New methods of approach related to the Riemann Hypothesis.

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Abstract. In this paper we develop techniques related to the Riemann Hypothesis that are based on the Taylor series of the Riemann Xi function, and the asymptotic behavior of $\xi^{(2n)}(\frac{1}{2})$.

1. Introduction.

In this article I present a formal treatment of a special limit process associated with an infinite series, such that when we add a new term of the series, some of the previous terms also slightly change, but in such h a manner that the limit can be precisely defined (mainly section 3 and theorem 2).

I also present a sufficient condition for an infinite series, in order to take only positive values (propositions 3 and 5, section 3).

In section 2 I present a theorem related to convexity that will be useful later in the article.

In section 4 I present a different approach, and theorem 3 presents Turan-type inequalities that are sufficient (if true) for the Riemann Hypothesis to be true.

The tools presented here allow us to attack Riemann's Hypothesis in a completely new manner. The presentation is informal but the results are clearly stated and the proofs given in full.

2. A theorem related to convexity.

Theorem 1. We consider the holomorphic function F(s) (that does not vanish identically) , where in general $s = \sigma + i \cdot t$, defined on the critical strip. We assume that F(s) satisfies the functional equation $F(s) = F(1-s)$. We also assume that the real function defined by $\varphi(\sigma) := |F(\sigma + i \cdot t)|^2$ is a convex function (as a function of σ) for $0 < \sigma < 1$ (and for any t fixed). Then the function F(s) has all its zeros on the critical line $Re(s) = \sigma = \frac{1}{2}$.

Proof. We assume that the function F(s) has a zero at $x + i \cdot t$, where $x < \frac{1}{2}$. Then from the functional equation $F(s) = F(1-s)$, the function also has a zero at $1 - x - i \cdot t$. Since the complex conjugate of $1 - x - i \cdot t$ is $1 - x + i \cdot t$, the function F(s) will also have a zero at $1 - x + i \cdot t$.

From the assumptions of the theorem, the function $\varphi(\sigma)$ is convex for $0 < \sigma < 1$. For any x_1 and x_2 we have:

$$\varphi(\sigma) \leq \frac{x_2 - \sigma}{x_2 - x_1} \cdot \varphi(x_1) + \frac{\sigma - x_1}{x_2 - x_1} \cdot \varphi(x_2) \text{ for } x_1 < \sigma < x_2.$$

We take x_1 to be the real part of a zero of F(s) , and $x_2 = 1 - x_1$ (which is the real part of another zero, for a fixed t). That means that $\varphi(\sigma) \leq 0$ for $x_1 < \sigma < x_2$. Since by definition $\varphi(\sigma)$ is nonnegative, that means that $\varphi(\sigma) \geq 0$ for $x_1 < \sigma < x_2$. The conclusion is that for that fixed t, we have $\varphi(\sigma) = |F(\sigma + i \cdot t)|^2 = 0$ for $x_1 < \sigma < x_2$.

We know that if a function is holomorphic in a region, and vanishes at all points of any smaller region included in the given region, or along any arc of a continuous curve in the region, then it must vanish identically (the identity theorem). Since we see that $F(\sigma + i \cdot t)$ vanishes on the segment joining the two zeroes of F(s), then the function $F(s)$ would have to vanish identically on the domain under consideration. We reached a contradiction, since we assumed that $F(s)$ does not vanish identically. Our assumption, that the function F(s) has a zero at $x + i \cdot t$, where $x < \frac{1}{2}$ is false.

The function F(s) has all its zeroes on the vertical $\mathrm{Re}(s) = \sigma = \frac{1}{2}$. The horizontal segment joining the two zeroes must collapse to a point. **QED.**

3. The main method of approach and basic calculations.

We consider the Riemann Xi function defined as:

$$\xi(s) := \frac{1}{2} \cdot s \cdot (s-1) \cdot \Gamma\left(\frac{1}{2}s\right) \cdot \pi^{-\frac{s}{2}} \cdot \zeta(s)$$

For the Riemann Xi function $\xi(s)$ we have the following series expansion:

$$\xi(s) = a_0 + a_2 \cdot (s-\tfrac{1}{2})^2 + a_4 \cdot (s-\tfrac{1}{2})^4 + a_6 \cdot (s-\tfrac{1}{2})^6 + \cdots \ldots \ldots \ldots, \qquad (1)$$

where all the coefficients a_{2n} are positive real numbers. This statement is proved in [1] , page 17.

We define the following functions. We define:

$$F_{2N}(s) := a_0 + a_2 \cdot (s-\tfrac{1}{2})^2 + a_4 \cdot (s-\tfrac{1}{2})^4 + a_6 \cdot (s-\tfrac{1}{2})^6 + \cdots \ldots .. + a_{2N} \cdot \left(s-\frac{1}{2}\right)^{2N} \qquad (2)$$

We have then: $|F_{2N}(\sigma + it)|^2 \rightarrow |\xi(\sigma + it)|^2$, when $N \rightarrow \infty$ (more general, we have $F_{2N}(\sigma + it) \rightarrow \xi(\sigma + it)$ when $N \rightarrow \infty$).

We also define (for a fixed t):

$$f_{2N}(\sigma) := |F_{2N}(\sigma + it)|^2 \ . \qquad (3)$$

In the following, we write $\beta = \sigma - \frac{1}{2}$.

We start with the identities:

$$(\beta + it)^2 = \binom{2}{0} \cdot \beta^2 - \binom{2}{2} \cdot t^2 + i \cdot \binom{2}{1} \cdot \beta t$$

$$(\beta + it)^4 = \binom{4}{0} \cdot \beta^4 - \binom{4}{2} \cdot \beta^2 t^2 + \binom{4}{4} \cdot t^4 + i \cdot (\binom{4}{1} \cdot \beta^3 t - \binom{4}{3} \cdot \beta t^3)$$

$(\beta+it)^6 = \binom{6}{0}\cdot\beta^6 - \binom{6}{2}\cdot\beta^4t^2 + \binom{6}{4}\cdot\beta^2t^4 + \binom{6}{6}\cdot t^6 + \ i\cdot(\binom{6}{1}\cdot\beta^5t - \binom{6}{3}\cdot\beta^3t^3 + \binom{6}{5}\cdot \beta t^5)$

$(\beta+it)^8 = \binom{8}{0}\cdot\beta^8 - \binom{8}{2}\cdot\beta^6t^2 + \binom{8}{4}\cdot\beta^4t^4 + \binom{8}{6}\cdot\beta^2t^6 + \binom{8}{8}\cdot t^8 + \ i\cdot\left(\binom{8}{1}\cdot\beta^7t - \binom{8}{3}\cdot \beta^5t^3 + \binom{8}{5}\cdot\beta^3\cdot t^5 - \binom{8}{7}\cdot\beta t^7\right)$

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It is clear how to continue this sequence of identities up to $(\beta+it)^{2N}$.

We write for the real and imaginary part of $F_{2N}(\sigma+it)$ as $\mathrm{Re}(F_{2N}(\sigma+it))$ and $\mathrm{Im}(F_{2N}(\sigma+it))$.

It is clear that we have:

$$\mathrm{Re}\big(F_{2N}(\sigma+it)\big) = B_0 + B_2\cdot\beta^2 + B_4\cdot\beta^4 + B_6\cdot\beta^6 + \cdots \ldots.. + B_{2N}\cdot\beta^{2N}\,. \quad (4)$$

$$\mathrm{Im}\big(F_{2N}(\sigma+it)\big) = B_1\cdot\beta + B_3\cdot\beta^3 + B_5\cdot\beta^5 + \cdots \ldots.. + B_{2N-1}\cdot\beta^{(2N-1)}\,. \quad (5)$$

The coefficients will depend on t and they will have the form:

$$B_0 = a_0 - a_2\cdot\binom{2}{2}\cdot t^2 + a_4\cdot\binom{4}{4}\cdot t^4 \pm \cdots \ldots.. + (-1)^N\cdot a_{2N}\cdot\binom{2N}{2N}\cdot t^{2N}$$

$$B_2 = a_2\cdot\binom{2}{0} - a_4\cdot\binom{4}{2}\cdot t^2 + a_6\cdot\binom{6}{4}\cdot t^4 \pm \cdots \ldots.. + (-1)^{N+1}\cdot a_{2N}\cdot\binom{2N}{2N-2}\cdot t^{2N-2}$$

$$B_4 = a_4\cdot\binom{4}{0} - a_6\cdot\binom{6}{2}\cdot t^2 + a_8\cdot\binom{8}{4}\cdot t^4 \pm \cdots \ldots.. + (-1)^{N+2}\cdot a_{2N}\cdot\binom{2N}{2N-4}\cdot t^{2N-4}$$

$$B_6 = a_6\cdot\binom{6}{0} - a_8\cdot\binom{8}{2}\cdot t^2 + a_{10}\cdot\binom{10}{4}\cdot t^4 \pm \cdots \ldots.. + (-1)^{N+3}\cdot a_{2N}\cdot\binom{2N}{2N-6}\cdot t^{2N-6}$$

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$$B_{2N-2} = a_{2N-2}\cdot\binom{2N-2}{0} - a_{2N}\cdot\binom{2N}{2}\cdot t^2$$

$$B_{2N} = a_{2N}\cdot\binom{2N}{0}$$

In general we have:

$$B_{2i} = \sum_{k=0}^{N}(-1)^{k+i}\cdot a_{2k}\cdot\binom{2k}{2k-2i}\cdot t^{2k-2i} \quad (6)$$

In the same way, the odd order coefficients will have the form:

$$B_1 = a_2\cdot\binom{2}{1}\cdot t - a_4\cdot\binom{4}{3}\cdot t^3 + a_6\cdot\binom{6}{5}\cdot t^5 \pm \cdots \ldots.. + (-1)^{N+1}\cdot a_{2N}\cdot\binom{2N}{2N-1}\cdot t^{2N-1}$$

$$B_3 = a_4\cdot\binom{4}{1}\cdot t - a_6\cdot\binom{6}{3}\cdot t^3 + a_8\cdot\binom{8}{5}\cdot t^5 \pm \cdots \ldots.. + (-1)^{N+2}\cdot a_{2N}\cdot\binom{2N}{2N-3}\cdot t^{2N-3}$$

$$B_5 = a_6\cdot\binom{6}{1}\cdot t - a_8\cdot\binom{8}{3}\cdot t^3 + a_{10}\cdot\binom{10}{5}\cdot t^5 \pm \cdots \ldots.. + (-1)^{N+3}\cdot a_{2N}\cdot\binom{2N}{2N-5}\cdot t^{2N-5}$$

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$$B_{2N-3} = a_{2N-2} \cdot \binom{2N-2}{1} \cdot t - a_{2N} \cdot \binom{2N}{3} \cdot t^3$$

$$B_{2N-1} = a_{2N} \cdot \binom{2N}{1} \cdot t$$

In general we have:

$$B_{2i+1} = \sum_{k=0}^{N} (-1)^{k+i+1} \cdot a_{2k} \cdot \binom{2k}{2k-2i-1} \cdot t^{2k-2i-1} \quad (7)$$

In relations (6) and (7) except for the usual conventions, we make the following conventions about the binomial coefficients:

Conventions. For $x, y > 0$ we have $\binom{0}{0} = \binom{x}{0} = 1$, $\binom{0}{-y} = \binom{x}{-y} = 0$, and if $x < y$ then $\binom{x}{y} = 0$.

Relations (6) and (7) can be written in the unified form:

$$B_m = \sum_{k=0}^{N} (-1)^{k+m-\left[\!\left[\frac{m}{2}\right]\!\right]} \cdot a_{2k} \cdot \binom{2k}{2k-m} \cdot t^{2k-m} \quad (8)$$

Here $\left[\!\left[\frac{m}{2}\right]\!\right]$ represents the integer part of $\frac{m}{2}$ and we use the conventions about the binomial coefficients above. Also m takes values from 0 to 2N. It would be better if we wrote $B_{m,2N}$ instead of B_m but we use the latter notation for simplicity.

We have then:

$$|F_{2N}(\sigma + it)|^2 = \left(B_0 + B_2 \cdot \beta^2 + B_4 \cdot \beta^4 + B_6 \cdot \beta^6 + \cdots \ldots + B_{2N} \cdot \beta^{2N}\right)^2 + (B_1 \cdot \beta + B_3 \cdot \beta^3 + B_5 \cdot \beta^5 + \cdots \ldots + B_{2N-1} \cdot \beta^{(2N-1)})^2$$

$$|F_{2N}(\sigma + it)|^2 = {B_0}^2 + \beta^2 \cdot \left({B_1}^2 + 2 \cdot B_0 \cdot B_2\right) + \beta^4 \cdot \left({B_2}^2 + 2 \cdot B_0 \cdot B_4 + 2 \cdot B_1 \cdot B_3\right) + \beta^6 \cdot \left({B_3}^2 + 2 \cdot B_0 \cdot B_6 + 2 \cdot B_1 \cdot B_5 + 2 \cdot B_2 \cdot B_4\right) + \cdots \ldots + \beta^{4N} \cdot {B_{2N}}^2 \quad (9)$$

We remember that $\beta = \left(\sigma - \frac{1}{2}\right)$.

From relation (8), we have then:

$$\frac{d^2}{d\sigma^2} f_{2N}(\sigma) = \frac{d^2}{d\sigma^2} |F_{2N}(\sigma + it)|^2 = 2 \cdot \left({B_1}^2 + 2 \cdot B_0 \cdot B_2\right) + 12 \cdot \left(\sigma - \frac{1}{2}\right)^2 \cdot \left({B_2}^2 + 2 \cdot B_0 \cdot B_4 + 2 \cdot B_1 \cdot B_3\right) + 30 \cdot \left(\sigma - \frac{1}{2}\right)^4 \cdot \left({B_3}^2 + 2 \cdot B_0 \cdot B_6 + 2 \cdot B_1 \cdot B_5 + 2 \cdot B_2 \cdot B_4\right) + \cdots \ldots + (4N) \cdot (4N-1) \cdot \left(\sigma - \frac{1}{2}\right)^{4N-2} \cdot {B_{2N}}^2 \quad (10)$$

From (10) we see that we have to calculate the quantities:

$$D'_{2n,2N} := \sum_{p+q=2n,\ 0 \le p,q \le 2N} B_p \cdot B_q.$$

Using relation (8) we have:

$$D'_{2n,2N} = \sum_{p+q=2n,\ \ 0\le p,q\le 2N} B_p \cdot B_q = \sum_{k=0}^{2N}\left(\sum_{i+j=k}\sum_{p+q=2n,\ \ 0\le p,q\le 2N}(-1)^{k-[\![\frac{p}{2}]\!]-[\![\frac{q}{2}]\!]} \cdot a_{2i} \cdot a_{2j} \cdot \binom{2i}{2i-p} \cdot \binom{2j}{2j-q}\right) \cdot t^{2k-2n} \qquad (11)$$

We also define the quantities:

$$D_{2n,2N} \coloneqq \sum_{p+q=2n} B_p \cdot B_q = \sum_{k=0}^{2N}\left(\sum_{i+j=k}\sum_{p+q=2n}(-1)^{k-[\![\frac{p}{2}]\!]-[\![\frac{q}{2}]\!]} \cdot a_{2i} \cdot a_{2j} \cdot \binom{2i}{2i-p} \cdot \binom{2j}{2j-q}\right) \cdot t^{2k-2n} \qquad (12)$$

We see that $0 \le 2n \le 4N$, and we have:

$$|F_{2N}(\sigma+it)|^2 = D'_{0,2N} + \beta^2 \cdot D'_{2,2N} + \beta^4 \cdot D'_{4,2N} + \beta^6 \cdot D'_{6,2N} + \cdots \ldots \ldots + \beta^{4N} \cdot D'_{4N,2N} \qquad (13)$$

We also note that when n is greater than N the quantities $D'_{2n,2N}$ will be incomplete (will not contain all its terms), but as N increases the number of complete $D'_{2n,2N}$'s in (13) will increase. The difference between $D'_{2n,2N}$ and $D_{2n,2N}$ is that in the third sum the condition $0 \le p,q \le 2N$ is discarded. For large N, the quantities $D'_{2n,2N}$ will be equal to $D_{2n,2N}$ for $2n \le 2N$, but will start to differ for $2n \ge 2N$. As N increases though, more and more terms in (13) will have their coefficients $D'_{2n,2N}$ equal to $D_{2n,2N}$.

We write D_{2n} for the quantities:

$$D_{2n} \coloneqq \sum_{k=0}^{\infty}\left(\sum_{i+j=k}\sum_{p+q=2n}(-1)^{k-[\![\frac{p}{2}]\!]-[\![\frac{q}{2}]\!]} \cdot a_{2i} \cdot a_{2j} \cdot \binom{2i}{2i-p} \cdot \binom{2j}{2j-q}\right) \cdot t^{2k-2n} \qquad (14)$$

We write then:

$$\mathbf{C_{2k,2n}} \coloneqq \sum_{\mathbf{i+j=k}}\sum_{\mathbf{p+q=2n}}(-\mathbf{1})^{\mathbf{k}-[\![\frac{p}{2}]\!]-[\![\frac{q}{2}]\!]} \cdot \mathbf{a_{2i}} \cdot \mathbf{a_{2j}} \cdot \binom{\mathbf{2i}}{\mathbf{2i-p}} \cdot \binom{\mathbf{2j}}{\mathbf{2j-q}} \qquad \mathbf{(15)}$$

Relation (14) can then be written as:

$$D_{2n} = \sum_{k=0}^{\infty} C_{2k,2n} \cdot t^{2k-2n} \qquad (16)$$

We note that in (16) we used the conventions mentioned before, in fact (16) can also be written:

$$\mathbf{D_{2n}} = \sum_{\mathbf{k=n}}^{\infty} \mathbf{C_{2k,2n}} \cdot \mathbf{t^{2k-2n}} \qquad \mathbf{(16)'}$$

We see that the important relation seems to be (15). This is a relation that involves only the coefficients a_{2n} that are involved in (1).

We also define $\boldsymbol{\alpha_{2n}} \coloneqq \mathbf{a_{2n}} \cdot (\mathbf{2n})! = \boldsymbol{\xi}^{(\mathbf{2n})}(\frac{\mathbf{1}}{\mathbf{2}})$. (17)

After these calculations and definitions we are ready to state the main theorem on which the rest of the results will be based.

Theorem 2. If for any $n \geq 0$ we have $\alpha_{2n} \leq (\ln(n+2))^{2n}$ (in other words, if $\xi^{(2n)}\left(\frac{1}{2}\right) \leq (\ln(n+2))^{2n}$), then the following series are absolutely convergent (also the series (16) and (16)') and the following relations are well defined and valid:

$$|\boldsymbol{\xi}(\boldsymbol{\sigma}+\mathbf{it})|^2 = \mathbf{D_0} + (\boldsymbol{\sigma}-\tfrac{1}{2})^2 \cdot \mathbf{D_2} + (\boldsymbol{\sigma}-\tfrac{1}{2})^4 \cdot \mathbf{D_4} + (\boldsymbol{\sigma}-\tfrac{1}{2})^6 \cdot \mathbf{D_6} + \cdots \ldots\ldots\ldots\ldots\ldots + \left(\boldsymbol{\sigma}-\tfrac{1}{2}\right)^{\mathbf{2n}} \cdot \mathbf{D_{2n}} + \cdots \ldots\ldots\ldots\ldots \quad \textbf{(18)}$$

$$\frac{\mathbf{d}}{\mathbf{d\sigma}} |\boldsymbol{\xi}(\boldsymbol{\sigma}+\mathbf{it})|^2 = \mathbf{2} \cdot \left(\boldsymbol{\sigma}-\tfrac{1}{2}\right) \cdot \mathbf{D_2} + \mathbf{4} \cdot \left(\boldsymbol{\sigma}-\tfrac{1}{2}\right)^{\mathbf{3}} \cdot \mathbf{D_4} + \mathbf{6} \cdot \left(\boldsymbol{\sigma}-\tfrac{1}{2}\right)^{\mathbf{5}} \cdot \mathbf{D_6} + \cdots \ldots\ldots\ldots\ldots + \mathbf{2n} \cdot \left(\boldsymbol{\sigma}-\tfrac{1}{2}\right)^{\mathbf{2n-1}} \cdot \mathbf{D_{2n}} + \cdots \ldots\ldots\ldots \quad \textbf{(19)}$$

$$\frac{\mathbf{d^2}}{\mathbf{d\sigma^2}} |\boldsymbol{\xi}(\boldsymbol{\sigma}+\mathbf{it})|^2 = \mathbf{2} \cdot \mathbf{D_2} + \mathbf{12} \cdot (\boldsymbol{\sigma}-\tfrac{1}{2})^2 \cdot \mathbf{D_4} + \mathbf{30} \cdot (\boldsymbol{\sigma}-\tfrac{1}{2})^4 \cdot \mathbf{D_6} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots + \mathbf{2n} \cdot (\mathbf{2n-1}) \cdot \left(\boldsymbol{\sigma}-\tfrac{1}{2}\right)^{\mathbf{2n-2}} \cdot \mathbf{D_{2n}} + \cdots \ldots\ldots \quad \textbf{(20)}$$

Proof. We start from relation (15) and we use the definition (17).

Relation (15) can then be written:

$$C_{2k,2n} = \sum_{i+j=k} \sum_{p+q=2n} (-1)^{k-[\![\frac{p}{2}]\!]-[\![\frac{q}{2}]\!]} \cdot \alpha_{2i} \cdot \alpha_{2j} \cdot \frac{1}{(2i)!} \cdot \frac{1}{(2j)!} \cdot \binom{2i}{2i-p} \cdot \binom{2j}{2j-q}$$

Now we calculate (we take into account the fact that $i+j=k$ and $p+q=2n$):

$$\frac{1}{(2i)!} \cdot \frac{1}{(2j)!} \cdot \binom{2i}{2i-p} \cdot \binom{2j}{2j-q} = \frac{1}{(2i-p)! \cdot (2j-q)!} \cdot \frac{1}{p! \cdot q!} = \frac{1}{(2k)!} \frac{(2k)!}{(2k-2n)! \cdot (2n)!} \cdot \frac{(2k-2n)!}{(2i-p)! \cdot (2j-q)!} \cdot \frac{(2n)!}{p! \cdot q!} = \frac{1}{(2k)!} \cdot \binom{2k}{2n} \cdot \binom{2k-2n}{2i-p} \cdot \binom{2n}{p}.$$

Using the relations above, we can write (15) in the form (where $i+j=k$ and $p+q=2n$):

$$\mathbf{C_{2k,2n}} = \frac{\mathbf{1}}{(\mathbf{2k})!} \cdot \binom{\mathbf{2k}}{\mathbf{2n}} \cdot \sum_{\mathbf{0} \leq \mathbf{i} \leq \mathbf{k}} \sum_{\mathbf{0} \leq \mathbf{p} \leq \mathbf{2n}} (-\mathbf{1})^{\mathbf{k}-[\![\frac{\mathbf{p}}{2}]\!]-[\![\frac{\mathbf{q}}{2}]\!]} \cdot \boldsymbol{\alpha_{2i}} \cdot \boldsymbol{\alpha_{2j}} \cdot \binom{\mathbf{2k-2n}}{\mathbf{2i-p}} \cdot \binom{\mathbf{2n}}{\mathbf{p}}. \quad \textbf{(21)}$$

From hypothesis we note that we have $\alpha_{2i} \cdot \alpha_{2j} \leq (\ln(i+2))^{2i} \cdot (\ln(j+2))^{2j} \leq (\ln(k+2))^{4k}$

From (21) we have then (for $s \geq 0$):

$$|C_{2n+2s,2n}| \leq \frac{(\ln(n+s+2))^{4n+4s}}{(2n)! \cdot (2s)!} \cdot \sum_{0 \leq i \leq n+s} \sum_{0 \leq p \leq 2n} \binom{2s}{2i-p} \cdot \binom{2n}{p} \leq \frac{(\ln(n+s+2))^{4n+4s}}{(2n)! \cdot (2s)!} \cdot 2^{2s-1} \cdot 2^{2n} = \frac{2^{2n}}{(2n)!} \cdot \frac{2^{2s-1}}{(2s)!} \cdot (\ln(n+s+2))^{4n+4s} \quad (22)$$

As a consequence, using (16) , (16)' and the relation above we have:

$$|D_{2n}| = \left| \sum_{s=0}^{\infty} C_{2n+2s,2n} \cdot t^{2s} \right| \leq \frac{2^{2n}}{(2n)!} \cdot \sum_{s=0}^{\infty} \frac{2^{2s-1}}{(2s)!} \cdot (\ln(n+s+2))^{4n+4s} \cdot t^{2s} = \frac{2^{2n-1}}{(2n)!} \cdot \sum_{s=0}^{\infty} \frac{2^{2s}}{(2s)!} \cdot (\ln(n+s+2))^{4n+4s} \cdot t^{2s} \quad (23)$$

From (22) we see that the series defined by (16) and (16)' are absolutely convergent (the coefficients $C_{2n+2s,2n}$ decrease very fast as a function of s), so the series on the right side of relations (18), (19) and (20) are well defined.

We write the proof for (18), the other two are similar. We consider the expression:

$$H(\sigma+it) = D_0 + (\sigma-\frac{1}{2})^2 \cdot D_2 + (\sigma-\frac{1}{2})^4 \cdot D_4 + (\sigma-\frac{1}{2})^6 \cdot D_6 + \cdots \ldots + \left(\sigma-\frac{1}{2}\right)^{2n} \cdot D_{2n} + \cdots \ldots \ldots \ldots \ldots$$

We know that for any $\varepsilon \geq 0$ there is a n_0 such that for all $n \geq n_0$ we have:

$$|D_0 - D'_{0,2n}| \leq \varepsilon, \ |D_2 - D'_{2,2n}| \leq \varepsilon, \ |D_4 - D'_{4,2n}| \leq \varepsilon, \ \ldots\ldots\ldots, \ |D_{4n} - D'_{4n,2n}| \leq \varepsilon, \ |D_{4n+2}| \leq \varepsilon, \ |D_{4n+4}| \leq \varepsilon, \ |D_{4n+6}| \leq \varepsilon, \ldots\ldots\ldots\ldots\ldots\ldots$$

We also know that $\left|\sigma - \frac{1}{2}\right| \leq \frac{1}{2}$. As a consequence, using (13) we can write:

$$|H(\sigma+it) - |F_{2N}(\sigma+it)|^2| \leq \varepsilon + \frac{1}{2^2} \cdot \varepsilon + \frac{1}{2^4} \cdot \varepsilon + \cdots \ldots + \frac{1}{2^{4n}} \cdot \varepsilon + \frac{1}{2^{4n+2}} \cdot \varepsilon + \cdots \ldots\ldots\ldots = \varepsilon \cdot \left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \cdots \ldots + \frac{1}{2^{4n}} + \frac{1}{2^{4n+2}} + \cdots \ldots\ldots\ldots\right) < 2\varepsilon .$$

In other words, we have proved that $|H(\sigma+it) - |F_{2N}(\sigma+it)|^2| \to 0$ as $N \to \infty$.

That basically means that $H(\sigma+it) = |\xi(\sigma+it)|^2$

Relations (19) and (20) can be proved in a similar manner. **QED.**

Proposition 1. We consider the analytic function f(s) defined on the strip $0 \leq \mathrm{Re}(s) \leq 1$, given by series of the type (Taylor series at $\frac{1}{2}$):

$$f(s) = b_0 + b_2 \cdot (s-\frac{1}{2})^2 + b_4 \cdot (s-\frac{1}{2})^4 + b_6 \cdot (s-\frac{1}{2})^6 + \cdots \ldots + b_{2n} \cdot \left(s-\frac{1}{2}\right)^{2n} + \cdots \ldots ,$$

We also define $\mu_{2n} := b_{2n} \cdot (2n)! = f^{(2n)}(\frac{1}{2})$

If for any $n \geq 0$, the coefficients b_{2n} are real and satisfy the relations $b_{2n} \geq 0$ and for any $n \geq 1$ we have :

$$f\left(\frac{1}{2}\right) \cdot f^{(2n)}\left(\frac{1}{2}\right) = f^{(2)}\left(\frac{1}{2}\right) \cdot f^{(2n-2)}\left(\frac{1}{2}\right) = f^{(4)}\left(\frac{1}{2}\right) \cdot f^{(2n-4)}\left(\frac{1}{2}\right) = \ldots\ldots = f^{(2u)}\left(\frac{1}{2}\right) \cdot f^{(2n-2u)}\left(\frac{1}{2}\right) = \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ,$$

then the function f(s) has all its zeros on the vertical $\mathrm{Re}(s) = \frac{1}{2}$.

Proof. We define, as in the case of the Xi function the following quantities (the calculations are similar):

$$C_{2k,2n} = \frac{1}{(2k)!} \cdot \binom{2k}{2n} \cdot \sum_{0 \leq i \leq k} \sum_{0 \leq p \leq 2n} (-1)^{k-\left[\!\left[\frac{p}{2}\right]\!\right]-\left[\!\left[\frac{q}{2}\right]\!\right]} \cdot \mu_{2i} \cdot \mu_{2j} \cdot \binom{2k-2n}{2i-p} \cdot \binom{2n}{p}$$

$D_{2n} = \sum_{k=n}^{\infty} C_{2k,2n} \cdot t^{2k-2n}$

Let's see how the expression $\left[\frac{p}{2}\right] + \left[\frac{p}{2}\right]$ behaves, for various values for n (where $p + q = 2n$).

For n = 0.

P	q	$\left[\frac{p}{2}\right] + \left[\frac{p}{2}\right]$
0	0	0

For n = 1.

P	q	$\left[\frac{p}{2}\right] + \left[\frac{p}{2}\right]$
0	2	1
1	1	0
2	0	1

For n = 2.

P	q	$\left[\frac{p}{2}\right] + \left[\frac{p}{2}\right]$
0	4	2
1	3	1
2	2	2
3	1	1
4	0	2

We note that when p is even, then $n - \left[\frac{p}{2}\right] - \left[\frac{q}{2}\right] = 0$

In general, we get the general pattern (the alternating even–odd when p takes values from 0 to 2n is important).

We consider $n \geq 1$.

We consider the case $2k = 2n$. In this case, the expression $2i - p$ in (21, or the similar relation applied for our function) can only take the value 0 (with the conventions mentioned before, all the other terms are 0).

In this case we have then:

$$C_{2n,2n} = \frac{1}{(2n)!} \cdot (\mu_0 \cdot \mu_{2n} \cdot \binom{2n}{0} + \mu_2 \cdot \mu_{2n-2} \cdot \binom{2n}{2} + \mu_4 \cdot \mu_{2n-4} \cdot \binom{2n}{4} + \cdots + \mu_{2n} \cdot \mu_0 \cdot \binom{2n}{2n}) .$$

We note that if all the quantities $\mu_{2i} \cdot \mu_{2j}$ with $i + j = k = n$, take a constant value $\mu_0 \cdot \mu_{2n} = \mu_2 \cdot \mu_{2n-2} = \mu_4 \cdot \mu_{2n-4} = \ldots = M_n$, then $C_{2n,2n}$ would take the value: $C_{2n,2n} = \frac{M_n}{(2n)!} \cdot \left(\binom{2n}{0} + \binom{2n}{2} + \binom{2n}{4} + \binom{2n}{6} + \cdots \ldots + \binom{2n}{2n} \right) = \frac{M_n \cdot 2^{2n-1}}{(2n)!}$.

We consider now the case $2k = 2n + 2$. In this case we have $0 \le 2i - p \le 2$.

In this case we have:

$$C_{2n+2,2n} = \frac{1}{(2n+2)!} \cdot \binom{2n+2}{2n} \cdot \left(-\left(\mu_0 \cdot \mu_{2n+2} \cdot \binom{2}{0} + \mu_2 \cdot \mu_{2n} \cdot \binom{2}{2}\right) \cdot \binom{2n}{0} + \mu_2 \cdot \mu_{2n} \cdot \binom{2}{1} \cdot \binom{2n}{1} - \left(\mu_2 \cdot \mu_{2n} \cdot \binom{2}{0} + \mu_4 \cdot \mu_{2n-2} \cdot \binom{2}{2}\right) \cdot \binom{2n}{2} + \mu_4 \cdot \mu_{2n-2} \cdot \binom{2}{1} \cdot \binom{2n}{3} + \cdots \ldots + \mu_{2n} \cdot \mu_2 \cdot \binom{2}{1} \cdot \binom{2n}{2n-1} - \left(\mu_{2n} \cdot \mu_2 \cdot \binom{2}{0} + \mu_{2n+2} \cdot \mu_0 \cdot \binom{2}{2}\right) \cdot \binom{2n}{2n} \right)$$

We note that if all the quantities $\mu_{2i} \cdot \mu_{2j}$ with $i + j = k = n + 1$ take a constant value $\mu_0 \cdot \mu_{2n+2} = \mu_2 \cdot \mu_{2n} = \mu_4 \cdot \mu_{2n-2} = \ldots = M_{n+1}$, then $C_{2n+2,2n}$ would be zero, because in this case $C_{2n+2,2n} = \frac{1}{(2n+2)!} \cdot \binom{2n+2}{2n} \cdot 2 \cdot M_{n+1} \cdot \left(-\binom{2n}{0} + \binom{2n}{1} - \binom{2n}{2} + \binom{2n}{3} \pm \cdots \ldots + \binom{2n}{2n-1} - \binom{2n}{2n} \right) = 0$.

We consider now the case $2k = 2n + 4$. In this case we have $0 \le 2i - p \le 4$.

In this case we have:

$$C_{2n+4,2n} = \frac{1}{(2n+4)!} \cdot \binom{2n+4}{2n} \cdot \left(\left(\mu_0 \cdot \mu_{2n+4} \cdot \binom{4}{0} + \mu_2 \cdot \mu_{2n+2} \cdot \binom{4}{2} + \mu_4 \cdot \mu_{2n} \cdot \binom{4}{4}\right) \cdot \binom{2n}{0} - \left(\mu_2 \cdot \mu_{2n+2} \cdot \binom{4}{1} + \mu_4 \cdot \mu_{2n} \cdot \binom{4}{3}\right) \cdot \binom{2n}{1} + \left(\mu_2 \cdot \mu_{2n+2} \cdot \binom{4}{0} + \mu_4 \cdot \mu_{2n} \cdot \binom{4}{2} + \mu_6 \cdot \mu_{2n-2} \cdot \binom{4}{4}\right) \cdot \binom{2n}{2} - \left(\mu_4 \cdot \mu_{2n} \cdot \binom{4}{1} + \mu_6 \cdot \mu_{2n-2} \cdot \binom{4}{3}\right) \cdot \binom{2n}{3} \pm \cdots \ldots - \left(\mu_{2n} \cdot \mu_4 \cdot \binom{4}{1} + \mu_{2n+2} \cdot \mu_2 \cdot \binom{4}{3}\right) \cdot \binom{2n}{2n-1} + \left(\mu_{2n} \cdot \mu_4 \cdot \binom{4}{0} + \mu_{2n+2} \cdot \mu_2 \cdot \binom{4}{2} + \mu_{2n+4} \cdot \mu_0 \cdot \binom{4}{4}\right) \cdot \binom{2n}{2n} \right).$$

We note that if all the quantities $\mu_{2i} \cdot \mu_{2j}$ with $i + j = k = n + 2$ take a constant value $\mu_0 \cdot \mu_{2n+4} = \mu_2 \cdot \mu_{2n+2} = \mu_4 \cdot \mu_{2n} = \ldots = M_{n+2}$, then $C_{2n+4,2n}$ would be zero, because in this case $C_{2n+4,2n} = \frac{1}{(2n+4)!} \cdot \binom{2n+4}{2n} \cdot 8 \cdot M_{n+2} \cdot \left(\binom{2n}{0} - \binom{2n}{1} + \binom{2n}{2} - \binom{2n}{3} \pm \cdots \ldots - \binom{2n}{2n-1} + \binom{2n}{2n} \right) = 0$.

The calculations for $C_{2n+6,2n}$, $C_{2n+8,2n}$, $\ldots\ldots$, $C_{2n+2s,2n}$, $\ldots\ldots$ can be done in a similar way. I will not write the general form because the calculations are similar (and the general form will look like (21)). We note that if all the quantities $\mu_{2i} \cdot \mu_{2j}$ with $i + j = k = n + s$ take a constant value $\mu_0 \cdot \mu_{2n+2s} = \mu_2 \cdot \mu_{2n+2s-2} = \mu_4 \cdot \mu_{2n+2s-4} = \ldots = M_{n+s}$, then $C_{2n+2s,2n}$ would be zero, for $s \ge 1$.

We reach the conclusion that $D_{2n} \ge 0$ for any $n \ge 1$.

From the relation (similar to (20) but applied for our function):

$$\frac{d^2}{d\sigma^2}\,|f(\sigma+it)|^2 = 2 \cdot D_2 + 12\cdot(\sigma-\tfrac{1}{2})^2\cdot D_4 + 30\cdot(\sigma-\tfrac{1}{2})^4\cdot D_6 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots +$$
$$2n\cdot(2n-1)\cdot\left(\sigma-\tfrac{1}{2}\right)^{2n-2}\cdot D_{2n} + \cdots\ldots\ldots.. ,$$

we see that $\frac{d^2}{d\sigma^2}\,|f(\sigma+it)|^2 \geq 0$ for $0 \leq \sigma \leq 1$.

Our function $|f(\sigma+it)|^2$, seen as a function of σ is a convex function, and from theorem 1 we conclude that all its zeros are on the vertical $\sigma = \frac{1}{2}$. **QED.**

Proposition 2. We consider the analytic function $f(s)$, where $s = \sigma + i\cdot t$, defined on the strip $0 \leq \mathrm{Re}(s) \leq 1$, given by series of the type (Taylor series at $\frac{1}{2}$):

$$f(s) = b_0 + b_2\cdot(s-\tfrac{1}{2})^2 + b_4\cdot(s-\tfrac{1}{2})^4 + b_6\cdot(s-\tfrac{1}{2})^6 + \cdots\ldots + b_{2n}\cdot\left(s-\tfrac{1}{2}\right)^{2n} + \cdots\ldots. ,$$

We define $\mu_{2n} \coloneqq b_{2n}\cdot(2n)! = f^{(2n)}(\frac{1}{2})$.

We define the quantities:

$$C_{2k,2n} = \frac{1}{(2k)!}\cdot\binom{2k}{2n}\cdot\sum_{0\leq i\leq k}\sum_{0\leq p\leq 2n}(-1)^{k-[\![\frac{p}{2}]\!]-[\![\frac{q}{2}]\!]}\cdot\mu_{2i}\cdot\mu_{2j}\cdot\binom{2k-2n}{2i-p}\cdot\binom{2n}{p}$$

$$D_{2n} = \sum_{k=n}^{\infty} C_{2k,2n}\cdot t^{2k-2n}$$

If for any $n \geq 0$ we have $\mu_{2n} \leq (\ln(n+2))^{2n}$, and if $D_{2n} \geq 0$ for any $n \geq 1$ and for any t, then our function has all its zeros on the vertical $\mathrm{Re}(s) = \frac{1}{2}$.

Proof. The proof is immediate from theorem 1 and the relation :

$$\frac{d^2}{d\sigma^2}\,|f(\sigma+it)|^2 = 2 \cdot D_2 + 12\cdot(\sigma-\tfrac{1}{2})^2\cdot D_4 + 30\cdot(\sigma-\tfrac{1}{2})^4\cdot D_6 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots +$$
$$2n\cdot(2n-1)\cdot\left(\sigma-\tfrac{1}{2}\right)^{2n-2}\cdot D_{2n} + \cdots\ldots\ldots..$$ **QED.**

Many other interesting results can be based on theorem 2, our main result.

Proposition 3. We consider the analytical function f(s) on the whole complex plane with the Taylor series at $\frac{1}{2}$ of the form:

$$f(s) = c_0 + c_2\cdot(s-\tfrac{1}{2})^2 + c_4\cdot(s-\tfrac{1}{2})^4 + c_6\cdot(s-\tfrac{1}{2})^6 + \cdots\ldots\ldots\ldots,$$

We also define:

$$\mu_{2n} \coloneqq c_{2n}\cdot(2n)! = f^{(2n)}(\tfrac{1}{2})$$

We consider the series:

$S(t) = b_0 + b_2 \cdot t^2 + b_4 \cdot t^4 + b_6 \cdot t^6 + \cdots \ldots \ldots \ldots$, where we take:

$b_0 = \mu_0{}^2$, $b_2 = -\frac{1}{2!} \cdot \left(\binom{2}{0} \cdot \mu_0 \cdot \mu_2 + \binom{2}{2} \cdot \mu_2 \cdot \mu_0 \right)$, $b_4 = \frac{1}{4!} \cdot \left(\binom{4}{0} \cdot \mu_0 \cdot \mu_4 + \binom{4}{2} \cdot \mu_2{}^2 + \binom{4}{4} \cdot \mu_4 \cdot \mu_0 \right)$, ……….. in general $b_{2k} = (-1)^k \cdot \frac{1}{(2k)!} \cdot \left(\binom{2k}{0} \cdot \mu_0 \cdot \mu_{2k} + \binom{2k}{2} \cdot \mu_2 \cdot \mu_{2k-2} + \cdots \ldots \ldots \ldots \binom{2k}{2k} \cdot \mu_{2k} \cdot \mu_0 \right)$

Then $S(t) \geq 0$ for any real t , in other words, S(t) takes only positive values. We note here that when the coefficients c_{2n} are positive, the series S(t) will have terms alternating in sign (and the proof of the proposition is not obvious).

Proof. The proof is immediate from:

$|f(\sigma + it)|^2 = D_0 + (\sigma - \frac{1}{2})^2 \cdot D_2 + (\sigma - \frac{1}{2})^4 \cdot D_4 + (\sigma - \frac{1}{2})^6 \cdot D_6 + \cdots \ldots \ldots \ldots \ldots \ldots + \left(\sigma - \frac{1}{2}\right)^{2n} \cdot D_{2n} + \cdots \ldots \ldots \ldots \ldots$, when observing that our series S(t) is exactly D_0. That means that:

$S(t) = D_0 = \left| f\left(\frac{1}{2} + it\right) \right|^2 \geq 0$. **QED.**

Proposition 4. (see reference [2] for the proof). Let ξ(z) be the Riemann Xi function and n a appositive integer. Then, as $n \rightarrow \infty$ we have:

$$\ln\left(\xi^{(2n)}\left(\frac{1}{2}\right)\right) = 2n \cdot \ln(\ln(n)) - 2\left(\ln 2 + \frac{1}{\ln(n)}\right) \cdot n + \frac{9}{4} \cdot \ln(2n) - \frac{3}{4} \cdot \ln(\ln(n)) + O(1). \qquad (24)$$

We also know that (Stirling):

$$\ln\big((2n)!\big) = \left(2n - \frac{1}{2}\right) \cdot \ln(2n) - 2n + \ln\sqrt{2\pi} + o(1). \qquad (25)$$

From (24) and (25) we conclude that:

$$\ln\frac{\xi^{(2n)}(\frac{1}{2})}{(2n)!} = -2n \cdot (\ln(2n) - \ln(\ln(n))) - 2n \cdot \left(\ln 2 + \frac{1}{\ln(n)} - 1\right) + \frac{11}{4} \cdot \ln(2n) - \frac{3}{4} \cdot \ln(\ln(n)) + O(1). \qquad (26)$$

From (24) we see that the conditions of theorem 2 are satisfied by the Riemann Xi function, so the Riemann Xi function indeed satisfied relations (18), (19) and (20).

Proposition 5. In relations (18), (19) and (20) the quantities D_{2n} (which depend on t, as defined by (14) and (16)) take positive values for all values of t, in other words $D_{2n}(t) \geq 0$ for all any t.

Incomplete Proof. We write D_{2n} in the form:

$D_{2n} = D_{2n}(t) = \sum_{s=0}^{\infty} C_{2n+2s,2n} \cdot t^{2s}$.

We will use proposition 3, and we claim that there are real numbers μ_0 , $\mu_2, \mu_4, \ldots \ldots .., \mu_{2n} \ldots \ldots$, such that the following system of equations is satisfied:

$C_{2n,2n} = \mu_0^{\,2}$; $C_{2n+2,2n} = -\frac{1}{2!} \cdot \left(\binom{2}{0} \cdot \mu_0 \cdot \mu_2 + \binom{2}{2} \cdot \mu_2 \cdot \mu_0\right)$; $C_{2n+4,2n} = \frac{1}{4!} \cdot \left(\binom{4}{0} \cdot \mu_0 \cdot \mu_4 + \binom{4}{2} \cdot \mu_2^{\,2} + \binom{4}{4} \cdot \mu_4 \cdot \mu_0\right)$; ……….; $C_{2n+2s,2n} = (-1)^s \cdot \frac{1}{(2s)!} \cdot \left(\binom{2s}{0} \cdot \mu_0 \cdot \mu_{2s} + \binom{2s}{2} \cdot \mu_2 \cdot \mu_{2s-2} + \cdots \ldots \ldots \ldots \binom{2s}{2s} \cdot \mu_{2s} \cdot \mu_0\right) \ldots \ldots \ldots \ldots \ldots$ (27)

The proof of the claim follows from recursively solving this system of equations.

$\mu_0 = \sqrt{C_{2n,2n}}$. We note that μ_0 is a real number because we can prove that $C_{2n,2n}$ is a positive quantity.

$$\mu_2 = \frac{1}{2\sqrt{C_{2n,2n}}} \cdot (-2 \cdot C_{\mathbf{2n+2},2n}) \quad (28)$$

$$\mu_4 = \frac{1}{2 \cdot C_{2n,2n} \cdot \sqrt{C_{2n,2n}}} \cdot \left(24 \cdot C_{\mathbf{2n},2n} \cdot C_{\mathbf{2n+4},2n} - 6 \cdot C^2_{\mathbf{2n+2},2n}\right) \quad (29)$$

$$\mu_6 = \frac{1}{2 \cdot C^2_{2n,2n} \cdot \sqrt{C_{2n,2n}}} \cdot \left(-720 \cdot C^2_{\mathbf{2n},2n} \cdot C_{\mathbf{2n+6},2n} + 360 \cdot C_{\mathbf{2n},2n} \cdot C_{\mathbf{2n+2},2n} \cdot C_{\mathbf{2n+4},2n} - 90 \cdot C^3_{\mathbf{2n+2},2n}\right) \quad (30)$$

………

In general we see that from the relations $C_{2n+2s,2n} = (-1)^s \cdot \frac{1}{(2s)!} \cdot \left(\binom{2s}{0} \cdot \mu_0 \cdot \mu_{2s} + \binom{2s}{2} \cdot \mu_2 \cdot \mu_{2s-2} + \cdots \ldots \ldots \ldots \binom{2s}{2s} \cdot \mu_{2s} \cdot \mu_0\right)$ we can recursively find the value of μ_{2s} as a function of the $C_{2n,2n}$, $C_{2n+2,2n}$, $C_{2n+4,2n}$, ……., $C_{2n+2k-2,2n}$, $C_{2n+2s,2n}$.

I also present the following known results which will probably be needed in the proof. First an inequality:

$\frac{2^{n \cdot H(\frac{r}{n})}}{n+1} \le \binom{n}{r} \le 2^{n \cdot H(\frac{r}{n})}$, where H(x) is the entropy

$$H(x) = -x \cdot \log(x) - (1-x) \cdot \log(1-x). \quad (31)$$

Second, we will need the following estimation (using (22)):

$$|(2s)! \cdot C_{2n+2s,2n}| \le (2s)! \cdot \frac{(\ln(n+s+2))^{4n+4s}}{(2n)! \cdot (2s)!} \cdot \sum_{0 \le i \le n+s} \sum_{0 \le p \le 2n} \binom{2s}{2i-p} \cdot \binom{2n}{p} \le \frac{2^{2n}}{(2n)!} \cdot 2^{2s-1} \cdot (\ln(n+s+2))^{4n+4s} . \quad (32)$$

The expressions that depend on n can be considered as a constant in this case, because we are interested in the absolute convergence of the series $\mu_0 + \frac{\mu_2}{2!} \cdot (z-\frac{1}{2})^2 + \frac{\mu_4}{4!} \cdot (z-\frac{1}{2})^4 + \frac{\mu_6}{6!} \cdot (z-\frac{1}{2})^6 + \cdots \ldots . + \frac{\mu_{2s}}{(2s)!} \cdot \left(z-\frac{1}{2}\right)^{2s} + \cdots \ldots ..$, so we are interested in the factor that depends on s.

In other words, the system of equations mentioned above always has a solution. In order to apply proposition 3 we only have to prove that the series $\mu_0 + \frac{\mu_2}{2!} \cdot (z-\frac{1}{2})^2 + \frac{\mu_4}{4!} \cdot (z-\frac{1}{2})^4 + \frac{\mu_6}{6!} \cdot (z-\frac{1}{2})^6 +$

$\cdots \ldots . + \frac{\mu_{2s}}{(2s)!} \cdot \left(z - \frac{1}{2}\right)^{2s} + \cdots \ldots ..$ is absolutely convergent on the whole complex plane (for any z). At this point I have not been able to reach a closed form expression for the coefficients involved in the expression for μ_{2s} (even if the calculations are straightforward and recursive). I leave this problem as a challenge for a mathematician (probably aided by some symbolic computation software) willing to finalize these calculations. I suspect that we need to consider the Taylor series of the Riemann's Xi function at other points on the critical line, in order to finalize the calculations (see conclusions).

If the calculations can be finalized, from proposition 3 we could conclude that all the quantities D_{2n} are positive, in other words $D_{2n}(t) \geq 0$ for all any t . **!!!.**

In the following, I will give an example that would also be a verification that the calculations (and theorem 2 in particular) are correct.

Example. We consider the values $b_{2n} = \frac{1}{(2n)!}$, $n \geq 0$ for the coefficients b_{2n} . Then we have :

$$\cosh\left(z - \frac{1}{2}\right) = 1 + \frac{1}{2!} \cdot (z - \tfrac{1}{2})^2 + \frac{1}{4!} \cdot (z - \tfrac{1}{2})^4 + \frac{1}{6!} \cdot (z - \tfrac{1}{2})^6 + \cdots \ldots \ldots \ldots ,$$

where we take $z = \sigma + i \cdot t$.

When we make the calculations we find :

$$|\cosh(\sigma - \tfrac{1}{2} + i \cdot t)|^2 = \frac{1}{4} \cdot \left(e^{2\left(\sigma - \frac{1}{2}\right)} + e^{-2\left(\sigma - \frac{1}{2}\right)}\right) + \frac{1}{2} \cdot (\cos^2 t - \sin^2 t)$$

We can then write:

$$|\cosh(\sigma - \tfrac{1}{2} + i \cdot t)|^2 = \frac{1}{2} \cdot \cosh(2 \cdot (\sigma - \tfrac{1}{2})) + \frac{1}{2} \cdot \cos(2t)) = \frac{1}{2} \cdot (1 + \cos(2t)) + \left(\sigma - \frac{1}{2}\right)^2 + \frac{1}{3} \cdot \left(\sigma - \frac{1}{2}\right)^4 + \cdots \ldots \ldots .$$

We see that in this case we have $D_0 = \frac{1}{2} \cdot (1 + \cos(2t))$, $D_2 = 1$, $D_4 = \frac{1}{3}$, and so on … ….If we use our formulas (15), (16), (16)' and (21) for this particular case, we see that we find the right values.

4. A different approach, Turan-type inequalities.

Theorem 3. Let $F(z)$ be a function of complex variable defined on the critical strip. We assume that F satisfies the following conditions.

1. $F(z)$ is a holomorphic function on the critical strip.
2. F satisfies the functional equation $F(z) = F(1 - z)$ for all z on the critical strip.
3. For any s real, $s > \frac{1}{2}$ and for any natural n , the n-th derivatives at s are positive $F^{(n)}(s) > 0$.
4. For any real s with $\frac{1}{2} < s < 1$ and for any natural numbers m, n with $m < n$, the following inequalities are satisfied:
   $F^{(m)}(s) \cdot F^{(n)}(s) > \frac{m}{n+1} \cdot F^{(m-1)}(s) \cdot F^{(n+1)}(s)$. Of course, we assume here that $m \geq 1$.

5. For any real s with $\frac{1}{2} < s < 1$ and for any natural number $n \geq 1$ the following inequalities are satisfied:
$(F^{(n)}(s))^2 > \frac{2n}{n+1} \cdot F^{(n-1)}(s) \cdot F^{(n+1)}(s)$

If conditions $1-5$ are satisfied, then all the zeros of the function F(z) are situated on the critical line $\mathrm{Re}(z) = \frac{1}{2}$.

Proof. We assume that there is a zero $z_0 = \sigma_0 + i \cdot t$ of the function F(z) with $\sigma_0 > \frac{1}{2}$. We will then reach a contradiction, and this will prove the theorem (the assumption that such a zero exists is false). We also note that from the functional equation for any zero with real part less than $\frac{1}{2}$ we would have a corresponding zero with real part greater than $\frac{1}{2}$, so it is sufficient to prove that there are no zeros with real part greater than $\frac{1}{2}$.

We consider the Taylor series of the function F(z) around the real point $\sigma_0 > \frac{1}{2}$. We take $z = \sigma + i \cdot t$.

$$F(z) = F(\sigma + i \cdot t) = F(\sigma_0) + F^{(1)}(\sigma_0) \cdot (\sigma - \sigma_0 + i \cdot t) + \frac{F^{(2)}(\sigma_0)}{2!} \cdot (\sigma - \sigma_0 + i \cdot t)^2 + \frac{F^{(3)}(\sigma_0)}{3!} \cdot (\sigma - \sigma_0 + i \cdot t)^3 + \frac{F^{(4)}(\sigma_0)}{4!} \cdot (\sigma - \sigma_0 + i \cdot t)^4 + \cdots \ldots \ldots \ldots \ldots \quad (32)$$

Relation (32) for $z_0 = \sigma_0 + i \cdot t$ will have the form:

$$0 = F(\sigma_0 + i \cdot t) = F(\sigma_0) + F^{(1)}(\sigma_0) \cdot (i \cdot t) + \frac{F^{(2)}(\sigma_0)}{2!} \cdot (i \cdot t)^2 + \frac{F^{(3)}(\sigma_0)}{3!} \cdot (i \cdot t)^3 + \frac{F^{(4)}(\sigma_0)}{4!} \cdot (i \cdot t)^4 + \cdots \ldots \ldots \ldots \ldots \quad (33)$$

We define the quantities: $a_n = \frac{F^{(n)}(\sigma_0)}{n!}$ for all $n \geq 0$. From condition 3 we conclude that $a_n > 0$ for all $n \geq 0$.

Relation (33) will become then:

$$0 = F(\sigma_0 + i \cdot t) = a_0 + a_1 \cdot (i \cdot t) + a_2 \cdot (i \cdot t)^2 + a_3 \cdot (i \cdot t)^3 + a_4 \cdot (i \cdot t)^4 + \cdots \ldots \ldots \ldots \ldots = (a_0 - a_2 \cdot t^2 + a_4 \cdot t^4 - a_6 \cdot t^6 + a_8 \cdot t^8 + \cdots \ldots) + i \cdot (a_1 \cdot t - a_3 \cdot t^3 + a_5 \cdot t^5 - a_7 \cdot t^7 + a_9 \cdot t^9 + \cdots \ldots) \quad (34)$$

From (34) we can write:

$$0 = |F(\sigma_0 + i \cdot t)|^2 = (a_0 - a_2 \cdot t^2 + a_4 \cdot t^4 - a_6 \cdot t^6 + a_8 \cdot t^8 + \cdots \ldots)^2 + (a_1 \cdot t - a_3 \cdot t^3 + a_5 \cdot t^5 - a_7 \cdot t^7 + a_9 \cdot t^9 + \cdots \ldots)^2 = a_0{}^2 + (a_1{}^2 - 2 \cdot a_0 \cdot a_2) \cdot t^2 + (a_2{}^2 - 2 \cdot a_1 \cdot a_3 + 2 \cdot a_0 \cdot a_4) \cdot t^4 + (a_3{}^2 - 2 \cdot a_2 \cdot a_4 + 2 \cdot a_1 \cdot a_5 - 2 \cdot a_0 \cdot a_6) \cdot t^6 + (a_4{}^2 - 2 \cdot a_3 \cdot a_5 + 2 \cdot a_2 \cdot a_6 - 2 \cdot a_1 \cdot a_7 + 2 \cdot a_0 \cdot a_8) \cdot t^8 + \cdots \ldots \ldots \quad (35)$$

In relation (35) the coefficient of the term containing t^{2n} will be $a_n{}^2 - 2 \cdot a_{n-1} \cdot a_{n+1} + 2 \cdot a_{n-2} \cdot a_{n+2} - 2 \cdot a_{n-3} \cdot a_{n+3} + \cdots \ldots \ldots + (-1)^n \cdot 2 \cdot a_0 \cdot a_{2n}$.

For $m < n$ the condition $a_m \cdot a_n > a_{m-1} \cdot a_{n+1}$ is equivalent to the condition $F^{(m)}(s) \cdot F^{(n)}(s) > \frac{m}{n+1} \cdot F^{(m-1)}(s) \cdot F^{(n+1)}(s)$. Also the condition $a_n{}^2 - 2 \cdot a_{n-1} \cdot a_{n+1} > 0$ is equivalent to the condition $(F^{(n)}(s))^2 > \frac{2n}{n+1} \cdot F^{(n-1)}(s) \cdot F^{(n+1)}(s)$.

Conditions 4 and 5 will imply that all the following quantities are positive (involved in the coefficients in relation (35)):

$a_n{}^2 - 2 \cdot a_{n-1} \cdot a_{n+1} > 0$; $2 \cdot a_{n-2} \cdot a_{n+2} - 2 \cdot a_{n-3} \cdot a_{n+3} > 0$; $2 \cdot a_{n-4} \cdot a_{n+4} - 2 \cdot a_{n-5} \cdot a_{n+5} > 0$; ... ... and so on.

That means that each term in the last series in (35) is positive, so the sum of these terms cannot be zero. That means that the assumption that $\sigma_0 + i \cdot t$ is a zero of F with $\sigma_0 > \frac{1}{2}$ is false. **QED.**

Observation 1. We note that conditions 4 and 5 in theorem 3 can be replaced with the following general condition (which is stronger, more restrictive):

For any real s with $\frac{1}{2} < s < 1$ and for any natural numbers m, n with $m \leq n$, the following inequalities are satisfied:

$F^{(m)}(s) \cdot F^{(n)}(s) > \frac{2m}{n+1} \cdot F^{(m-1)}(s) \cdot F^{(n+1)}(s)$. Of course, we assume here that $m \geq 1$.

Under these two conditions theorem 3 holds with a very similar proof.

Observation 2. We know that Riemann's Xi function satisfies conditions 1. and 2. In reference [3] Mark Coffey proves that condition 3 is also satisfied. We are left to prove that Riemann's Xi function also satisfies conditions 4 and 5 (or a weaker version that makes the coefficients in (35) positive).

In reference [2] Mark Coffey proves the following proposition.

Proposition 6. For real s and $j \longrightarrow \infty$ we have:

$$\xi^{(j)}(s) = \frac{j \cdot (j-1)}{2^{j-1}} \cdot \frac{(j-2)^{\frac{s-1}{2}}}{(\ln(j-2))^{\frac{s}{2}}} \cdot \left(1 + (-1)^j \cdot \left(\frac{\ln(j-2)}{j-2}\right)^{s-\frac{1}{2}}\right) \cdot \left(\ln\left(\frac{j-2}{\pi}\right) - \ln\left(\ln\left(\frac{j-2}{\pi}\right)\right) + o(1)\right)^{j-\frac{3}{2}} \cdot \exp\left(-\frac{j-2}{\ln(j-2)}\right) . \quad (36)$$

Observation 3. We could use Coffey's result, and using relation (36) we can try to prove that conditions 4 and 5 from theorem 3 are satisfied by the Riemann's Xi function. The calculations are complex, but this would prove that the Riemann Hypothesis is true. We also note that conditions (4) and (5) from theorem 3 are not exactly the Turan inequalities, that is the reason why I called them Turan – type inequalities.

5. **Conclusions.**

We do not claim that we proved that the Riemann Hypothesis is true, at this point. We do emphasize the following points. Using the asymptotic results of Mark Coffey (proposition 4) we can prove that Riemann's Xi function satisfies the conditions from theorem 2. That means that relations (18), (19) and (20) are valid. Using proposition 3, we sketched the incomplete proof in proposition 5 that all the quantities $D_{2n}(t)$ are positive for any value of t (the absolute convergence of the series $\mu_0 + \frac{\mu_2}{2!} \cdot (z-\frac{1}{2})^2 + \frac{\mu_4}{4!} \cdot (z-\frac{1}{2})^4 + \frac{\mu_6}{6!} \cdot (z-\frac{1}{2})^6 + \cdots \ldots + \frac{\mu_{2s}}{(2s)!} \cdot \left(z-\frac{1}{2}\right)^{2s} + \cdots \ldots$ has to be clearly established). From relation (20) we see that the function $|\xi(\sigma + it)|^2$ (seen as a function of σ) is convex, and from theorem 1 we conclude that Riemann's Xi function has all its zeros (on the strip) on the vertical $\sigma = \frac{1}{2}$. We could then conclude that Riemann's Hypothesis is true.

There is a second way to approach the problem that avoids convexity , but using the reformulation of Riemann's Hypothesis (in reference [4]). We see from relation (19) that if all the quantities $D_{2n}(t)$ are positive, then the function $|\xi(\sigma + it)|^2$, seen as a function of σ is decreasing for $\sigma < \frac{1}{2}$ and increasing for $> \frac{1}{2}$. Using the result in [4] this would be a proof of Riemann's Hypothesis.

We also note that the method described in this paper can be applied for the Taylor series of the Riemann's Xi function around other points. In [3] the derivatives $\xi^{(j)}(\frac{1}{2} + i \cdot t)$ (with t real) of the Riemann's Xi function on the critical line are calculated, and we can give asymptotic estimations (actually upper bounds) for these derivatives similar to those known for $\xi^{(2j)}(\frac{1}{2})$. By considering the Taylor series of the Riemann's Xi function around points $\frac{1}{2} + i \cdot t$, where t is very close to the imaginary part of a zero and using the methods described in this article, the calculations can be managed properly. Basically, we work with a Taylor series developed around a point (on the critical line) close to a zero (in imaginary parts), and we prove that the particular zero under consideration can only have real part equal to $\frac{1}{2}$. The problem that I could not solve in this article is the absolute convergence of the series that appears in proposition 5 (for any value for the imaginary part t). The corresponding problem that we will have when considering the Taylor series around points on the critical line (following similar methods as here) will involve proving the absolute convergence of a series only for arbitrary small values of the imaginary part t, and this is a considerably easier problem (even if the initial calculations are more complex since in this case the odd order derivatives will not be zero). In this case, the problem that I could not solve in this article (the absolute convergence of the series mentioned in proposition 5) might be properly managed.

Related to section 4, and the Turan – type inequalities, the problem here is to use Coffey's result, proposition 6 in order to prove that conditions 4 and 5 from theorem 3 are satisfied. That would be a proof that Riemann's Hypothesis is indeed true. To start, we can prove that conditions 4 and 5 are satisfied asymptotically.

As a general conclusion, I suspect that this is a matter of complex calculations, and this is the conclusion of my article.

References:

1. H. M. Edwards, *"Riemann's Zeta Function",* Dover 2001.
2. Mark W. Coffey, *"Asymptotic Estimation of $\xi^{(2n)}(\frac{1}{2})$: On a Conjecture of Farmer and Rhoades"*, Mathematics of Computation, volume 78, 2000, pages 1147 – 1154.
3. Mark W. Coffey, *"Relations and positivity results for the derivatives of the Riemann Xi function"*, Journal of Computational and Applies Mathematics, 166, 2004, pages 525 – 534.
4. Jonathan Sondow and Cristian Dumitrescu, *"A Monotonicity Property of the Riemann's Xi Function and a Reformulation of Riemann's Hypothesis "*, Periodica Mathematica Hungarica, March 2010, volume 60, issue 1, pp 37-40.

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