^k - I_1^0|}{|I_1^0|}, RI_2^k = \frac{|I_2^k - I_2^0|}{|I_2^0|}, RI_3^k = \frac{|I_3^k - I_3^0|}{|I_3^0|}, k = 0, 1, 2, \ldots, N \tag{4.1}$$

where

$$I_1^k = h\sum_{j=0}^{N-1}(A_x u_j^k) = h\sum_{j=0}^{N-1}\frac{u_{j+1}^k + u_j^k}{2}$$



(a) (b)

Figure 1: The numerical solution of the LEP scheme with $\tau = 0.125$ and $h = 0.1$. (a) Propagation of solitary wave from $t = 0$ to $t = 60$. (b) The conservation properties of the LEP scheme.

is the discrete global mass;

$$I_2^k = \frac{h}{2}\sum_{j=0}^{N-1}((A_x u_j^k)^2 + \sigma(\delta_x u_j^k)^2) = \frac{h}{2}\sum_{j=0}^{N-1}(\frac{u_{j+1}^k + u_j^k}{2})^2 + \frac{\sigma}{h}(u_{j+1}^k - u_j^k)$$

is the discrete global momentum and

$$I_3^k = \frac{h}{3}\sum_{j=0}^{N-1}(\varepsilon(A_x u_j^k)^3 + 3(A_x u_j^k)^2) = \frac{h}{3}\sum_{j=0}^{N-1}(\varepsilon(\frac{u_{j+1}^k + u_j^k}{2})^3 + 3(\frac{u_{j+1}^k + u_j^k}{2})^2$$

is the discrete global energy.

Now we carry out three numerical experiments to test the theoretical analysis on the difference solutions.

Example 4.1. (One-solitary wave)

We consider the RLW equation with initial boundary conditions

$$u(x, 0) = 3c\,sech^2(k(x - x_0)), u(a, t) = u(b, t),$$

where $\varepsilon = \sigma = 1$, $c = 0.3$, $x_0 = 0$.

Computations are done with grid number $N = 400$, space step $h = 0.125$ temporal step $\tau = 0.1$ and spatial step $h = 0.01$, $-40 \le x \le 60$. The relative error of mass, energy and momentum are computed by using local structure preserving schemes LMP and LEP.

Figure 1 display the results of single soliton obtained by using the LEP scheme. As can be seen from Figure 1(a), the propogation of solitary wave over time interval [0,60] is travelling from left to right as required and the shape of the solution is preserved accurately. Figure 1(b) show that the relative mass and energy are conserved to the machine accuracy. We can also see that the error of energy growth linearly. Figure 2 show the solitary wave, mass and energy error from $t = 0$ to $t = 60$ by using the LMP scheme. These results coincide with the theoretical analysis.

Example 4.2. (Two-solitary wave)

In the following simulations, we will study interaction of two positive solitary waves.

$$u(x, 0) = 3c_1 sech^2(k_1(x - x_1)) + 3c_2 sech^2(k_2(x - x_2)), \tag{4.2}$$

where $\varepsilon = \sigma = 1$, $c_1 = 0.2, c_2 = 0.1$, $x_1 = -177, x_2 = -147$.

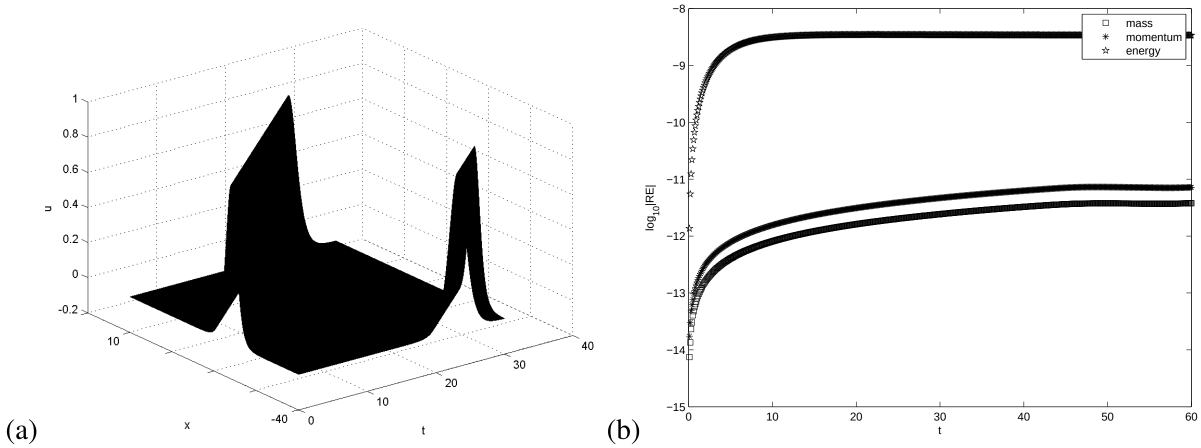


Figure 2: The numerical solution of the LMP scheme with $\tau = 0.125$ and $h = 0.1$. (a) Propagation of solitary wave from $t = 0$ to $t = 60$. (e) The conservation properties of the LMP scheme.

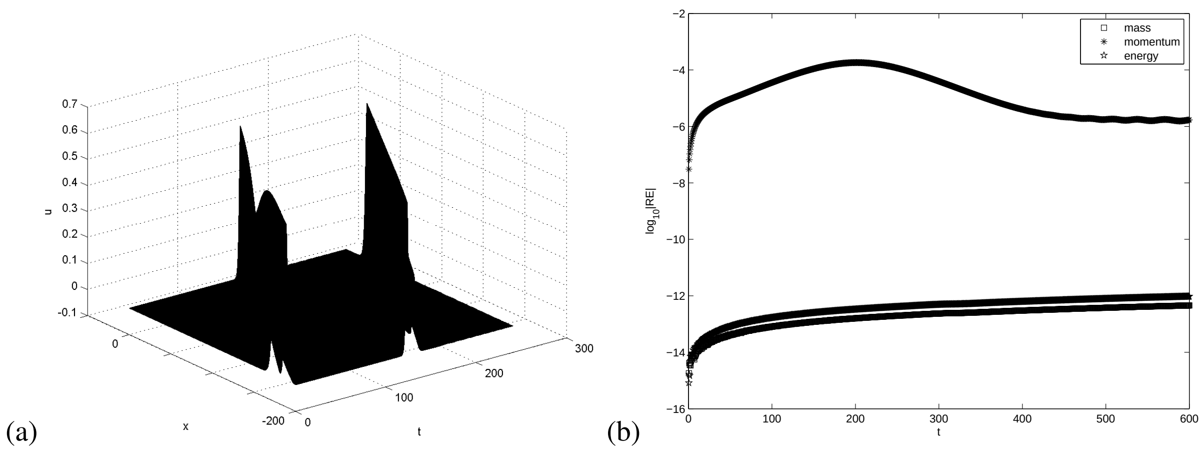


Figure 3: The numerical solution of the LEP scheme with $\tau = 0.5$ and $h = 0.5$. (a) Propagation of solitary wave from $t = 0$ to $t = 600$. (b) The conservation properties of the LEP scheme.

First we apply the LEP scheme to the solitary waves with initial condition (4.2) over time interval [0,600]. Computations are carried out with grid number $N = 1200$, temporal step $\tau = 0.5$ and spatial step $h = 0.5$, $-200 \leq x \leq 200$. The interaction process of two solitary obtained by using the LEP scheme can be viewed in Figure 3(a). The quantities mass, energy and momentum versus time are depicted in Figure 3(b). Then by using the LMP scheme, Figure 4(a) and Figure 4(b) show the solitary wave and conservation properties respectively.

Example 4.3. (Three-solitary wave)

$$u(x,0) = 3c_1 sech^2(k_1(x - x_1)) + 3c_2 sech^2(k_2(x - x_2)) + 3c_3 sech^2(k_3(x - x_3)),$$

where $\varepsilon = \sigma = 1, c_1 = 1, c_2 = 0.5, c_3 = 0.25, x_1 = 0, x_2 = 18, x_3 = 35$.

Computations are done with grid number $N = 400$, space step $h = 0.1$ temporal step $\tau = 0.05$ and spatial step $h = 0.1$, $-40 \leq x \leq 110$.

Figure 5 provide the results of three-solitary wave with the time step $\tau = 0.05$ and the space step $h = 0.1$ by using the LEP scheme. As can be seen from Figure 5(a), the propogation of solitary wave over

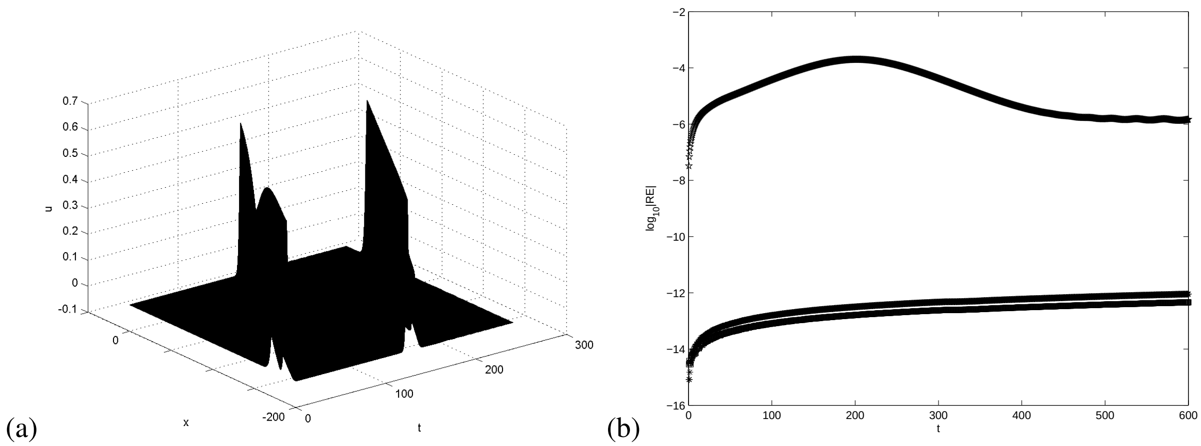


Figure 4: The numerical solution of the LMP scheme with $\tau = 0.5$ and $h = 0.5$. (a) Propagation of solitary wave from $t = 0$ to $t = 600$. (b) The conservation properties of the LMP scheme.

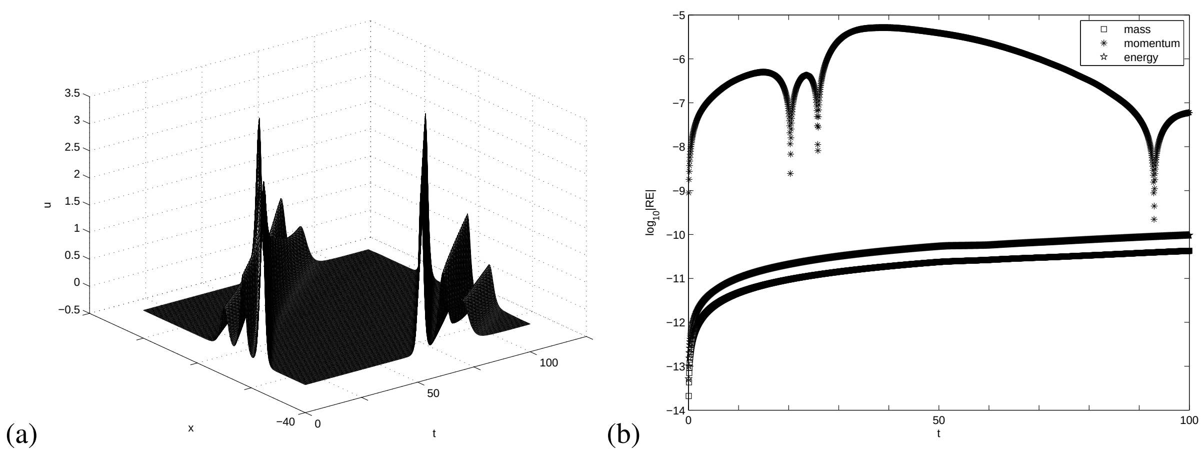


Figure 5: The numerical solution of the LEP scheme with $\tau = 0.05$ and $h = 0.1$. (a) Propagation of solitary wave from $t = 0$ to $t = 100$. (b) The conservation properties of the LEP scheme.

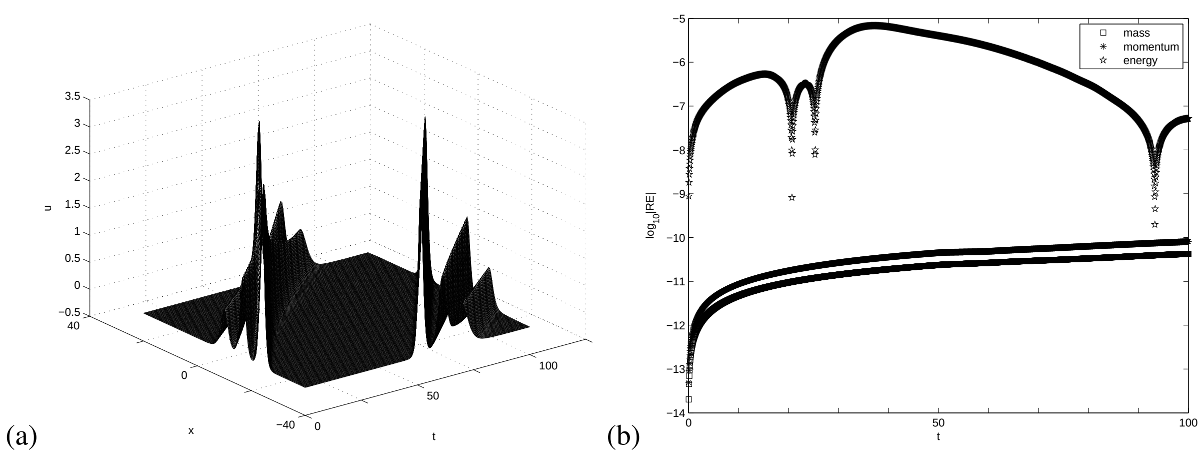


Figure 6: The numerical solution of the LMP scheme with $\tau = 0.1$ and $h = 0.05$. (a) Propagation of solitary wave from $t = 0$ to $t = 100$. (b) The conservation properties of the LMP scheme.

time interval [0,100] is travelling from left to right as required and the shape of the solution is preserved accurately. Figure 5(b) show that the relative mass and energy are conserved to the machine accuracy. The error of energy growth linearly. Figure 6 show the three-solitary wave obtained by using the LEP scheme, mass momentum and energy error from $t = 0$ to $t = 100$.

We can draw a clear conclusion from the numerical results that the LEP and LMP schemes provides highly accurate numerical solutions and preserves the local mass, energy and momentum to machine accuracy.

5. Conclusions

The local/global conservation laws, such as symplectic and multisymplectic conservation laws, local energy and momentum conservation laws, usually play an important role in physics and applications for PDEs. In this work, we have proposed two local structure preserving algorithms to simulate the RLW equation. The two schemes are conservative schemes, which not only preserve discrete local momentum / energy but also preserve the local mass precisely. The merit of the two schemes is that with suitable boundary conditions, for example with periodic boundary conditions, this algorithm conserve the global mass and momentum / energy precisely. Numerical results indicate that the present scheme can well simulate different solitary wave behaviors of the RLW equation in long term computation and also show excellent performance in preserving geometry structure.

Acknowledgements

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