Pricing Rent-to-Own Options with a Barrier Level: Taking Housing Contracts as an example

Abstract

In this study, a formula is developed for pricing a rent-to-own option in which a barrier level is embedded, where the tenant has the right to buy a specified rental house during the life of the contract. Through a boundary integral method, we derive a closed-form approximate solution of the rent-to-own option. This study finds several characteristics through a sequence of numerical analyses and provides an available method for pricing rent-to-own options taking housing contracts with a barrier level as an example.

Keywords: rent-to-own option, boundary integral method, Green's function.

1. Introduction

Recently, many option characteristics have been embedded in residential lease contracts. These options give tenants some rights to exercise renewal, to purchase, to cancel, or to break contracts. In fact, these contingent rights can change the future terms of residential lease contracts. Thus, future payoffs to both tenants and landlords are both contingent on and dependent on the underlying asset prices. Residential lease contracts with embedded options change the initial rent payments to a right or obligation to renew, to break, or to expand the terms of the lease, as compared to standard residential lease contracts with no embedded options. One of commonly embedded options in residential lease contracts is a kind of purchase option, which gives the tenant the right to buy the predetermined real estate at a specified price. We call this option a rent-to-own option. The right to own such houses increases the value of residential lease contracts.

However, the simple embedded options referenced above, such as renewals, breaking the lease, or purchase options appear frequently in real estate markets, and some additional special options have also been developed recently, such as an option to impose barrier constraints. A barrier option is a path-dependent option, which comes in various frameworks, but its major characteristic is that this type of option is either started or terminated based on the underlying price bumping a predetermined level during pre-specified time intervals.

This study investigates a special embedded option in residential lease contracts, which is one of the rent-to-own options for rental real estate by which tenants can exercise the right to buy the rental real estate early when the house price surpasses a predetermined benefit value. In addition, when the house price touches the pre-specified barrier level, the contract writer terminates the option right and pays a specific number of rebates to the contract owner. Therefore, the rent-to-own option with a barrier constraint will limit the tenant's benefits. Residential lease contract rents embedded with a contingent purchase option with a barrier level are higher than those in standard residential lease contracts without a purchase option, but are lower than that of standard residential lease contracts embedded with a purchase option without any barrier constraint. Furthermore, the embedded rent-to-own option with a barrier level in a residential lease contract can limit loss to the writer since the effect of a barrier is contingent on the purchase option. On the other hand, residential lease contracts with a barrier level can reduce the cost of rental payments made by tenants and also limit price risks for the writers.

Barrier rent-to-own options differ from standard rent-to-own options embedded in residential lease contracts in several ways: First, standard rent-to-own options provide an additional right to the tenant to buy the specified real estate without limitations. On the contrary, an option with a barrier contingency will limit the benefits to the contract owner since barrier rent-to-own options typically have a limit payoff (i.e., the difference between the barrier price and strike price) when the owners exercise the contracts. Therefore, the barrier rent-to-own option can reduce tenant costs. Second, a clause containing the option contract with a barrier contingency is also favorable for option writers since it reduces price risks related to high exercise values because the underlying house price can go up infinitely.

In short, although residential lease contracts embedded with option features appear to be one of the most common forms available in real estate markets, it is not currently possible to understand the complicated structure of barrier levels in practical contracts. Our investigation is an attempt to value the barrier rent-to-own options embedded in residential lease contracts and to analyze the price characteristics of embedded options.

Obviously, a rent-to-own option with a barrier level is similar to the up-and-out barrier options in financial derivatives. Generally, up-and-out barrier options are designed as one form of financial derivatives. However, the embedded barrier rent-to-own option in residential lease contracts is one of a number of real options that provides an opportunity for property owners to limit their losses because the tenant will purchase the house if the price surpasses the barrier level. In addition, the standard barrier option provides rights for the option owners to enter or to quit the market. The barrier rent-to-own option in residential lease contracts provides tenants with an opportunity to purchase a rental real estate at a desirable price, where the option writer pays a specific amount of rebate to the option holder and terminates the option right early. In this situation, the rent payment for the tenants of residential lease contracts with a barrier rent-to-own option are lower than the rent paid in the case of residential lease contracts with a standard rent-to-own option.

Several researchers have dealt with the problem of developing appropriate valuation methods for different kinds of residential lease contracts with embedded options (e.g., Grenadier 1995; Buetow and Albert 1998; Albert and Beutow 2000; Clapham 2003; Bellalah et al. 2002). Simple embedded rent-to-own options in residential lease contracts can be modeled using the Black-Scholes partial differential equation (PDE). Therefore, we can use many traditional techniques for pricing embedded options, such as the lattice method, the finite difference method, and real option methods. Grenadier (1995), and Buetow and Albert (1998) analyzed residential lease contracts with embedded options using a real option and an explicit finite difference method. They estimated the value of embedded options in residential lease contracts that gave tenants the right to renew the lease or to purchase the assets at a rent tied to a market index. Extending the discussion of Buetow and Albert (1998), Clapham

(2003) provided an analytical solution within a partial differential equations framework and compared renewal options for different lease lengths. A further problem in the valuations of the embedded options is that some embedded options come with a complicated payoff structure. We can use several traditional methods for pricing simple embedded options, but we cannot easily value the more complicated embedded options. Thus, we attempt a more feasible method by which to value embedded options, especially with regard to rent-to-own options with a barrier level.

In the present study, the formula of the barrier rent-to-own option embedded in the residential lease contracts is derived through a Green's function and the boundary integral method in the partial differential equation (PDE) framework. Compared to several traditional methods, such as the lattice numerical analysis, the binomial tree model, and risk neutral measures, the boundary integral method provides a more efficient and more feasible technique to obtain a closed-form approximate solution for the derivatives under consideration. This method has not been applied in the valuation of residential lease contracts and in real option analysis. In addition, the method can be easily used to solve the problem of option pricings with a complex payoff structure. Specifically, the method is appropriate to solve American-type options with a barrier level. Several studies on financial issues have used a boundary integral method to analyze the prices of derivatives (e.g., Farto and Vázquez 2005; Cartea and del-Castillo-Negrete 2007; Ben-Ameur et al. 2007; Hernandez-Martinez et al. 2011). For example, using the boundary integral method in dynamic programming procedures, Ben-Ameur et al. (2007) priced options embedded in bonds and documented that their procedure is both efficient and accurate.

The remainder of this study proceeds as follows: The following section reviews the embedded options in residential lease contracts, specifically with regard to rent-to-own options. Section 3 develops the valuation method for the rent-to-own options with a barrier level embedded in residential lease contracts. Section 4 provides a numerical analysis of the impact of some important factors on the option value. The last section offers some conclusions.

2. Related Literature

Residential lease contracts with option features have been considered by several previous researchers (Shilling et al. 1985; Buetow and Albert 1998; Bellalah 2002; Benjamin and Chinloy 2004; Clapham 2004; Iskandar et al. 2017). Several studies typically have mentioned several extra rights or obligations for tenants (Smith and Wakeman 1985; Crosby et al. 2003; Gómez-Padilla et. al. 2021; Iskandar and Husniah 2017). Crosby et al. (2003) pointed out that some contracts contain the right to break the lease, to review, to renew, or to sublet. From both the owner's and the tenant's perspective, Vimpari (2018) suggests that the advancements in option pricing could provide to value-building adaptability and help determine optimal initial rent. Smith and Wakeman (1985) analyzed the option mechanisms to renew, to purchase, or to cancel in some lease contracts. Currently, in addition to the options referenced above, complicated types of options have been embedded in residential lease contracts.

These more complicated embedded options in residential lease contracts have payoff structures that are similar to standard financial options. For example, a renewal option for the tenants can be evaluated as a financial call option, while an option to break contracts can be priced as a put option (Grenadier 1995; Buetow and Albert 1998). An option to renew a contract provides the tenant with a bargaining position when the contract expires. An option to break or cancel contracts gives the right for the tenants to abandon rental contracts. In addition, rent reviews also can be one option since they give the right to the landlord to decide whether to increase the rental fee.

The rent-to-own option is one of important real options in residential lease contracts. Rental contracts or lease contracts provide the use of a specific piece of real estate over a fixed period, in which the tenant receives the lease benefits, and the landlord receives the rent streams. Smith (1979) first introduced a means by which to use a financial option method to describe and formulate rental contracts. Grenadier (1995) developed a unified framework for pricing various lease or rental contracts through a real option approach. Using the phrase "structure of rent rates" to analyze specific cases involving lease contracts, Grenadier (1995) determined equilibrium rent rates for various embedded options in leasing contracts. Furthermore, using a real options approach, Bellalah (2002) valued lease contracts under incomplete information. Therefore, several researchers have used financial option pricing methods for residential lease contract embedded options. Specifically, applying a theoretical option model, Benjamin and Chinloy (2004) discussed several issues concerning retail leases, such as the base rent, a contingency payment, and rent sharing between landlords and tenants. In addition, considering stochastic externalities and tenant defaults, Cho and Shilling (2007) solved a PDE (partial differential equation) for retail shopping center lease contracts through a real option model. Jaggia et. al. (2019) developed a model to explore the pricing mechanism of a rent-to-own contract that points out the contract's embedded options and several bundled services.

Numerous studies have succeeded in valuing embedded options in financial contracts; however, few studies have given numerical or closed-form solutions for embedded options in residential lease contracts. Buetow and Albert (1998) used a numerical method to price renewal options or purchase options for residential lease contracts in which the house price or rent was tied to a specific price index. Furthermore, extending the analysis of Buetow and Albert (1998), Clapham (2003) derived analytical derivatives of embedded lease options. Specifically, Clapham (2003) also compared various renewal options for different lease lengths in Grenadier's (1995) analytical framework.

Using several traditional valuation methods or numerical analyses, it is possible to price simple embedded rent-to-own options in residential lease contracts; however, a complicated form of embedded rent-to-own options in payoff structures cannot be easily derived using these traditional valuation methods. A boundary integral method along with Green's function has been proved to have advantages in terms of both efficiency and accuracy for complex option structures. For example, modeling the valuation of options embedded in bond contracts, Ben-Ameur et al. (2007) suggested a simple, efficient approximation method for options with a complex structure, such as Bermudan or American-style derivatives.

The current study differs from previous studies in that we discuss a more complicated residential lease contract embedded a rent-to-own option with a barrier level. This study attempts to derive the valuation for a rent-to-own option with a barrier level embedded in a residential lease contract. Our analysis can offer a method with valuable applications in real estate practice. In addition, owing to the barrier level, our pricing model can provide a closed-form approximate solution for this complicated problem through Green's function and the boundary integral method.

3. Methodology

This section discusses the valuation of residential rent-to-own options with a pre-specified barrier level. Under Black-Scholes economics, the underlying price of a house follows a geometric Brownian motion. The rent-to-own option value is a function of the underlying house price and the time. Therefore, the rent-to-own option in residential lease contracts satisfies the partial differential equation (PDE) as follows:

$$\frac{1}{2}\sigma^2 H^2 C_{HH}(H,t) + r\,H\,C_H(H,t) + C_t(H,t) = r\,C(H,t)\,, \tag{1}$$

where *C* denotes the value of the rent-to-own option with a barrier level embedded in the residential lease contract; *H* denotes the underlying house price; *σ* represents the volatility of the underlying house return; *r* is the risk-free interest rate, and *t* is the time.

To obtain the right to purchase a house at a good price, a tenant will want to pay a higher rent than a general quote. The premium of the residential lease contract with a purchase right is equivalent to the rent-to-own option value. On the other hand, we set a barrier level *B* to control risk to the rent-to-own option writer in the lease contract at the same time. Therefore, the contract holder owns a right to exercise the right to purchase the house early at a desirable price *K* when the house price *H* is higher than the pre-specified price *K* in the contract life. However, when the house price reaches the pre-specified barrier level *B*, the rent-to-own option writer terminates the tenant's right to early purchase and pays a rebate to the option holder. On the contrary, the contract holder loses the lease premium specified in the contract when the house price remains below the early purchase price *K*.

Since equation (1) is a non-homogeneous PDE, we can use a set of variable transformations (2) to simplify it.

$$\begin{cases} \tau = T - t \\ x = \ln H + (r - \dfrac{\sigma^2}{2})\tau\,, \\ v(x,\tau) = e^{r\cdot\tau} C(H,\tau) \end{cases} \tag{2}$$

Then, Equation (1) is further simplified into a linear homogeneous heat equation as follows:

$$\frac{\sigma^2}{2} v_{xx} = v_\tau . \tag{3}$$

We introduce a composite Green's function as follows:

$$G(x,\tau;\bar{x},\bar{\tau}) = \frac{1}{\sqrt{2\pi\sigma^2(\bar{\tau}-\tau)}} \left\{ \exp\left[-\frac{(x-\bar{x})^2}{2\sigma^2(\bar{\tau}-\tau)} \right] - \exp\left[-\frac{(x-2B^*(\bar{\tau})+\bar{x})^2}{2\sigma^2(\bar{\tau}-\tau)} + \alpha \right] \right\} H(\bar{\tau}-\tau), \tag{4}$$

where $B^*(\bar{\tau})$ is the transformed barrier level at the time to maturity $\bar{\tau}$, and $B^*(\bar{\tau}) = \ln B + (r - \frac{\sigma^2}{2})\bar{\tau}$. α is a constant value, and $\alpha = \frac{2\left(r-\frac{\sigma^2}{2}\right)\left(B^*(\bar{\tau})-\bar{x}\right)}{\sigma^2}$.

Since time (t) is transformed into the time to maturity (τ), the terminal payoff structures now become the initial condition of Equation (3). We define the initial condition as follows:

$$v_0(x,0) = \begin{cases} 0 & ,\text{if } e^x < K \\ e^x - K & ,\text{if } e^x \ge K \end{cases} \tag{5}$$

During the contract period, the rent-to-own option writer can terminate the early purchase right and pay a rebate to the option holder when the underlying house price reaches the pre-specified barrier level B. It will be the boundary condition defined as follows:

$$v_b(B^*(\tau),\tau) = (B-K)\cdot e^{r\cdot\tau} . \tag{6}$$

The heat equation will be a well-post problem. Therefore, the valuation of the residential rent-to-own option with a barrier level can be represented as a closed-form approximate solution as follows:

$$C(H_0,\tau_0)=$$

$$e^{-r\cdot\tau_0}\left(\int_{-\infty}^{B^*(0)} v_0(x,0)G(x,0;x_0,\tau_0)dx-\frac{\sigma^2}{2}\int_0^{\tau_0} v_b(B^*(\tau),\tau)G_x(B^*(\tau),\tau;x_0,\tau_0)d\tau\right), \quad (7)$$

where x_0 is the transformation of the current house price H_0; $x_0=\ln H_0+(r-\frac{\sigma^2}{2})\tau_0$, and τ_0 is the current time t_0 to maturity of the contract, $\tau_0=T-t_0$.

4. Numerical Examples

In this section, we take various numerical examples to discuss the sensitivities of the risk-free rate, the volatility of the underlying house return, the time to maturity of the contract life, and the barrier level, respectively. Our purpose is to observe how the barrier rent-to-own option values change with specific decision factors.

The initial values of these parameters are as follows: The current underlying house price is assumed to be \$200,000. The early-exercise purchase price is the same (\$200,000). The barrier level is \$240,000. That is, when the house price reaches the pre-determined barrier price, the rent-to-own option writer will terminate the early purchase right and pay a rebate to the option holder. In this example, we set the rebate as the difference between barrier level *B* and the early exercise price *K*. The risk-free rate is one percent. The volatility of the underlying house price is fifteen percent. The time to maturity of the contract is one year.

Table 1 shows the effects of the risk-free rate to the value of barrier rent-to-own options embedded in a residential lease contract, where it can be observed that the embedded option value of the contract increases with the risk-free rate since an increase in the interest rate will reduce the cost of early purchase. Alternatively, the increments of the embedded option value are almost double, even triple, that of the risk-free rates.

<Insert Table 1>

Under various current house prices, Figure 1 shows the embedded rent-to-own option value of contracts with different levels of volatility for the underlying house price return. There is a positive relation between the embedded option value and the volatility of the underlying house price return. The embedded option value with a low volatility of underlying house price return increases significantly when the current house price is at a relatively high level. Alternatively, the valuation of the embedded option with high volatility increases at a more even rate than is the case for the one with low volatility.

<Insert Figure 1>

Figure 2 represents the tendency of the embedded rent-to-own option value of the contract under various levels of volatility for the underlying house price return when the embedded option is at-the-money. In the figure, it can be observed that the embedded option value increases quickly when the volatility of the house price return is lower than about 30%. However, the scenario is opposite if the volatility of house price return is higher than about 40%. Therefore, the effects of low volatility in an underlying house price return on the embedded option value is more significant than that of high volatility since high volatility will increase the probability that a high underlying house price will reach the pre-specified barrier level, and the payoff is almost sure to be determined at the same time.

<Insert Figure 2>

Figure 3 shows the effect of the time to maturity to the embedded rent-to-own option value. The embedded option value is positively related to the time to maturity, and the sensitivity of time to option value is not obvious. Specifically, when increasing the time to maturity of the contract, the partial increments of the embedded rent-to-own option value decrease.

<Insert Figure 3>

Finally, we discuss the relation between the embedded rent-to-own option value and the pre-determined barrier level in Figure 4. The embedded option value increases with the barrier level. When the barrier price is high enough, the payoff is similar to a plain vanilla call option with no boundary constraint. Therefore, there exists a horizontal asymptote as well as the fact that the barrier price is at a much higher level.

<Insert Figure 4>

5. Conclusions

Residential rent-to-own options with a pre-specified barrier level are a type of path-dependent option. The contract holder can buy the rental house at a favorable price during the contract life. At the same time, the contract writer will set a pre-specified barrier level to control for exposure to risk. In our study, we introduce an efficient, accurate method to obtain a closed-form approximate solution to price residential rent-to-own options with a barrier level.

We find that the embedded option value varies significantly when the volatility of the rental house price return is low. The effects of the risk-free rate and the time to maturity on the embedded option values are small. The price of the embedded rent-to-own option of the contract with a high pre-specified barrier level approaches the price of a vanilla option without any barrier constraint. This study clearly explains the value characteristics of rent-to-own options with a barrier level embedded in a real estate lease contract.

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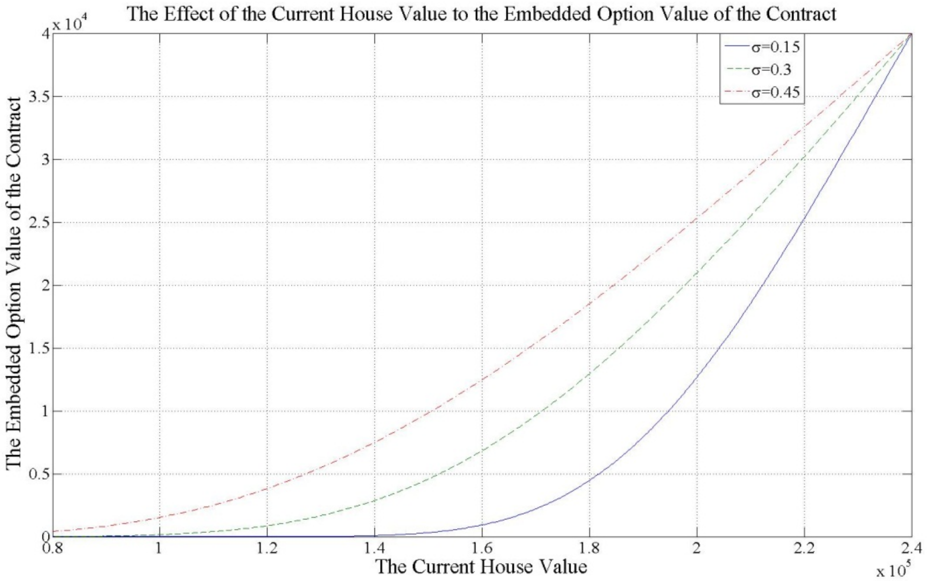
Table 1. The effect of the risk-free rate on the embedded option value of a contract

risk-free rate	The embedded option value
0.500%	$12273.00
0.625%	$12371.19
0.750%	$12469.64
0.875%	$12568.34
1.000%	$12667.30
1.125%	$12766.50
1.250%	$12865.93
1.375%	$12965.59
1.500%	$13065.49

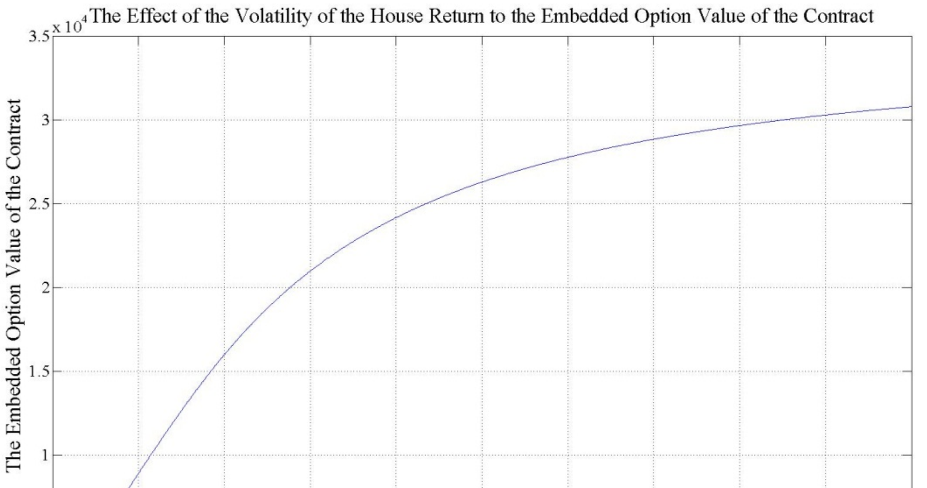
The parameter settings: The current house price is $200,000. The early-exercise purchase price is the same ($200,000). The barrier is $240,000. The volatility of the house price return is fifteen percent. The time to maturity of the contract is one year.

Figure 1. The effect of the current house value on the embedded option value of the contract



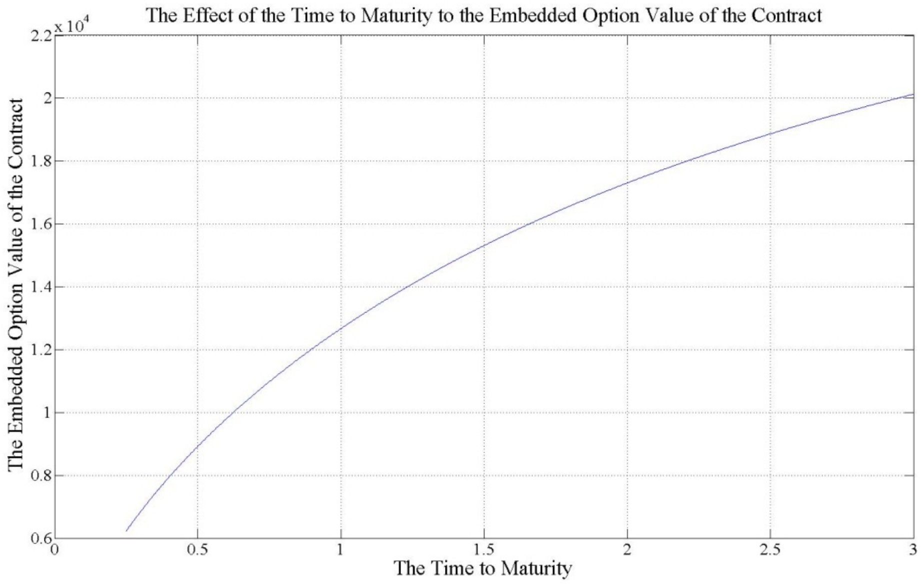
The parameter settings: The early-exercise purchase price is the same ($200,000). The barrier price is $240,000. The risk-free rate is one percent. The time to maturity is one year.

Figure 2. The effect of the volatility of house price return on the embedded option value of the contract



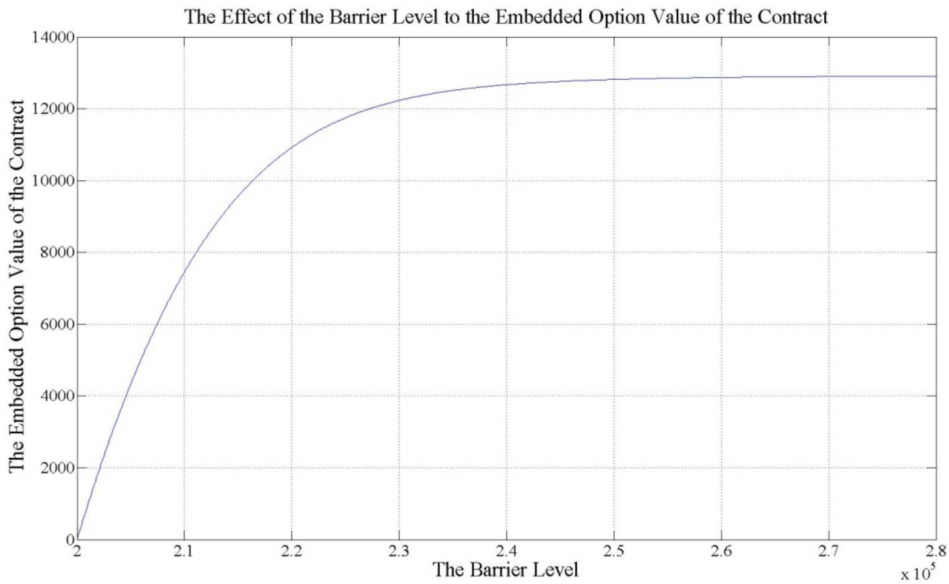
The parameter settings: The current house price is $200,000. The early-exercise purchase price is the same ($200,000). The barrier price is $240,000. The risk-free rate is one percent. The time to maturity is one year.

Figure 3. The effect of the time to maturity on the embedded option value of the contract



The parameter settings: The current house price is $200,000. The early-exercise purchase price is the same ($200,000). The barrier price is $240,000. The risk-free rate is one percent. The volatility of the house price return is fifteen percent.

Figure 4. The effect of the barrier level on the embedded option value of the contract



The parameter settings: The current house price is $200,000. The early-exercise purchase price is the same ($200,000). The risk-free rate is one percent. The volatility of the house price return is fifteen percent. The time to maturity of the contract life is one year.