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Pricing Exotic Power Options with a Brownian-Time-Changed Variance Gamma Process on a Dividend-Paying Stock[*]

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Abstract

This paper proposes a Brownian time change for modeling stochastic volatility and combines it with a drifted variance gamma process in deriving explicit pricing methods for exotic power options in the presence of jumps and volatility clustering. In view of the changeful payoff structure of exotic options, the underlying stock is assumed to pay constant dividends on a continuous basis. In-depth analysis of properties of the time change as well as the time-changed process is focused on characteristic functions, which facilitate pricing via Fourier transform. The pricing mechanism of plain-vanilla options is discussed as a basis for pricing power options. Also, Monte-Carlo simulation techniques are studied through time discretization while empirical analysis is conducted on real financial markets. My objective is to study the theoretical elements of this stochastic-volatility model and its interesting advantages in the context of power option pricing.

KEY WORDS: Brownian time change; Variance gamma process; Power options; Characteristic function

1 Introduction

The Black-Scholes model based on standard Brownian motion and normal distribution proposed in 1973 [5] has been one of the most classical and successful models in option pricing theory. It is arguably the pathfinder of continuous-time pricing for a wide class of financial derivatives. Despite this, the model has inevitable drawbacks by assuming constant volatility and failing to incorporate the conspicuous asymmetric leptokurtic feature of financial returns. In connection with this, many studies have existed over the recent decades to make pertinent improvement as much as possible. Instead of multifarious complicated diffusion-based semimartingales, the use of a Lévy process has been advantageous as it directly defines an infinitely divisible probability distribution whose skewness and excess kurtosis are easily adjustable, as well as is usually able to describe volatility smile by introducing discontinuities aimed to capture short-term large jumps. Related works include Madan and Seneta (1990) [14] and Carr *et al* (2002) [7], in which the renowned variance gamma model and CGMY model were respectively introduced. Nevertheless, imperfections still exist in traditional Lévy models in that they assume by construction deterministic volatility evolution and thus have proven ineffectual for volatility cluster effect.

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In a discrete-time environment, such effect has been thoroughly modeled by heteroscedasticity (GARCH) models, whereas under continuous time, two solutions exist in general - one directly constructs a stochastic process for the volatility coefficient while the other randomizes the time structure to distort financial returns frequencies. A comprehensible advantage of the latter, which has a common appellation of a time change, is increased analytical tractability when adapted to a wide range of Lévy processes. To name a few, Carr *et al* (2003) [8] initially utilized a mean-reverting process proposed by Cox *et al* (1985) [10] as a stochastic time change, while Barndorff-Nielsen and Shepard (2001) [4] studied non-Gaussian Ornstein-Uhlenbeck processes for this purpose. It is worth mentioning that these time changes possess characteristic functions in explicit form.

In this paper, time change is modeled under a rather convenient structure. I choose a drifted integrated squared Brownian motion due to its controllable stochastic level and, importantly, tractability with a closed-form characteristic function. In addition, allowing for its relative diversity and simplicity in simulation, a drifted variance gamma process is selected as the primary model for fluctuations in financial returns. As a consequence, a time-changed variance gamma process is constructed under which the pricing of derivatives can be analyzed. In all, such a model is capable of capturing not only asymmetric leptokurtic feature but also volatility clustering, and I expect it to yield desirable modeling outcomes when applied to real financial time series.

Recently, with the exception of standard European and American options, a large number of new financial derivatives have emerged in the international financial market. Among them, power option is one of the new exotic options. A power option is in general a European-style derivative that provides the option holder with a leveraged or distorted payoff. There are currently two types of power options. Asymmetric power options have payoff at maturity based on the price of an underlying asset raised to a specific power in excess of the option strike price, while symmetric power options simply place a power effect on the original payoff. These options are structurally closely related to typical plain-vanilla options. Also, due to its relative rarity at present, research of power options is critically significant in both theoretical aspect and practical area and is thus the main focus of this paper.

The remainder of this paper is organized as follows. In Section 2 a stochastic time change is formally defined with a Brownian integral, whose statistical features are analyzed subsequently. Section 3 then provides a brief review on the key concepts pertaining to gamma and variance gamma processes. In Section 4 the time change is combined with a drifted variance gamma process and the outcome is used for analyzing evolution of the price of a risky asset. Later, Section 5 discusses in depth specific characteristic-function-based pricing methods for plain-vanilla and power-type options using the new model, while Section 6 explains some simulation techniques as an alternative way of pricing. Finally, empirical analysis is conducted in Section 7 where plain-vanilla pricing results are compared to real market prices and power option prices are given as numerical examples. Some crucial conclusions are drawn in Section 8. Several selected proofs are shown in Appendices that follow.

2 Construction of Business Time

To begin with, let me clarify that this paper assumes a continuous-time environment with $t \geq 0$, and to distinguish it from stochastic business time, t is expressly referred to as the calendar time admitting no randomness.

In general, there are several requirements for the construction of a time change. As a particular type of time, a business time process should be nonnegative and monotonically increasing, and to well cater to continuous-time modeling, it is required to be continuous, i.e., have no jumps, and better increasing in a smooth manner. The latter property can be realized by constructing a purely continuous instantaneous activity rate process, while the former simply necessitates positivity of such a process.

Let $W \equiv (W_t)$ be a standard Brownian motion, which is normally distributed. Rather intuitively, a qualified candidate for the instantaneous activity rate can be found by simply squaring it⟨1⟩, or W^2. On this basis, I can define a stochastic business time $B \equiv (B_t)$ as an integrated squared Brownian motion plus a drift.

$$B_t := mt + v \int_0^t W_s^2 \mathrm{d}s \tag{2.1}$$

for $m \geq 0$ and $v > 0$. In differential form, this is written as

$$\mathrm{d}B_t = \left(m + vW_t^2\right)\mathrm{d}t \tag{2.2}$$

with $B_0 = 0$ a.s., which shows that B is of finite total variation and, indeed, nonnegative and smoothly strictly increasing, with smoothness understood from the existence of $\mathrm{d}B_t/\mathrm{d}t$, $\forall t$. Commenting on the convexity of B yet makes no sense as W is nowhere differentiable with respect to t. An advantage of this time construction, apart from simplicity, is that it can be flexibly adjusted towards or away from the calendar time t. While m maintains the consistency of B with respect to t, v controls the extent of stochastic volatility. In particular, as $v \searrow 0$ and $m \nearrow 1$, $B_t \to t$, $\forall t$.

The distribution of B is quite elusive as the integral in (2.1) is at bottom an infinite Riemann sum of correlated chi-square distributed random variables. Despite this, its characteristic function can still be found in the following closed form,

$$\psi_{B|t}(u) := \mathbb{E}\left[e^{\mathrm{i}uB_t}\right] = e^{\mathrm{i}mtu}\sqrt{\sec\sqrt{2\mathrm{i}vt^2u}} \tag{2.3}$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. Appendix A gives in a convenient way a detailed proof of this function, thanks to the analytical tractability of chi-squared and gamma distributions. By using (2.3) the four crucial moments of B are readily available.

$$\mathbb{E}[B_t] = -\mathrm{i}\frac{\mathrm{d}\ln\psi_{B|t}(u)}{\mathrm{d}u}\bigg|_{u\to 0} = mt + \frac{vt^2}{2}$$

⟨1⟩As implied from Appendix A, the square impact gives rise to an analytical characteristic function of B, while other possible choices, such as an absolute value operator, fail in general.

$$
\begin{aligned}
\mathrm{Var}[B_t] &= (-\mathrm{i})^2 \frac{\mathrm{d}^2 \ln \psi_{B|t}(u)}{\mathrm{d}u^2}\bigg|_{u\to 0} = \frac{v^2 t^4}{3} \\
\mathrm{Skew}[B_t] &= \frac{(-\mathrm{i})^3}{\sqrt{(\mathrm{Var}[B_t])^3}} \frac{\mathrm{d}^3 \ln \psi_{B|t}(u)}{\mathrm{d}u^3}\bigg|_{u\to 0} = \frac{8\sqrt{3}}{5} \approx 2.7713 > 0 \\
\mathrm{Kurt}[B_t] &= 3 + \frac{(-\mathrm{i})^4}{(\mathrm{Var}[B_t])^2} \frac{\mathrm{d}^4 \ln \psi_{B|t}(u)}{\mathrm{d}u^4}\bigg|_{u\to 0} = \frac{513}{35} \approx 14.6571 > 3
\end{aligned} \tag{2.4}
$$

It can be seen that B has time-dependent mean and variance, but is unconditionally skewed to the right and leptokurtic. This business time choice, also referred to as a Brownian time change, will play an essential role in creating stochastic volatility in the pricing models to appear later on.

3 Gamma and Variance Gamma Processes

In this section, I will synthesize some essential concepts related to gamma processes and variance gamma processes, which crucially underly price modeling afterwards. These processes have been thoroughly studied in such papers as Madan and Seneta (1990) [14] and Madan *et al* (1998) [15].

3.1 Gamma Processes

A gamma process $G \equiv (G_t)_{t\geq 0}$ is defined as a purely discontinuous Lévy process admitting a gamma law. I.e., $G_0 = 0$ a.s., and G has independent and stationary increments such that for any $h > 0$, $G_{t+h} - G_t \overset{\text{law}}{=} G_h \sim \mathrm{Gamma}(ah, b)$ where $a > 0$ and $b > 0$. To be precise, a gamma process with parameters (a, b) has the following density.

$$
f_{G|t}(x) = \frac{b^{at}}{\Gamma(at)} x^{at-1} e^{-bx}, \quad x > 0 \tag{3.1.1}
$$

where $\Gamma(\cdot)$ stands for the gamma function. Jumps of G are driven by its Lévy measure,

$$
\nu_G(\mathrm{d}x) = x^{-1} a e^{-bx} \mathbf{1}_{\{x>0\}} \tag{3.1.2}
$$

where $\mathbf{1}_{\{\cdot\}}$ denotes the indicator function which takes value 1 if the argument is true and 0 otherwise. Since $\int_{\mathbb{R}\setminus\{0\}} \nu_G(\mathrm{d}x) = \infty$ and $\int_{\mathbb{R}\setminus\{0\}} |x|\nu_G(\mathrm{d}x) < \infty$, it is said that the gamma process has an infinite jump arrival rate, i.e., infinitely many jumps over any finite time interval, but is of finite total variation. Notice that the characteristic function of (G_t) is given by

$$\psi_{G|t}(u) := \mathbb{E}\left[e^{\mathrm{i}uG_t}\right] = \left(1 - \frac{\mathrm{i}u}{b}\right)^{-at} \tag{3.1.3}$$

By the Lévy-Khintchine representation (see Papapantoleon (2000) [16]), such a function is infinitely divisible, and on its basis the four crucial moments of G are $\mathbb{E}[G_t] = at/b$, $\mathrm{Var}[G_t] = at/b^2$, $\mathrm{Skew}[G_t] = 2/\sqrt{at} > 0$ and $\mathrm{Kurt}[G_t] = 3 + 6/(at) > 3$ for $t > 0$. Clearly, the gamma distribution is asymmetric with a fat right tail.

3.2 Variance Gamma Processes

A variance gamma process can be defined in three distinct ways. It has a preliminary and natural relationship with a variance gamma distribution. This is a three-parameter distribution with $a > 0$, $\theta \in \mathbb{R}$, and $\sigma > 0$, and has the following characteristic function,

$$\psi_{\mathrm{VarGamma}}(u) = \left(1 - \frac{\mathrm{i}\theta u}{a} + \frac{\sigma^2 u^2}{2a}\right)^{-a} \tag{3.2.1}$$

which is also infinitely divisible, and thus gives rise to a variance gamma process as a Lévy process. A variance gamma process $H^{(0)} \equiv \left(H_t^{(0)}\right)$ can hence be defined by its Lévy properties - $H_0^{(0)} = 0$ a.s., and $H^{(0)}$ has independent and stationary increments such that for any $h > 0$, $H_{t+h}^{(0)} - H_t^{(0)} \overset{\text{law}}{=} H_h^{(0)} \sim \mathrm{VarGamma}(ah, \theta h, \sigma\sqrt{h})$.

$H^{(0)}$ is known to be yet another purely discontinuous process whose jumps are governed by the Lévy measure

$$\nu_{H^{(0)}}(\mathrm{d}x) = \frac{a}{|x|}\left(\frac{\theta x - |x|\sqrt{2a\sigma^2 + \theta^2}}{\sigma^2}\right)\mathrm{d}x \tag{3.2.2}$$

which indicates an infinite jump arrival rate and finite total variation. In some cases an additional drift may be necessary to generate a persistent trend for modeling. A drifted variance gamma process $H \equiv (H_t)$ with drift parameter $\mu \in \mathbb{R}$ is formally⟨2⟩ defined as

$$H_t := \mu t + \theta G_t + \sigma W'_{G_t} \tag{3.2.3}$$

where (W'_t) is another standard Brownian motion and (G_t) is a special gamma process with parameters (a, a). On an important note, W, G, and W' are mutually independent. This definition sees the variance gamma process as a gamma-time-changed Brownian motion with drift⟨3⟩. The

⟨2⟩This is the formal definition of a variance gamma process as presented in Madan and Seneta (1990) [14].

⟨3⟩Early discussed in Clark (1973) [9], a positive non-decreasing Lévy process of finite total variation can be used as a particular time change, a.k.a. a subordinator, to convert a Brownian motion into a purely discontinuous process. This type of time change, however, admits discontinuities and is not suitable for modeling business time.

characteristic function of H is presented by directly introducing a drift impact.

$$\psi_{H|t}(u) := \mathbb{E}\left[e^{\mathrm{i}uH_t}\right] = e^{\mathrm{i}\mu t u}\left(1 - \frac{\mathrm{i}\theta u}{a} + \frac{\sigma^2 u^2}{2a}\right)^{-at} \tag{3.2.4}$$

The additional drift only changes the mean of H, while the other central moments are unaffected.

$$\begin{aligned}
\mathbb{E}[H_t] &= -\mathrm{i}\frac{\mathrm{d}\ln\psi_{H|t}(u)}{\mathrm{d}u}\bigg|_{u=0} = (\mu+\theta)t \\
\mathrm{Var}[H_t] &= (-\mathrm{i})^2\frac{\mathrm{d}^2\ln\psi_{H|t}(u)}{\mathrm{d}u^2}\bigg|_{u=0} = \left(\frac{\theta^2}{a}+\sigma^2\right)t \\
\mathrm{Skew}[H_t] &= \frac{(-\mathrm{i})^3}{\sqrt{(\mathrm{Var}[H_t])^3}}\frac{\mathrm{d}^3\ln\psi_{H|t}(u)}{\mathrm{d}u^3}\bigg|_{u=0} = \frac{(2\theta^3+3a\theta\sigma^2)}{(\theta^2+a\sigma^2)^2}\sqrt{\frac{\theta^2+\sigma^2}{at}} \\
\mathrm{Kurt}[H_t] &= 3 + \frac{(-\mathrm{i})^4}{(\mathrm{Var}[H_t])^2}\frac{\mathrm{d}^4\ln\psi_{H|t}(u)}{\mathrm{d}u^4}\bigg|_{u=0} = \frac{3((at+2)\theta^4+2a(at+2)\theta^2\sigma^2+a^2(at+1)\sigma^4)}{at(\theta^2+a\sigma^2)^2}
\end{aligned} \tag{3.2.5}$$

The variance gamma process observably has all time-variant moments, which partially explain its flexibility in practice. In concrete, θ places an impact on all the moments and largely controls the level of asymmetry, noting that $\mathrm{Skew}[H_t] = 0$ if $\theta = 0$; σ mainly defines the level of volatility or fluctuations; the gamma parameter a has a primary control over the leptokurtic feature, because $\lim_{a\searrow 0}\mathrm{Kurt}[H_t] = \infty$, whereas $\lim_{a\to\infty}\mathrm{Kurt}[H_t] = 3$, which becomes mesokurtic.

Moreover, Madan *et al* (1998) [15] proved, by decomposing its characteristic function, that a variance gamma process without drift can also be regarded as the difference of two independent gamma processes. In fact,

$$H_t^{(0)} \equiv H_t - \mu t = G_t^{(1)} - G_t^{(2)} \tag{3.2.6}$$

where $G^{(1)}$ and $G^{(2)}$ are 2 independent gamma processes with parameters $(a, 1/\ell_+)$ and $(a, 1/\ell_-)$, respectively, for which

$$\ell_\pm = \sqrt{\frac{\theta^2}{4a^2}+\frac{\sigma^2}{2a}} \pm \frac{\theta}{2a} > 0 \tag{3.2.7}$$

From another perspective, notice that the Lévy measure (3.2.2) can be reformatted into the following piecewise function,

$$\nu_{H^{(0)}}(\mathrm{d}x) = \begin{cases} x^{-1}ae^{x/\ell_+}\mathrm{d}x, & x>0 \\ |x|^{-1}ae^{x/\ell_-}\mathrm{d}x, & x<0 \end{cases} \tag{3.2.8}$$

thus pointing to the difference of two independent gamma processes' Lévy measures.

4 Model Definition

After well establishing the business time structure and the variance gamma process, a time-changed process can now be constructed to implicitly model the evolution of the price of a risky financial asset. In this section, the model will be rigorously defined and its properties will be analyzed concretely as the foundation of pricing financial derivatives.

4.1 A Time-Changed Process

Be a stochastic process $X \equiv (X_t)$ defined by Brownian-time-changing a drifted variance gamma process, namely,

$$X_t := H_{B_t} \tag{4.1.1}$$

with $X_0 = 0$ a.s. Indeed, X well reserves the jump size pattern of H, as defined by its Lévy measure, while it is the timing of jumps that has been changed in the presence of B. Whenever B increases faster than t, or $\mathrm{d}B_t/\mathrm{d}t > 1$, jumps occur more frequently than otherwise, and hence the volatility structure is automatically randomized. As noted before, the closer B is to t, the less stochastic the volatility level becomes.

The characteristic functions of H and B already known in (2.3) and (3.2.4), the characteristic function of X can be easily derived based on the tower property of expectations.

$$\begin{aligned}\psi_{X|t}(u) := \mathbb{E}\big[e^{\mathrm{i}uX_t}\big] &= \mathbb{E}\big[\mathbb{E}\big[e^{\mathrm{i}uH_{B_t}}|B_t\big]\big] \\ &= \mathbb{E}\left[e^{\mathrm{i}\mu uB_t}\left(1-\frac{\mathrm{i}\theta u}{a}+\frac{\sigma^2u^2}{2a}\right)^{-aB_t}\right] \\ &= \psi_{B|t}\left(\mu u+\mathrm{i}a\ln\left(1-\frac{\mathrm{i}\theta u}{a}+\frac{\sigma^2u^2}{2a}\right)\right)\end{aligned} \tag{4.1.2}$$

To be precise, I have

$$\psi_{X|t}(u) = e^{\mathrm{i}m\mu tu}\left(1-\frac{\mathrm{i}\theta u}{a}+\frac{\sigma^2u^2}{2a}\right)^{-mat}\sqrt{\sec\sqrt{2\mathrm{i}vt^2\left(\mu u+\mathrm{i}a\ln\left(1-\frac{\mathrm{i}\theta u}{a}+\frac{\sigma^2u^2}{2a}\right)\right)}} \tag{4.1.3}$$

which immediately implies the following crucial moments.

$$\begin{aligned}\mathbb{E}[X_t] &= -\mathrm{i}\frac{\mathrm{d}\ln\psi_{X|t}(u)}{\mathrm{d}u}\bigg|_{u\to0} = (\mu+\theta)\left(mt+\frac{vt^2}{2}\right) \\ \mathrm{Var}[X_t] &= (-\mathrm{i})^2\frac{\mathrm{d}^2\ln\psi_{X|t}(u)}{\mathrm{d}u^2}\bigg|_{u\to0} \\ &= \frac{t}{6a}(6m(\theta^2+a\sigma^2)+v\theta^2t((3+2vat^2)+4v\mu a\theta t^2+a(2v\mu^2t^2+3\sigma^2)))\end{aligned} \tag{4.1.4}$$

Besides, the skewness and kurtosis come in considerably lengthy but elementary forms.

$$
\begin{aligned}
\mathrm{Skew}[X_t] &= \frac{(-\mathrm{i})^3}{\sqrt{(\mathrm{Var}[X_t])^3}} \frac{\mathrm{d}^3 \ln \psi_{X|t}(u)}{\mathrm{d}u^3}\bigg|_{u\to 0} \\
&= \sqrt{6}(60m\theta^3 + 2tv(\theta^3(15+vat^2(15+8vat^2)) + 3v\mu a\theta^2 t^2(5+8vat^2) + 24v^2\mu^2 a^2\theta t^4 \\
&\qquad + 8v^2\mu^3 a^2 t^4) + 15a(6m\theta + v\sigma^2 t(\theta(3+2vat^2) + 2v\mu a t^2))) \\
&\qquad \Big/ \Big(5\sqrt{at(6m(\theta^2+a\sigma^2) + vt(\theta^2(3+2vat^2)) + 4v\mu a\theta^2 t^2 + a(2v\mu^2 t^2 + 3\sigma^2))^3}\Big) \\
\mathrm{Kurt}[X_t] &= 3 + \frac{(-\mathrm{i})^4}{(\mathrm{Var}[X_t])^2} \frac{\mathrm{d}^4 \ln \psi_{X|t}(u)}{\mathrm{d}u^4}\bigg|_{u\to 0} \\
&= 3\big(1260m^2 at(\theta^2 + a\sigma^2)^2 + 420m(\theta^2(\theta^2(6+vat^2(3+2vat^2)) + 4v^2\mu a^2\theta t^4 \\
&\qquad + 2v^2\mu^2 a^2 t^4) + 2a\sigma^2(\theta^2(6+vat^2(3+vat^2)) + 2v^2\mu a^2\theta t^4 + v^2\mu^2 a^2 t^4) \\
&\qquad + 3a^2\sigma^4(1+vat^2)) + vt(\theta^4(1260+vat^2(1855+36vat^2(49+19vat^2))) \\
&\qquad + 8v\mu a\theta^3 t^2(140+9vat^2(49+38vat^2)) + 24v\mu a^2\theta t^2(114v^2\mu^2 at^4 \\
&\qquad + 7\sigma^2(10+21vat^2)) + 18\mu^2 a\theta^2(2v^2\mu^2 at^4(49+114vat^2) + 7\sigma^2(20 \\
&\qquad + vat^2(25+14vat^2))) + 3a(228v^3\mu^4 at^6 + 588v^2\mu^2 a\sigma^2 t^4 + 35\sigma^4(6+7vat^2))))\big) \\
&\qquad \Big/ \big(35at(6m(\theta^2+a\sigma^2) + vt(\theta^2(3+2vat^2) + 4v\mu a\theta t^2 + a(2v\mu^2 t^2 + 3\sigma^2)))^2\big)
\end{aligned}
\tag{4.1.5}
$$

It can be seen that all these moments of X explicitly depend on time t, involving not only the parameters of the business time B but those of the variance gamma process H as well. In fact, it is not difficult to notice that, although the variance gamma parameters a, θ, and σ still have significant influence on the kurtosis, skewness, and variance, such influence displays more uncertainty due to the business time parameters m and v. A superficial comment, nevertheless, is that any moment of X approaches the corresponding one of H as $m \nearrow 1$ and $v \searrow 0$ simultaneously, while it tends towards that of B as $t \to \infty$. In short, X is so far characterized with both time-dependent asymmetric leptokurtic feature and stochastic volatility structure.

4.2 Real-World and Risk-Neutral Evolutions of Stock Price

Now consider a frictionless continuous-trading financial market, i.e., no transaction costs are present. The market information set or filtration is $(\mathscr{F}_t)_{t\geq 0}$ and the real-world probability measure is $\mathbb{P}$. Denote the price process of a risky asset, typically a stock, by $S \equiv (S_t)$. Suppose the initial stock price is $S_0 > 0$, and that under $\mathbb{P}$, S evolves according to the following geometric process.

$$S_t = S_0 e^{X_t} \tag{4.2.1}$$

This is an ordinary exponential of X, which is defined as in (4.1.1). A reason behind this construction is the effect of continuous compounding over continuous time. To this end, notice that $(\ln S_t = \ln S_0 + X_t)$ is nothing but the log price process whose shape is exactly the same as X's. The real-world characteristic function of $\ln S$ is simply

$$\psi_{\ln S|t}(u) := \mathbb{E}\left[e^{\mathrm{i}u\ln S_t}\right] = e^{\mathrm{i}u\ln S_0}\psi_{X|t}(u) \tag{4.2.2}$$

Let r be the risk-free rate and d the stock's dividend yield, both continuously compounded and assumed to be constant. As always, for pricing financial derivatives it is necessary to eliminate the existence of arbitrage by finding a measure $\mathbb{P}^*$ under which the discounted post-dividend stock price process $\left(e^{-(r-d)t}S_t\right)$ becomes a local martingale. Under such a measure the stock should have a mean log return equal to $r-d$. In this case, since the density function of X is not explicitly known, I can simply let the risk-neutral or $\mathbb{P}^*$-evolution of S be given by

$$S_t = \frac{S_0 e^{(r-d)t+X_t}}{\mathbb{E}\left[e^{X_t}\right]} = \frac{S_0 e^{(r-d)t+X_t}}{\psi_{X|t}(-\mathrm{i})} \tag{4.2.3}$$

To give a short proof of the local martingale property, note that the business time B is in itself an adapted process, i.e., (B_t) is $\mathscr{F}_t$-measurable, and by its strictly increasing property that $B_t - B_s > 0$ a.s. $\forall 0 \le s < t$, $(X_t \equiv H_{B_t})$ is also $\mathscr{F}_t$-adapted. On the other hand, in light of the Lévy property of H, for any $0 \le s < t$, one can claim that the increment $X_t - X_s \equiv H_{B_t} - H_{B_s}$ is independent from the information set at time s, $\mathscr{F}_s$. Therefore,

$$\mathbb{E}^*\left[e^{-(r-d)t}S_t|\mathscr{F}_s\right] = S_0\mathbb{E}\left[\frac{e^{X_t}}{\psi_{X|t}(-\mathrm{i})}|\mathscr{F}_s\right] = \frac{S_0 e^{X_s}}{\psi_{X|s}(-\mathrm{i})}\mathbb{E}\left[\frac{e^{X_t-X_s}}{\mathbb{E}\left[e^{X_t-X_s}\right]}\right] = \frac{S_0 e^{X_s}}{\psi_{X|s}(-\mathrm{i})} = e^{-(r-d)s}S_s \tag{4.2.4}$$

where $\mathbb{E}\left[e^{X_t-X_s}\right] \neq \psi_{X|t-s}(-\mathrm{i})$ in general because X has nonstationary increments.

Under $\mathbb{P}^*$, (4.2.3) can be specified as

$$S_t = S_0 e^{(r-d-m\mu)t+X_t}\left(\frac{2a-2\theta-\sigma^2}{2a}\right)^{mat}\sqrt{\cos\sqrt{2vt^2\left(\mu - a\ln\frac{2a-2\theta-\sigma^2}{2a}\right)}} \tag{4.2.5}$$

for which an implicit requirement is that the cosine function return a nonnegative value⟨4⟩, or

$$\frac{2a-2\theta-\sigma^2}{2a} > 0 \quad \text{and} \quad vt^2\left(\mu - a\ln\frac{2a-2\theta-\sigma^2}{2a}\right) \le \frac{\pi^2}{8} \tag{4.2.6}$$

As will be seen in Section 7, this condition is rarely violated in practice. By consulting (4.2.3), the risk-neutral characteristic function of the log price process $\ln S$ can be expressed in terms of the characteristic function of X.

$$\psi^*_{\ln S|t}(u) := \mathbb{E}^*\left[e^{\mathrm{i}u\ln S_t}\right] = e^{\mathrm{i}u((r-d)t+\ln S_0)}\psi_{X|t}(u)(\psi_{X|t}(-\mathrm{i}))^{-\mathrm{i}u} \tag{4.2.7}$$

This will be useful to pricing financial derivatives.

⟨4⟩Notice that $\mu - a\ln((2a-2\theta-\sigma^2)/(2a))$ can take negative values as the cosine or secant function is positive on the imaginary axis.

5 Exotic Power Option Pricing

In this section I will analyze in depth the pricing mechanism of European-style exotic power options when the underlying stock follows a geometric Brownian-time-changed variance gamma process as in (4.2.1) and (4.2.3). Discussion of plain-vanilla options is necessary to conduce the analysis of power options.

5.1 Plain-Vanilla Options

Consider a European plain-vanilla call option with strike price $K > 0$ and time to maturity $T > 0$. Its standard payoff is written as

$$C_T = (S_T - K)^+ \tag{5.1.1}$$

where $(\cdot)^+$ is identical to $\max\{\cdot, 0\}$. Under the risk-neutral measure $\mathbb{P}^*$, the option price today is the expected discounted payoff,

$$\begin{aligned} C_0 \equiv C_0(S_0, K, r, d, T; m, v, \mu, a, \theta, \sigma) &= \mathbb{E}^*[e^{-rT}(S_T - K)^+] \\ &= e^{-rT}\mathbb{E}^*[S_T \mathbf{1}_{\{S_T > K\}}] - Ke^{-rT}\mathbb{E}^*[\mathbf{1}_{\{S_T > K\}}] \end{aligned} \tag{5.1.2}$$

A semicolon has been used to separate the explicit parameters that are observable from the market from the implicit that are only obtainable by means of statistical modeling.

Many works have so far existed to discuss explicit pricing methods for plain-vanilla options when the characteristic function, but not the density, of the underlying stock's log price process is known in the context of risk-neutrality. For instance, using the fact that characteristic functions are even in its real and imaginary parts, Bakshi and Madan (2000) [2] derived a very straightforward formula for evaluating the expectations in (5.1.2). As a consequence,

$$C_0 = S_0 e^{-dT}\Pi_1 - Ke^{-rT}\Pi_2 \tag{5.1.3}$$

where the associated in-the-money probabilities are given by

$$\begin{aligned} \Pi_1 &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left\{\frac{e^{-\mathrm{i}u\ln K}\psi^*_{\ln S|T}(u - \mathrm{i})}{\mathrm{i}u\psi^*_{\ln S|T}(-\mathrm{i})}\right\}\mathrm{d}u \\ \Pi_2 &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left\{\frac{e^{-\mathrm{i}u\ln K}\psi^*_{\ln S|T}(u)}{\mathrm{i}u}\right\}\mathrm{d}u \end{aligned} \tag{5.1.4}$$

$\Re\{\cdot\}$ denoting the real part operator. To briefly explain, Π_1 results from choosing the random variable S_T as a numéraire, because $\mathbb{E}^*\left[e^{-(r-d)T}S_T/S_0\right] = 1$. In general, these integrals can be numerically evaluated with high efficiency, to which classical truncation methods such as Simpson's rule also apply.

For a similar plain-vanilla put option with standard payoff

$$P_T = (K - S_T)^+ \tag{5.1.5}$$

its price is directly implied by the call price through put-call parity. The version including continuous dividend yield can be found in Guo and Su (2006) [12].

$$P_0 = C_0 + Ke^{-rT} - S_0 e^{-dT} \tag{5.1.6}$$

5.2 Asymmetric Power Options

Also called leveraged options, asymmetric power options are designed to grant the option holder a leveraged view on a specific underlying stock or its volatility. Such an option's payoff becomes nonlinear by raising the stock price to a fixed power. A leveraged call option has the following payoff at maturity,

$$C_T^{(\mathrm{ap})} = \left(S_T^p - K\right)^+ \tag{5.2.1}$$

where $p > 0$ is a predetermined power coefficient. Allowing for the magnificent impact of leverage, p can hardly exceed 2 in practice.

According to (4.2.3), the $\mathbb{P}^*$-evolution of the powered stock price $S^p \equiv \left(S_t^p\right)$ is given by

$$S_t^p = \frac{S_0^p e^{p(r-d)t + pX_t}}{(\psi_{X|t}(-\mathrm{i}))^p} \tag{5.2.2}$$

To transform the stochastic part into a well-compensated martingale form, let a function be defined on (p,t) as

$$\begin{aligned}\varphi(p,t) &:= \frac{1}{t}(\ln \psi_{X|t}(-\mathrm{i}p) - p \ln \psi_{X|t}(-\mathrm{i})) \\ &= ma\left(\ln \frac{2a - 2p\theta - p^2\sigma^2}{2a} - p \ln \frac{2a - 2\theta - \sigma^2}{2a}\right) \\ &\quad + \frac{1}{t} \ln \sqrt{\sec \sqrt{2vt^2\left(p\mu - a \ln \frac{2a - 2p\theta - p^2\sigma^2}{2a}\right)}} \\ &\quad + \frac{p}{t} \ln \sqrt{\cos \sqrt{2vt^2\left(\mu - a \ln \frac{2a - 2\theta - \sigma^2}{2a}\right)}}\end{aligned} \tag{5.2.3}$$

which is real finite as long as

$$\frac{2a - 2p\theta - p^2\sigma^2}{2a} > 0 \quad \text{and} \quad vt^2\left(p\mu - a \ln \frac{2a - 2p\theta - p^2\sigma^2}{2a}\right) \le \frac{\pi^2}{8} \tag{5.2.4}$$

in addition to (4.2.6). This function allows (5.2.2) to be conveniently rewritten as

$$S_t^p = \frac{S_0^p e^{(r - d_p(t))t + pX_t}}{\psi_{X|t}(-\mathrm{i}p)} \tag{5.2.5}$$

by constructing a power-dependent dividend yield function $d_p(t)$ as

$$d_p(t) = pd - (p-1)r - \varphi(p,t) \tag{5.2.6}$$

Notice that it is the randomness in B that leads to the time dependence of d_p, because the last two terms of φ would not exist if B happened to be t. As a result of φ, d_p varies with t as long as $p \neq 1$; obviously, $\lim_{p \searrow 0} d_p = r$ and $d_1 = d$, regardless of t. To highlight, d_p is not to be misunderstood as a deterministic process - it does not vary over any trading period, but rather, is fixed once the finite time length is known.

To this end, S^p can be viewed as the price of another stock whose continuous uncertainty is governed by a new time-changed process pX and pays a new dividend d_p. pX understandably has the same structure as X, since the power p does not affect my choice of B, yet resulting in a new drifted variance gamma process pH with parameters $(p\mu, a, p\theta, p\sigma)$, which is merely a direct implication from (3.2.3).

Therefore, conditioned on S^p, the asymmetric power call possesses a payoff structure of plain-vanilla type, and by modifying the parameters in the call price function (5.1.2), I can write

$$C_0^{(\text{ap})} = C_0\big(S_0^p, K, r, d_p(T), T; m, v, p\mu, a, p\theta, p\sigma\big) \tag{5.2.7}$$

and price through characteristic function as before. Furthermore, put-call parity also holds for the price of a similar put with terminal payoff $P_T^{(\text{ap})} = \big(K - S_T^p\big)^+$, i.e.,

$$P_0^{(\text{ap})} = C_0^{(\text{ap})} + Ke^{-rT} - S_0^p e^{-d_p(T)T} \tag{5.2.8}$$

5.3 Symmetric Power Options

A symmetric power option is an exotic option whose payoff at maturity is raised to an agreed-upon power. Another appellation for this class is powered options. Instead of a leverage effect, the power here is aimed at distorting the option payoff, which would in turn affect the option value. Starting from a powered call, the one-time payoff is

$$C_T^{(\text{sp})} = \big((S_T - K)^+\big)^p \tag{5.3.1}$$

for some $p > 0$. In fact, when $0 < p < 1$ the effect is a minus or shrinking, and when $p > 1$ the effect is a plus or magnifying. Most of the time, p takes values no larger than 3.

Because of the distorting effect on both S_T and K, pricing methods for this type are not as easy as for asymmetric power options. In fact, by using binomial expansion for the powered difference, the option payoff can be expressed in terms of

$$C_T^{(\text{sp})} = S_T^p \left(1 - \frac{K}{S_T}\right)^p \mathbf{1}_{\{S_T > K\}} = \sum_{k=0}^{\infty} \binom{p}{k} S_T^p \left(-\frac{K}{S_T}\right)^k \mathbf{1}_{\{S_T > K\}} \tag{5.3.2}$$

which converges for $S_T/K > 1$. Notice that this ensures that the option have a positive intrinsic value at maturity and is exactly the condition for option exercise. The call price hence follows as

$$C_0^{(\mathrm{sp})} = \mathbb{E}^*\left[e^{-rT}C_T^{(\mathrm{sp})}\right] = \sum_{k=0}^{\infty}\left(\begin{array}{c} p \\ k \end{array}\right)(-K)^k\mathbb{E}^*\left[e^{-rT}S_T^{p-k}\mathbf{1}_{\{S_T>K\}}\right] \tag{5.3.3}$$

Analogous to the asymmetric power case, in each of these expectations the stock price is raised to a power $p-k$, $\forall k$. However, it is logical to think of S^{p-k} as the price of yet another stock paying a constant dividend only for $p-k \geq 0$. On the other hand, the moment generating function $\psi_{X|T}(-\mathrm{i}(p-k))$ will understandably fail to exist for some sufficiently large k, by which using this powered stock price as a numéraire with martingale property under the risk-neutral setting becomes problematic. To this end, change of numéraire only applies to a finite number of k with $k \leq \lfloor p \rfloor$, $\lfloor\cdot\rfloor$ denoting the floor function, and, of course, subject to the constraints (4.2.6) and (5.2.4); for large values of k the expectations remain to be evaluated as direct integrals. Following this idea, the call price comes as a piecewise summation⟨5⟩.

$$C_0^{(\mathrm{sp})} = \sum_{k=0}^{\lfloor p \rfloor}\left(\begin{array}{c} p \\ k \end{array}\right)(-K)^k S_0^{p-k}e^{-d_{p-k}(T)T}\tilde{\Pi}_{p-k} + K^p e^{-rT}\sum_{k=\lfloor p \rfloor+1}^{\infty}\left(\begin{array}{c} p \\ k \end{array}\right)(-1)^k\mathbf{I}_{1,p-k} \tag{5.3.4}$$

For the first part, by way of changing numéraire, each in-the-money probability is given by

$$\tilde{\Pi}_{p-k} = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty}\Re\left\{\frac{e^{-\mathrm{i}u\ln K}\psi^*_{\ln S|T}(u-\mathrm{i}(p-k))}{\mathrm{i}u\psi^*_{\ln S|T}(-\mathrm{i}(p-k))}\right\}\mathrm{d}u, \quad 0 \leq k \leq \lfloor p \rfloor \tag{5.3.5}$$

and the second part relies on the supplementary integrals specified as

$$\mathbf{I}_{1,p-k} = \frac{1}{\pi}\int_0^{\infty}\Re\left\{\frac{e^{-\mathrm{i}u\ln K}\psi^*_{\ln S|T}(u)}{\mathrm{i}u-p+k}\right\}\mathrm{d}u, \quad k \geq \lfloor p \rfloor + 1 \tag{5.3.6}$$

Alternatively, it can be further shown that the call price has the following equivalent expression.

$$\begin{aligned} C_0^{(\mathrm{sp})} = &\sum_{k=0}^{\lfloor p \rfloor}\left(\begin{array}{c} p \\ k \end{array}\right)(-K)^k S_0^{p-k}e^{-d_{p-k}(T)T}\tilde{\Pi}_{p-k} \\ &+ K^p e^{-rT}(-1)^{1+\lfloor p \rfloor}\left(\begin{array}{c} p \\ 1+\lfloor p \rfloor \end{array}\right)\frac{1}{\pi}\int_0^{\infty}\Re\left\{\frac{e^{-\mathrm{i}u\ln K}\psi^*_{\ln S|T}(u)\Upsilon(p,\mathrm{i}u)}{1+\mathrm{i}u-p+\lfloor p \rfloor}\right\}\mathrm{d}u \end{aligned} \tag{5.3.7}$$

where the new function $\Upsilon(\cdot,\cdot)$ makes use of the well-known generalized Gauss hypergeometric function in such a way that

$$\Upsilon(p,\mathrm{i}u) = {}_3\mathbf{F}_2\left(\begin{array}{c} 1 \\ 1-p+\lfloor p \rfloor \\ 1+\mathrm{i}u-p+\lfloor p \rfloor \end{array};\begin{array}{c} 2+\lfloor p \rfloor \\ 2+\mathrm{i}u-p+\lfloor p \rfloor \end{array};1\right) \tag{5.3.8}$$

⟨5⟩Despite an infinite series, my personal experience suggests that an upper bound of 100 will typically do fine, by which computational efficiency is guaranteed. Also, the binomial coefficient $\left(\begin{array}{c} p \\ k \end{array}\right) := \prod_{j=1}^{k}(p-j+1)/k!$ is generalized for arbitrary $p > 0$ and $k \in \mathbb{N}$.

The series representation and some crucial properties of ${}_3\mathbf{F}_2$ can be found in Abramowitz and Stegun (1972) [1]. All detailed derivations of these pricing formulae are shown in Appendix B. Notably, the purpose of (5.3.7) with (5.3.8) is to enhance computational accuracy for packages, such as Mathematica® by Wolfram Research, Inc. (2015) [18], in which the hypergeometric function family is well established.

To emphasize, given $k \leq p$, d_{p-k} represents the constant dividend yield of the stock as powered by $p-k$; in the special case where $p \in \mathbb{N}_{++}$, whenever $p = k$ the option position becomes perfectly hedged or risk-free, as $d_0 \equiv r$.

For a similar symmetric power put, put-call parity does not work for all $p \neq 1$ as the standard payoff structure is distorted. Likewise, binomially expanding the option payoff as

$$P_T^{(\mathrm{sp})} = \left((K-S_T)^+\right)^p = K^p \sum_{k=0}^{\infty} \begin{pmatrix} p \\ k \end{pmatrix} \left(-\frac{S_T}{K}\right)^k \mathbf{1}_{\{S_T<K\}} \tag{5.3.9}$$

which is convergent for $S_T/K < 1$, implies the following expression for its price.

$$P_0^{(\mathrm{sp})} = \mathbb{E}^*\left[e^{-rT} P_T^{(\mathrm{sp})}\right] = K^p e^{-rT} \left(\Pi_2^{\complement} + \sum_{k=1}^{\infty} \begin{pmatrix} p \\ k \end{pmatrix} (-1)^k \mathbf{I}_{2,k}\right) \tag{5.3.10}$$

where $\Pi_2^{\complement} = 1 - \Pi_2$ as in (5.1.4) and each supplementary integral is given by

$$\mathbf{I}_{2,k} = \frac{1}{\pi} \int_0^{\infty} \Re\left\{\frac{e^{-\mathrm{i}u\ln K}\psi^*_{\ln S|T}(u)}{k-\mathrm{i}u}\right\} \mathrm{d}u, \quad k \geq 1 \tag{5.3.11}$$

Indeed, the symmetric power put pricing formulae are simpler compared to those for the symmetric power call. No change of numéraire is necessary because the stock price S now is powered by $k \in \mathbb{N}$ and the integral $\mathbf{I}_{2,k}$ holds true for $k > 0$, while for Π_2 to exist recall that only (4.2.6) needs to be in force.

Similarly, (5.3.10) with (5.3.11) can be alternatively simplified into

$$P_0^{(\mathrm{sp})} = K^p e^{-rT} \left(\frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \Re\left\{e^{-\mathrm{i}u\ln K}\psi^*_{\ln S|T}(u)\mathbf{B}(1+p,-\mathrm{i}u)\right\} \mathrm{d}u\right) \tag{5.3.12}$$

provided that (4.2.6) is well met, thanks to Euler's Beta function $\mathbf{B}(\cdot,\cdot)$⟨6⟩. This expression significantly facilitates numerical computation by transforming the infinite series into a simple function. Again, see Appendix B for detailed proof.

It is not difficult to check that, other things equal, $C_0^{(\mathrm{sp})} \equiv C_0^{(\mathrm{ap})} \equiv C_0$ and $P_0^{(\mathrm{sp})} \equiv P_0^{(\mathrm{ap})} \equiv P_0$ when and only when $p = 1$, and so "symmetry" is not a special case of "asymmetry" in describing power options.

⟨6⟩ Defined as $\mathbf{B}(x,y) := \Gamma(x)\Gamma(y)/\Gamma(x+y)$, this function is easily evaluated with most standard packages and a built-in function in Mathematica®.

6 Monte-Carlo Simulation

As mentioned in the very beginning, one of the important reasons behind combining a Brownian time change with a variance gamma process is that the resulting processes are easy to simulate, because of the relative simplicity of underlying distributions. This section thus expatiates on the simulation techniques of each of the processes analyzed above as well as discusses the associated pricing logic as a comparison to characteristic function pricing in Section 5. The simulation is primarily realized through time discretization in the absence of path dependence.

Let us start from the business time construction. By way of discretization, I construct $N+1$ discrete time points $\{0\}\cup\{n\Delta\}_{n=1,2,\dots,N}$ for a certain time interval $[0,T]$, where $\Delta = T/N$ is the quadrature magnitude. Based on the Lévy property of the standard Brownian motion, W can be approximated by $\hat{W}\equiv\left(\hat{W}_{n\Delta}\right)$ with the following recursion,

$$\hat{W}_0 = 0 \rightsquigarrow \hat{W}_n = \hat{W}_{(n-1)\Delta} + \omega_n \tag{6.1}$$

where $(\omega_n) \sim$ i.i.d.Normal$(0,\Delta)$. Notice that from its definition (2.1), B is at bottom a Riemann integral of W and starts from 0. Thus, denoting an estimator by $\hat{B}\equiv\left(\hat{B}_{n\Delta}\right)$, simple quadrature rule can be applied to obtain

$$\hat{B}_0 = 0 \rightsquigarrow \hat{B}_{n\Delta} = \hat{B}_{(n-1)\Delta} + m\Delta + v\Delta\hat{W}^2_{n\Delta} \tag{6.2}$$

The above relations directly imply that, at a given $T = N\Delta$, the estimator $\hat{B}_{N\Delta}$ is asymptotically unbiased towards B_T in that

$$\begin{aligned}\mathbb{E}\left[\hat{B}_{N\Delta}\right] &= \mathbb{E}\left[mN\Delta + v\sum_{n=1}^{N}\Delta\hat{W}^2_{n\Delta}\right] = mN\Delta + v\Delta\sum_{n=1}^{N}\sum_{k=1}^{n}\mathbb{E}\left[\omega_k^2\right] \\ &= mN\Delta + \frac{v\Delta^2 N(N+1)}{2} = mT + \frac{vT^2}{2} + \frac{vT^2}{2N}\end{aligned} \tag{6.3}$$

which tends to $\mathbb{E}[B_T] = mT + vT^2/2$ as $N\to\infty$. Also, the mean squared error (MSE) of $\hat{B}_{N\Delta}$ is calculated as the sum of its squared bias and variance.

$$\begin{aligned}\mathrm{MSE}\left[\hat{B}_{N\Delta}\right] &\equiv \left(\mathbb{E}\left[\hat{B}_{N\Delta} - B_T\right]\right)^2 + \mathrm{Var}\left[\hat{B}_{N\Delta}\right] \\ &= \left(\frac{v\Delta^2 N}{2}\right)^2 + v^2\Delta^2\mathrm{Var}\left[\sum_{n=1}^{N}\hat{W}^2_{n\Delta}\right] \\ &= \frac{v^2\Delta^4 N^2}{4} + v^2\Delta^2\left(\sum_{n_1,n_2=1}^{N}((n_1\vee n_2)\Delta + (n_1\wedge n_2)\Delta)(n_1\wedge n_2)\Delta\right) \\ &= \frac{v^2\Delta^4 N(5N^3+10N^2+10N+2)}{12}\end{aligned} \tag{6.4}$$

where the second equality follows from the independent and stationary increment property of $\hat{W}$ along with W. This result is of order $\mathrm{O}(N^4\Delta^4)\equiv\mathrm{O}(T^4)$, which is the same as that of $\mathrm{Var}[B_T] =$

$v^2T^4/3$, whereas it asymptotically deviates from the true variance by $5v^2N^4\Delta^4/12 - v^2T^4/3 = v^2T^4/12$, as $N \to \infty$.

In an attempt to simulate the time-changed variance gamma process X, the calendar time frame needs to be modified by $\{0\} \cup \{\hat{B}_{n\Delta}\}_{n=1,2,\ldots,N}$, which gives rise to stochastic volatility. On this basis, let $\hat{X} \equiv (X_{n\Delta})$ be an approximation of X, and then it can be generated in three ways equivalently, according to Subsection 3.2.

Firstly, the Lévy property of variance gamma process implies the following recursive relation,

$$\hat{X}_0 = 0 \rightsquigarrow \hat{X}_{n\Delta} = \hat{X}_{(n-1)\Delta} + \mu\left(\hat{B}_{n\Delta} - \hat{B}_{(n-1)\Delta}\right) + \eta_n \tag{6.5}$$

where $(\eta_n) \sim \mathrm{VarGamma}\Big(a\big(\hat{B}_{n\Delta} - \hat{B}_{(n-1)\Delta}\big), \theta\big(\hat{B}_{n\Delta} - \hat{B}_{(n-1)\Delta}\big), \sigma\sqrt{\hat{B}_{n\Delta} - \hat{B}_{(n-1)\Delta}}\Big)$ are independent random variables. This approach is yet time-consuming because it is difficult to sample from a variance gamma distribution due to its density's complexity; again, see Madan *et al* (1998).

Secondly, by the formal definition (3.2.3), it follows that

$$\hat{X}_0 = 0 \rightsquigarrow \hat{X}_{n\Delta} = \hat{X}_{(n-1)\Delta} + \mu\left(\hat{B}_{n\Delta} - \hat{B}_{(n-1)\Delta}\right) + \theta\gamma_n + \sigma\sqrt{\gamma_n}\,\omega'_n \tag{6.6}$$

where $(\gamma_n) \sim \mathrm{Gamma}\big(a\big(\hat{B}_{n\Delta} - \hat{B}_{(n-1)\Delta}\big), a\big)$ is a sequence of independent variables⟨7⟩ and $(\omega'_n) \sim$ i.i.d.Normal$(0,1)$. Just as in theory, (γ_n) and (ω'_n), $\forall n$, are independent from each other.

For the third approach, which adopts a decomposition in terms of gamma processes, I can write

$$\hat{X}_0 = 0 \rightsquigarrow \hat{X}_{n\Delta} = \hat{X}_{(n-1)\Delta} + \mu\left(\hat{B}_{n\Delta} - \hat{B}_{(n-1)\Delta}\right) + \gamma_n^{(1)} - \gamma_n^{(2)} \tag{6.7}$$

where $\big(\gamma_n^{(i)}\big) \sim \mathrm{Gamma}\big(a\big(\hat{B}_{n\Delta} - \hat{B}_{(n-1)\Delta}\big), 1/\ell_\pm\big)$, $i \in \{1,2\}$, are taken to be two mutually independent sequences of independent random variables, with $\ell_\pm$ given in (3.2.7). Because of the simplicity of gamma and normal distributions relative to a variance gamma, the last two approaches are much more preferable. Obviously, as $m \nearrow 1$ and $v \searrow 0$, $\hat{B}_{n\Delta} \to n\Delta$, $\forall n$, and all of (6.5) to (6.7) will yield estimators of the original variance gamma process H.

Also, $\hat{X}_{N\Delta}$ is asymptotically unbiased towards X_T since, by (6.6) and the tower property,

$$\mathbb{E}\big[\hat{X}_{N\Delta}\big] = (\mu+\theta)\mathbb{E}\big[\hat{B}_{N\Delta}\big] = (\mu+\theta)\left(mT + \frac{vT^2}{2} + \frac{vT^2}{2N}\right) \tag{6.8}$$

which tends towards $\mathbb{E}[X_T] = (\mu+\theta)(mT + vT^2/2)$ as $N \to \infty$. Clearly, the existence of bias is only a result of the randomness in B. On the other hand, the estimator's mean squared error is obtained by consulting the law of total variance.

$$\begin{aligned}
\mathrm{MSE}\big[\hat{X}_{N\Delta}\big] &= \left((\mu+\theta)\frac{v\Delta^2 N}{2}\right)^2 + \mathbb{E}\big[\mathrm{Var}\big[\hat{X}_{N\Delta}|\hat{B}_{N\Delta}\big]\big] + \mathrm{Var}\big[\mathbb{E}\big[\hat{X}_{N\Delta}|\hat{B}_{N\Delta}\big]\big] \\
&= (\mu+\theta)^2\frac{v^2\Delta^4N^2}{4} + \left(\frac{\theta}{a} + \sigma\right)\mathbb{E}\big[\hat{B}_{N\Delta}\big] + (\mu+\theta)^2\mathrm{Var}\big[\hat{B}_{N\Delta}\big]
\end{aligned}$$

⟨7⟩Generating independent gamma random variables is straightforward for most standard packages, though the gamma generator using standard uniform distributions proposed by Johnk (1964) [13] is yet another acceptable approach provided that $a\Delta \leq 1$.

$$= \frac{1}{12}N\Delta\Bigg(3v^2(\mu+\theta)^2N\Delta^3 + v^2(\mu+\theta)^2\Delta^3(5N^3+10N^2+7N+2) \\ + \frac{6(2m+v\Delta(1+N))(\theta+a\sigma)}{a}\Bigg) \tag{6.9}$$

which still has the same order as $\mathrm{Var}[X_T]$, or $\mathrm{O}(T^4)$.

Based on $\hat{X}$, under the real-world probability measure $\mathbb{P}$, an estimator of the log stock price at a given maturity date T, $\ln S_T$, can be created by

$$\ln \hat{S}_{N\Delta} = \ln S_0 + \hat{X}_{N\Delta} \tag{6.10}$$

for $N\Delta = T$, which is immediately asymptotically unbiased and has the same mean squared error as $\hat{X}_{N\Delta}$'s. Regarding computational effort, given $T > 0$, generating one trajectory of $\hat{W}$ requires N operations, and so simulating one sample path of B necessitates $N + \sum_{n=1}^{N} n = N(N+3)/2 \sim \mathrm{O}(N^2)$ operations. Obtaining an estimate $\hat{X}_{N\Delta}$ of X_T therefore requires computational effort of order $\mathrm{O}(N^2)$ based on the increments of $\hat{B}$. For practical purposes, usually $M \gg 1$ paths are simulated, in which case an aggregate computational effort of $\mathrm{O}(MN^2)$ is needed.

Once M estimators of $\ln S_T$, denoted by $\ln \hat{S}^{(j)}_{N\Delta}$, $j = 1,2,\dots,M$, are obtained, the stock price estimators can be put into the risk-neutral setting via a simple multiplier. I.e., under $\mathbb{P}^*$, the compensated estimators are given by

$$\hat{S}^{(j)}_{N\Delta} = \frac{S_0 e^{(r-d)T + \hat{X}^{(j)}_{N\Delta}}}{\frac{1}{M}\sum_{j=1}^{M} e^{\hat{X}^{(j)}_{N\Delta}}}, \quad j = 1,2,\dots,M \tag{6.11}$$

as an approximation of (4.2.3). With this, the payoff of a particular type of option can be correspondingly estimated by properly discounting the average resulting payoff from each simulated path under $\mathbb{P}^*$. To be precise,

$$\hat{C}_0 = \frac{e^{-rT}}{M}\sum_{j=1}^{M}\big(\hat{S}^{(j)}_{N\Delta} - K\big)\mathbf{1}_{\{\hat{S}^{(j)}_{N\Delta} > K\}}, \quad \hat{P}_0 = \frac{e^{-rT}}{M}\sum_{j=1}^{M}\big(K - \hat{S}^{(j)}_{N\Delta}\big)\mathbf{1}_{\{\hat{S}^{(j)}_{N\Delta} < K\}} \tag{6.12}$$

and, similarly, for exotic power options with power $p > 0$,

$$\hat{C}^{(\mathrm{ap})}_0 = \frac{e^{-rT}}{M}\sum_{j=1}^{M}\big(\hat{S}^{(j)p}_{N\Delta} - K\big)\mathbf{1}_{\{\hat{S}^{(j)p}_{N\Delta} > K\}}, \quad \hat{P}^{(\mathrm{ap})}_0 = \frac{e^{-rT}}{M}\sum_{j=1}^{M}\big(K - \hat{S}^{(j)p}_{N\Delta}\big)\mathbf{1}_{\{\hat{S}^{(j)p}_{N\Delta} < K\}} \\ \hat{C}^{(\mathrm{sp})}_0 = \frac{e^{-rT}}{M}\sum_{j=1}^{M}\big(\hat{S}^{(j)}_{N\Delta} - K\big)^p\mathbf{1}_{\{\hat{S}^{(j)}_{N\Delta} > K\}}, \quad \hat{P}^{(\mathrm{sp})}_0 = \frac{e^{-rT}}{M}\sum_{j=1}^{M}\big(K - \hat{S}^{(j)}_{N\Delta}\big)^p\mathbf{1}_{\{\hat{S}^{(j)}_{N\Delta} < K\}} \tag{6.13}$$

These approximations are to be used as an alternative to the analytical pricing results using characteristic functions. Advantageously, they can provide some insights into the evolution of each process visually.

To give a graphical illustration, suppose $m = 0.76$, $v = 0.4$, $\mu = -0.17$, $a = 66.45$, $\theta = 0.28$, $\sigma = 0.36$, and $T = 1$ with $N = 1000$. Figure 1 displays a simulated sample path for each of the standard Brownian motion W, business time B, and time-changed variance gamma process $X \equiv \ln S - \ln S_0$.

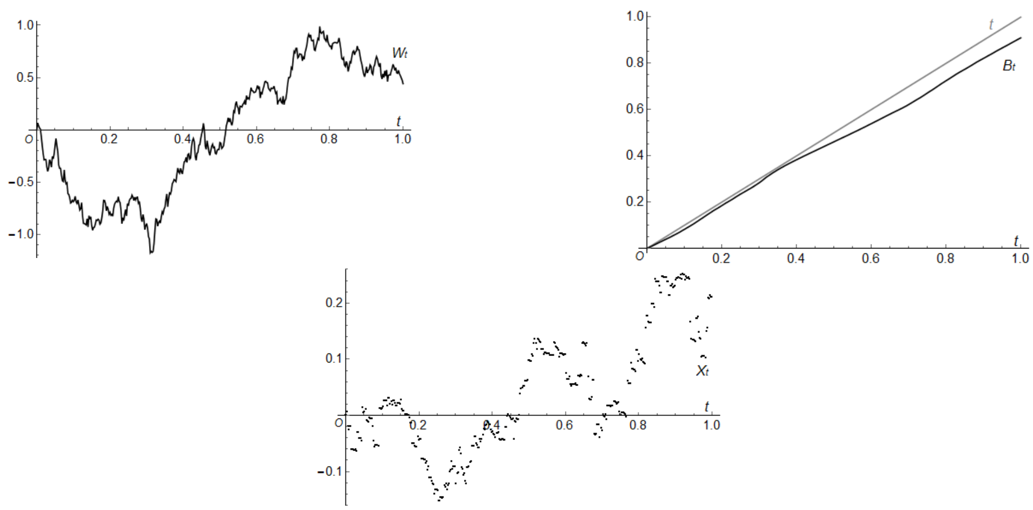


Figure 1: Simulated sample paths of (W_t), (B_t), and (X_t)

Apparently, B is smoothly increasing beside positivity due to the almost-sure continuity of W, while X is observationally a purely discontinuous process.

7 Empirical Analysis

In this section, the Brownian-time-changed variance gamma model will be applied to real financial time series and market prices of standard options. Theoretical prices will be compared to the true prices to indicate the model's overall fitting degree. Further comparison will be made with the case under calendar time or without drift and the Black-Scholes model with normality. For exotic power options, numerical examples will be given solely based on the time-changed model, on purpose of providing some insights into the validity of business-time pricing when the power takes different values.

7.1 Data Modeling

I choose the daily closing prices of Standard & Poor 500 Index (SP500) over the recent year from July 2, 2015 to July 1, 2016 (collected from *Yahoo Finance*) as the study object, with a total number of 253 observations, denoted by $\check{S}_i$, for $i = 1, 2, \ldots, 253$, in proper order. To obtain a stationary series, the daily log returns are accordingly calculated as

$$\check{R}_i = \ln \check{S}_i - \ln \check{S}_{i-1}, \quad i = 2, 3, \ldots, 253 \tag{7.1.1}$$

which consist of 252 observations. Figures 2 and 3 below present the respective series of the closing prices ($) and log returns.

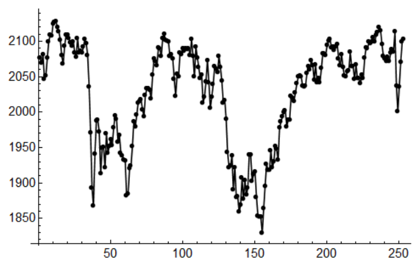


Figure 2: Series of $\left(\check{S}_i\right)$

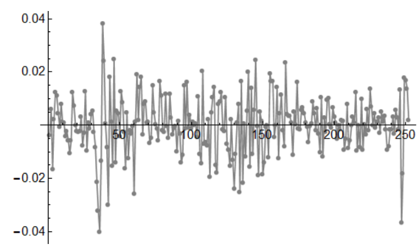


Figure 3: Series of $\left(\check{R}_i\right)$

Sample statistics of the log returns include insignificant mean of 4.96924×10^{-5}, standard deviation of 0.010847, skewness of -0.30999 and kurtosis of 4.39189. This indicates a leptokurtic feature with left skewness. Also, trends of clustering can be easily noticed in Figure 3, which signify that stochastic volatility does exist in the returns of SP500. These phenomena can already be well captured by the time-changed variance gamma process.

In the absence of a density function, performing maximum likelihood estimation becomes unrealistic, and so for convenience purposes the model parameters are estimated under the method of moments, which is covered in Bowman and Shenton (1998) [6]. Under this scheme, parameter estimation is realized by minimizing the level to which the model moments deviate from the data moments, and the number of sample raw moments should reasonably match the number of parameters for consistency. In this connection, the first six raw moments of the log returns are found in Table 1.

$\overline{\check{R}_i}$	$\overline{\check{R}_i^2}$	$\overline{\check{R}_i^3}$	$\overline{\check{R}_i^4}$	$\overline{\check{R}_i^5}$	$\overline{\check{R}_i^6}$
4.96924×10^{-5}	1.172×10^{-4}	-3.7583×10^{-7}	6.02476×10^{-8}	-5.71495×10^{-10}	5.72786×10^{-11}

Table 1: Sample raw moments of $\left(\check{R}_i\right)$

As mentioned before, four models will be tested for comparison, including the Black-Scholes geometric Brownian motion model, whose characteristic function is well known in explicit form.

For each one the estimation scheme involves solving the following least square problem,

$$\min_{\mathscr{P}} \left\{ \sum_{n=1}^{|\mathscr{P}|} \left(\overline{\check{R}_i^n} - (-\mathrm{i})^n \frac{\mathrm{d}^n \psi_{X|\frac{1}{252}}(u)}{\mathrm{d}u^n} \bigg|_{u\to 0} \right)^2 \right\} \tag{7.1.2}$$

where $|\mathscr{P}|$ stands for the cardinality or number of parameters in the constrained parameter set $\mathscr{P}$. In the business-time variance gamma model case, for instance, $\mathscr{P} = \{m \geq 0, v > 0, \mu, a > 0, \theta, \sigma > 0\}$ and $|\mathscr{P}| = 6$. Also notice that there are 252 trading days in a calendar year. The optimal parameter set, $\mathscr{P}^\star$, contains all the parameters needed for option pricing. By using Mathematica®, Table 2 below summarizes the parameter estimation under each model.

Black-Scholes	non-drift variance gamma	variance gamma	Brownian-time-changed variance gamma
-	-	-	$m^\star = 0.452847$
-	-	-	$v^\star = 0.299871$
$\mu^\star = 0.0272894$	-	$\mu^\star = 2.64113$	$\mu^\star = 0.738514$
-	$a^\star = 633.306$	$a^\star = 630.536$	$a^\star = 631.116$
-	$\theta^\star = 0.0125224$	$\theta^\star = -2.6286$	$\theta^\star = -0.710898$
$\sigma^\star = 0.171854$	$\sigma^\star = 0.171853$	$\sigma^\star = 0.136282$	$\sigma^\star = 0.253637$
[0.0156001]	[0.0780005]	[0.140401]	[3.12002]

Table 2: Summary of parameter estimation

Enclosed in square brackets are the associated CPU time in seconds⟨8⟩. Notably, the two location parameters μ and θ have a theoretical offsetting effect which explains the significant difference among estimates under different models. In particular, the drift estimate $\mu^\star$ is useless for Black-Scholes pricing as it is entirely replaced by the risk-free rate r under risk-neutrality.

Under the business-time parameter estimates, the daily log returns' implied mean, standard deviation, skewness and kurtosis are 4.96922×10^{-5}, 0.0108258, -0.310869 and 5.70631, respectively, according to the formulae in (4.1.4) and (4.1.5). It can be seen that the parameters X well describe the asymmetric leptokurtic feature, though they have slightly overestimated the kurtosis. On the contrary, under calendar time, the variance gamma model fails to explain the asymmetry by giving a skewness estimate of 0.00547943 without drift, while with drift it makes an overestimation by giving -1.0124. Needless to say, the normal model is always symmetric and mesokurtic.

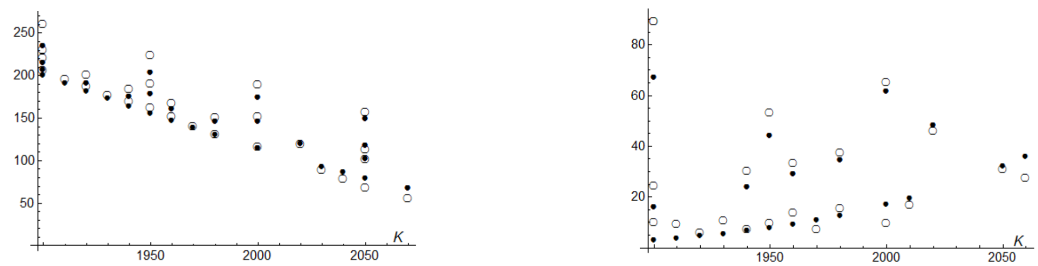
7.2 Standard Option Prices

Prices of standard (European-style plain-vanilla) options are collected from *Market Watch*, quoted as of July 1, 2016. Mid-prices are calculated by averaging the bid and ask prices in pairs. In general, I select options with strikes ranging from \$1950 to \$2070 expiring in August 2016, October 2016, December 2016, and Jun 2017, with respective maturities of $35/252$, $80/252$, 0.5, and 1

⟨8⟩The minimization program is run by a personal computer with an Intel Core i5-4300U and 4GB RAM.

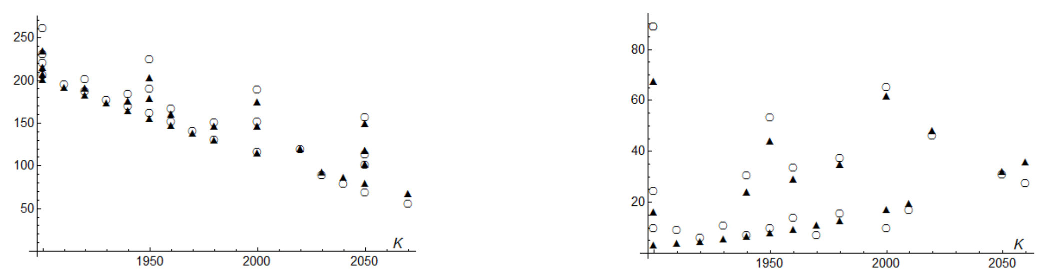
year. On July 1, 2015, SP500 closed at \$2102.95. This generates a total of 50 prices for actively trading option contracts, including 29 calls and 21 puts. On that day, the one-year risk-free rate was at 0.45% and the SP500 dividend yield was at 2.09% per annum. Thus, I assume $r = 0.0045$ and $d = 0.0209$ on the annual basis.

By plugging these values into the pricing formulae, Figures 4 through 7 plot the model prices versus the market mid-prices (\$). Empty circles are used for true prices while solid marks stand for model prices.



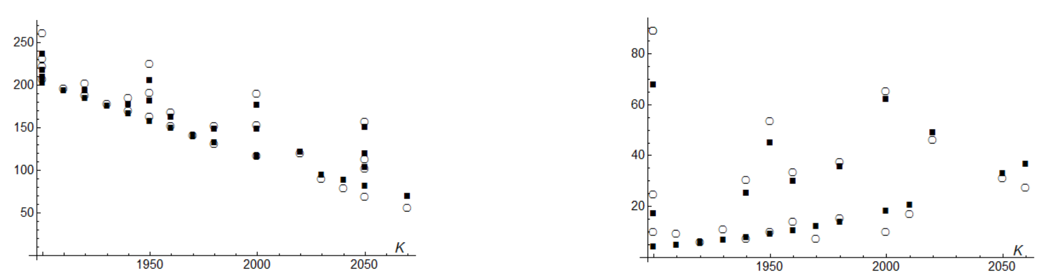
Calls Puts

Figure 4: Black-Scholes model prices vs market prices



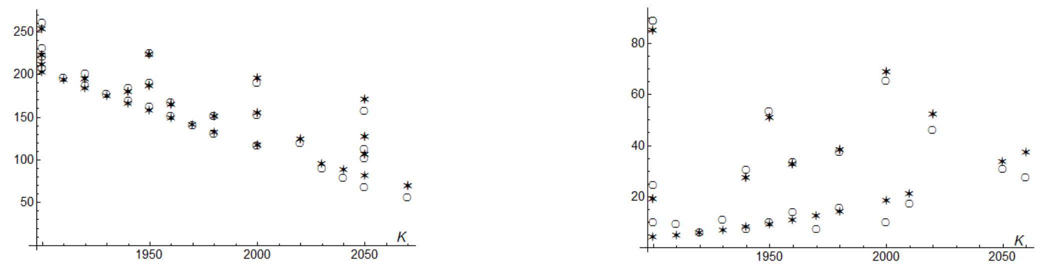
Calls Puts

Figure 5: Non-drift variance gamma model prices vs market prices



Calls Puts

Figure 6: Variance gamma model prices vs market prices



Calls Puts

Figure 7: Brownian-time-changed variance gamma model prices vs market prices

In each plot, there are observably four strings of option prices, The uppermost string contains the prices with the longest expiry, while the lowest represents the shortest expiry. Due to certain liquidity issues, the prices may fluctuate at some low level with respect to increasing strikes.

In general, deviations prevail because historical data are taken into account in place of calibration, and recent price trends are only able to partially reflect market expectations. Despite this, the four models uniformly fit better for short-expiry options than for long-expiry ones. In comparison, the variance gamma models visually improve from the Black-Scholes for short-term options by introducing large jumps, while the business time structure appears to increase accuracy for long-term options with stochastic volatility.

To provide a more rigorous comparison, I calculate the average relative percentage error for each model. Denote by $\check{C}_{i,0}$ and $\check{P}_{j,0}$ the market prices of the standard call and put options, for $i = 1, 2, \ldots, 29$ and $j = 1, 2, \ldots, 21$, as mentioned before. These errors are calculated separately for calls and puts as below.

$$\mathrm{E}_C = \frac{1}{29} \sum_{i=1}^{29} \left| \frac{C_{i,0}}{\check{C}_{i,0}} - 1 \right| \quad \text{and} \quad \mathrm{E}_P = \frac{1}{21} \sum_{j=1}^{21} \left| \frac{P_{j,0}}{\check{P}_{j,0}} - 1 \right| \tag{7.2}$$

The following table displays the respective errors in percentages under the four models.

	Black-Scholes	non-drift variance gamma	variance gamma	Brownian-time-changed variance gamma
E_C	1.78667%	1.78295%	1.74167%	1.58334%
E_P	6.72582%	6.70562%	6.30003%	5.84279%

Table 3: Average relative percentage errors

It is not surprising that these errors decline with the increase of the number of parameters. After all, an increased adaptability to the financial data's distribution pattern basically results in a better fit for market expectations in a relative sense. The purpose of comparing different modeling results mainly lies in illustrating the necessity and validity of short-term large jumps as well as long-term stochastic volatility.

7.3 Power Option Prices

Since exotic power option contracts are typically traded over-the-counter, no market prices are accessible. For this reason, I use solely the business-time variance gamma model and stick to the parameter estimation $\mathscr{P}^{\star} \equiv \{m^{\star}, v^{\star}, \mu^{\star}, a^{\star}, \theta^{\star}, \sigma^{\star}\}$ in the previous subsection while fixing the strike price and expiry at $K = \$2050$ and $T = 1$, respectively, aimed at better explaining the power impact. Also, $S_0 = \$2102.95$, $r = 0.0045$ and $d = 0.0209$ unchanged.

Notice that for asymmetric power options, strike prices necessarily need to be adjusted to the same order of magnitude as the powered stock price. Here I simply raise the original strike $K = \$2050$ to the power p to achieve this effect. For asymmetric power options there is no need to change K.

In a piecewise manner, Figures 8 and 9 respectively plot the sensitivity to power of asymmetric and symmetric power option prices (\$).

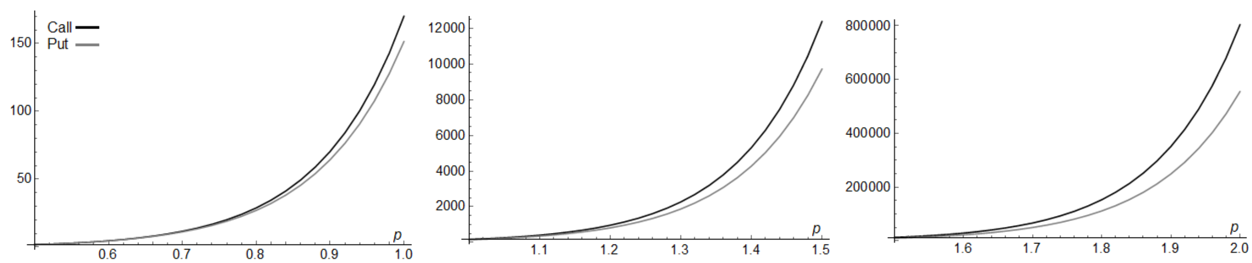


Figure 8: Asymmetric power option price sensitivity to power

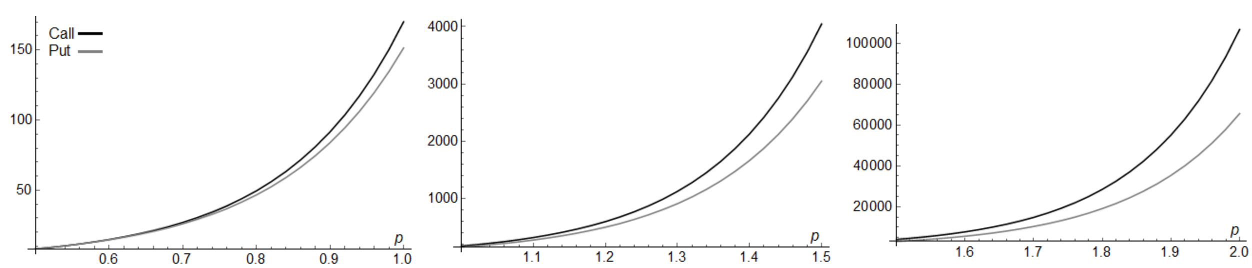


Figure 9: Symmetric power option price sensitivity to power

To accelerate computation, I have adopted the infinite series method (5.3.4) for plotting symmetric power call prices and the beta function method (5.3.12) for symmetric power put prices. Thanks to put-call parity, it typically takes 0.0156001 second to compute a pair of put-call prices for an asymmetric power option using Mathematica®, regardless of p. On the other hand, an increase in p can decelerate computation for pricing symmetric power options under the infinite series methods, but runtime is acceptably around 0.4 second for a single call or put; under the Gauss hypergeometric and beta functions methods which eliminate systematic errors, however, computing a symmetric call price requires 5 to 6 seconds while computing a similar put price only needs 0.0156001 second. In particular, for $p = 1.5$, computation gives an asymmetric call price of \$12379.7 and an asymmetric put price of \$9712.79; using (5.3.4) and (5.3.10), the symmetric power call and put prices are obtained to be \$4046.57 and \$3049.42, while using (5.3.7) and (5.3.12) the prices are \$4046.89 and \$3049.74, from which it is seen that the series methods inevitably produce small errors.

Based on the figures, the impact of power on the option prices is significantly large. All the sensitivity curves slope upward and grow exponentially. Compared to symmetric power options, the prices of asymmetric power options appear to be more sensitive to p. A simple explanation is that, by assuming power-adjusted asymmetric power option strikes, a power difference exceeds a powered difference if and only if the power gets larger than 1. When $p = 1$ in particular, the options become plain-vanilla and their prices coincide.

Moreover, it is of interest to study the pricing via Monte-Carlo simulation. Continuing with the same parameter set, I choose a quadrature size $\Delta = 1/400$ and three groups of simulation sample sizes - $M = 100, 500, 1000$. Table 4 displays the simulation-based pricing results (\$) for $p = 0.5, 1, 1.5, 2$ expressly, using (6.13). These results are compared with the formula-based numerical results and their respective absolute relative percentage errors are also calculated using $\left|\hat{C}_0^{(\mathrm{ap}),(\mathrm{sp})}/C_0^{(\mathrm{ap}),(\mathrm{sp})} - 1\right|$ and $\left|\hat{P}_0^{(\mathrm{ap}),(\mathrm{sp})}/P_0^{(\mathrm{ap}),(\mathrm{sp})} - 1\right|$.

Asymmetric power options								
	Formula-based		Simulation-based					
			$M = 100$		$M = 500$		$M = 1000$	
p	$C_0^{(\mathrm{ap})}$	$P_0^{(\mathrm{ap})}$	$\hat{C}_0^{(\mathrm{ap})}$	$\hat{P}_0^{(\mathrm{ap})}$	$\hat{C}_0^{(\mathrm{ap})}$	$P_0^{(\mathrm{ap})}$	$\hat{C}_0^{(\mathrm{ap})}$	$\hat{P}_0^{(\mathrm{ap})}$
0.5	1.75785	1.77448	1.67228 (4.87%)	1.67585 (5.56%)	1.78726 (1.67%)	1.83633 (3.49%)	1.76147 (0.21%)	1.7783 (0.22%)
1	170.059	151.4	162.291 (4.57%)	143.633 (5.13%)	174.305 (2.50%)	155.646 (2.80%)	170.198 (0.08%)	151.54 (0.09%)
1.5	12379.7	9712.79	11864.6 (4.16%)	9252.38 (4.74%)	12811.8 (3.49%)	9930.49 (2.24%)	12369.5 (0.08%)	9712.58 (0.00%)
2	803940	555183	774764 (3.63%)	530840 (4.38%)	841634 (4.69%)	565046 (1.78%)	801534 (0.30%)	554778 (0.07%)
Symmetric power options								
	Formula-based		Simulation-based					
			$M = 100$		$M = 500$		$M = 1000$	
p	$C_0^{(\mathrm{sp})}$	$P_0^{(\mathrm{sp})}$	$\hat{C}_0^{(\mathrm{sp})}$	$\hat{P}_0^{(\mathrm{sp})}$	$\hat{C}_0^{(\mathrm{sp})}$	$P_0^{(\mathrm{sp})}$	$\hat{C}_0^{(\mathrm{sp})}$	$\hat{P}_0^{(\mathrm{sp})}$
0.5	8.15967	8.2358	7.8862 (3.35%)	8.01634 (2.66%)	8.07665 (1.02%)	8.32861 (1.13%)	8.18723 (0.34%)	8.21919 (0.20%)
1	170.059	151.4	162.291 (4.57%)	143.633 (5.13%)	174.305 (2.50%)	155.646 (2.80%)	170.198 (0.08%)	151.54 (0.09%)
1.5	4046.89	3049.74	3936.84 (2.72%)	2802.01 (8.12%)	4408.74 (8.94%)	3241.32 (6.28%)	4014.98 (0.79%)	3066.75 (0.56%)
2	106698	65557	109369 (2.50%)	58053.9 (11.45%)	126984 (19.01%)	73104.1 (11.51%)	103721 (2.79%)	66534.9 (1.49%)

Table 4: Pricing results via simulation and error analysis

The absolute relative percentage errors (rounded to a basis point) are given in the little parentheses below the prices. Using the same program, simulating 100 sample paths takes about 4 seconds to complete, and so simulating 1000 paths results in a large computational effort of 40 seconds approximately, which exactly grows in an arithmetic manner. It is clearly seen that the relative errors generally decline with increased simulation sample sizes, which in some sense fits into the asymptotical unbiasedness of the estimators of the log price process. Nevertheless, due to its superior computational efficiency, pricing through characteristic functions is highly preferred over simulation.

8 Concluding Remarks

In this paper a Brownian time change is constructed to randomize time structure and thus model stochastic volatility in finance. Composed of a nonnegative drift and a quadratic Brownian integral, this time change has only two parameters which reciprocally control the randomness of volatility. The business time is associated with a drifted variance gamma process for financial modeling, and the resulting process is able to incorporate jumps, has an asymmetric leptokurtic feature, and flexibly describe volatility clustering, as well as is tractable with characteristic functions and very easy to simulate. After establishing the stock price dynamics under the real-world and risk-neutral measures, the time-changed model is used for option pricing. By using its uncomplicated characteristic function, pricing for plain-vanilla options is considerably efficient. An asymmetric power option can be regarded as a plain-vanilla option on a new powered price stock and so follows the same pricing mechanism. Also, I find that symmetric power options can be priced in two approaches, one with infinite series expansion and the other with some advanced functionals, the latter eliminating certain systematic errors. Compared to asymmetric power options, the pricing of symmetric power options takes significantly more time.

In discussing the Monte-Carlo simulation of this time-changed process, a general time discretization is used. In proper order, the business time, variance gamma process with drift, and stock price process can be simulated conveniently. It is confirmed that the estimator of the log stock price at a fixed time point is asymptotically unbiased, and therefore pricing through simulation is readily available.

Afterwards, one-year SP500 daily data are taken into account for empirical modeling. Since the model's density function is not explicitly known, the method of moments has been used to estimate the parameters. With market mid-prices of standard options obtained, by comparing the model's fitting degree with those of another three calendar-time or purely continuous models it is observed that while discontinuities are necessary for short-term large fluctuations, stochastic volatility seems to be needed in the long run. In addition, based on the numerical pricing of power options, it is directly concluded that the power impact is enormous or the option prices show very high sensitivity to the power taken. Furthermore, when pricing options via simulation, a larger sample size leads to less absolute relative percentage error while requiring more computational effort that grows quasi-arithmetically.

Of course, the Brownian time change can presumably work well with other types of Lévy processes, such as the normal inverse Gaussian process and the CGMY process. See, further, Barndorff-Nielsen (1995) [3]. Pricing other exotic options is understandably realistic under the time-changed process. However, imperfections of the model still exist. In reality, neither risk-free rates nor dividends are constant numbers; they are in themselves stochastic processes. Therefore, constructing specific interest rate and dividend yield models can better fit into time-variant market expectations. On the other hand, this would unavoidably impede the analytical tractability of the pricing mechanism, which should instead have reliance on other computational methods.

Appendix A - Proof of Characteristic Function of Business Time

In an attempt to prove (2.3), note that the only source of randomness in the business time (B_t) is the integrated squared Brownian motion, $I \equiv (I_t) := \left(\int_0^t W_s^2 \mathrm{d}s\right)$. Since the square impact makes W^2 no longer a Lévy process, analysis should have recourse to certain decomposition in order to construct uncorrelated variables. The Karhunen-Loève theorem (see, e.g., Ghanem and Spanos (1991) [11]) provides a useful canonical orthogonal representation for W. Fixing the time interval $[0,t]$, since for any $s>0$, $\mathbb{E}[W_t]=0$, $\mathbb{E}\left[W_t^2\right]=t$, and $\mathrm{Cov}[W_s,W_t]=s\wedge t$, which is a continuous function in time and can be used as a Mercer kernel, W admits a decomposition that

$$W_s = \sum_{k=1}^{\infty} Z_k g_k(s), \quad s \geq 0 \tag{A.1}$$

where convergence is understood in $\mathfrak{L}^2$-norm; $Z_k = \int_0^t W_s g_k(s)\mathrm{d}s \sim \mathrm{Normal}(0,\lambda_k)$, $\forall k$, are pairwise uncorrelated random variables; g_k, $\forall k$, are eigenfunctions forming an orthonormal basis with corresponding eigenvalues $\lambda_k > 0$. In this case, the integral I can be expressed as

$$I_t = \int_0^t \left(\sum_{k=1}^{\infty} Z_k g_k(s)\right)^2 \mathrm{d}s = \sum_{k=1}^{\infty} Z_k^2 \int_0^t g_k^2(s)\mathrm{d}s = \sum_{k=1}^{\infty} Z_k^2 \tag{A.2}$$

Given t, I_t happens to be the sum of a sequence of weighted uncorrelated chi-squared random variables. Since $Z_k^2 \sim \lambda_k \chi^2(1) \overset{\text{law}}{=} \mathrm{Gamma}(1/2, 1/(2\lambda_k))$ for any k[9], the distribution of I_t is indeed

[9] The chi-squared distribution has a natural relationship with the gamma distribution family by positive scaling. Refer to Walck (2007) [17].

equivalent to a sum of uncorrelated gamma random variables. Recall the characteristic function of a gamma random variable; see (3.1.3). Then the characteristic function of I conditioned on t is expressed as

$$\psi_{I|t}(u) := \mathbb{E}\left[e^{\mathrm{i}uI_t}\right] = \mathbb{E}\left[e^{\mathrm{i}u\sum_{k=1}^{\infty} Z_k^2}\right] = \prod_{k=1}^{\infty} \mathbb{E}\left[e^{\mathrm{i}uZ_k^2}\right] = \prod_{k=1}^{\infty} (1-2\mathrm{i}\lambda_k u)^{-\frac{1}{2}} \tag{A.3}$$

According to the Karhunen-Loève theorem, the eigenfunctions and eigenvalues are found by solving a homogeneous Fredholm integral equation of the second kind, namely,

$$\lambda_k g_k(u) = \int_0^t g_k(s)(s \wedge u)\mathrm{d}s \tag{A.4}$$

Separating the integral for the minimum function results in

$$\lambda_k g_k(u) = \int_0^u s g_k(s)\mathrm{d}s + \int_u^t u g_k(s)\mathrm{d}s \tag{A.5}$$

By the Leibniz rule, differentiating twice both sides gives rise to the ordinary differential equation (ODE) below.

$$\lambda_k g_k''(u) = -g_k(u) \tag{A.6}$$

subject to the boundary conditions that $g_k(0) = 0$ and $g_k'(t) = 0$. This equation has nontrivial solutions

$$g_k(u) = \sqrt{\frac{2}{t}} \sin \frac{u}{\sqrt{\lambda_k}}, \quad k \in \mathbb{N}_{++} \tag{A.7}$$

if and only if the eigenvalues admit the form

$$\lambda_k = \frac{4t^2}{\pi^2(2k-1)^2}, \quad k \in \mathbb{N}_{++} \tag{A.8}$$

with t given. Clearly, $\sum_{k=1}^{\infty} \lambda_k = t^2/2 < \infty$.

According to Abramowitz and Stegun (1972) [1], the cosine function has a very famous product representation,

$$\cos x = \prod_{k=1}^{\infty} \left(1 - \frac{4x^2}{\pi^2(2k-1)^2}\right) \tag{A.9}$$

Owing to this expansion, along with (A.8), (A.3) can be conveniently reduced into

$$\psi_{I|t}(u) = \sqrt{\left(\prod_{k=1}^{\infty}\left(1 - \frac{8\mathrm{i}t^2u}{\pi^2(2k-1)^2}\right)\right)^{-1}} = \sqrt{\sec\sqrt{2\mathrm{i}t^2u}} \tag{A.10}$$

Therefore, (2.3) becomes clear as, for $m \geq 0$ and $v > 0$,

$$\mathbb{E}\left[e^{\mathrm{i}u(mt+vI_t)}\right] = e^{\mathrm{i}mtu}\sqrt{\sec\sqrt{2\mathrm{i}vt^2u}} \tag{A.11}$$

Appendix B - Proof of Selected Pricing Formulae for Symmetric Power Options

Recall that in (5.3.3) the price of a symmetric power call involves a series of expectations, which I denote by $\mathscr{E}_{1,p-k} := \mathbb{E}^*\left[e^{-rT}S_T^{p-k}\mathbf{1}_{\{S_T>K\}}\right]$, for $k \in \mathbb{N}$. Given $k \leq \lfloor p \rfloor$, $p-k \geq 0$, and change of numéraire is meaningful as every powered stock price S^{p-k} can be thought to have the following $\mathbb{P}^*$-evolution,

$$S_t^{p-k} = \frac{S_0^{p-k}e^{(r-d_{p-k}(t))t+(p-k)X_t}}{\psi^*_{\ln S|t}(-\mathrm{i}(p-k))}, \quad k \leq \lfloor p \rfloor \tag{B.1}$$

In this case $\psi^*_{\ln S|t}(-\mathrm{i}(p-k))$ and $d_{p-k}(t)$ exist provided that (5.2.4) is true, and $\left(e^{-(r-d_{p-k}(t))t}S_t^{p-k}\right)$ is indeed a local martingale under $\mathbb{P}^*$. Hence, choosing S^{p-k}, for $k \leq \lfloor p \rfloor$, as numéraires directly leads to the in-the-money probabilities given in (5.3.5). Notice that there is substantially no difference between the handling method here and that for asymmetric power options.

On the other hand, if $k \geq \lfloor p \rfloor + 1$, the stock power becomes negative and makes no practical sense. Then, denote by $f^*_{\ln S|T}$ the density function of $\ln S_T$ under $\mathbb{P}^*$, and applying inverse Fourier transform it follows that

$$\begin{aligned}
\mathscr{E}_{1,p-k} &= e^{-rT}\int_{\mathbb{R}} e^{(p-k)x}\mathbf{1}_{\{x>\ln K\}}f^*_{\ln S|T}(x)\mathrm{d}x \\
&= e^{-rT}\int_0^\infty e^{(p-k)(y+\ln K)}\frac{1}{\pi}\int_0^\infty \Re\left\{e^{-\mathrm{i}u(y+\ln K)}\psi^*_{\ln S|T}(u)\right\}\mathrm{d}u\mathrm{d}y \\
&= \frac{K^{p-k}e^{-rT}}{\pi}\int_0^\infty \Re\left\{e^{-\mathrm{i}u\ln K}\psi^*_{\ln S|T}(u)\int_0^\infty e^{(p-k-\mathrm{i}u)y}\mathrm{d}y\right\}\mathrm{d}u \\
&= \frac{K^{p-k}e^{-rT}}{\pi}\int_0^\infty \Re\left\{\frac{e^{-\mathrm{i}u\ln K}\psi^*_{\ln S|T}(u)}{\mathrm{i}u-p+k}\right\}\mathrm{d}u
\end{aligned} \tag{B.2}$$

where the second equality follows from changing variable with $y = x - \ln K$ and the third is a result of Fubini's theorem for iterated integrals; note that integration is exchangeable only for $p-k<0$. Plugging (B.2) into (5.3.3) yields (5.3.4) through (5.3.6).

Further, by changing the order of integration and summation, the second sum in (5.3.4) becomes

$$K^p e^{-rT}\frac{1}{\pi}\int_0^\infty \Re\left\{e^{-\mathrm{i}u\ln K}\psi^*_{\ln S|T}(u)\sum_{k=\lfloor p \rfloor+1}^{\infty}\begin{pmatrix} p \\ k \end{pmatrix}\frac{(-1)^k}{\mathrm{i}u-p+k}\right\}\mathrm{d}u \tag{B.3}$$

The kth term of the sum in (B.3) can be rearranged by doing some product tricks as follows.

$$\begin{aligned}
&\frac{(-1)^{\lfloor p \rfloor+k}}{\mathrm{i}u-p+\lfloor p \rfloor+k}\begin{pmatrix} p \\ \lfloor p \rfloor+k \end{pmatrix} \\
=\ &\frac{\prod_{j=1}^{\lfloor p \rfloor+k}(-p+j-1)}{(\mathrm{i}u-p+\lfloor p \rfloor+k)(\lfloor p \rfloor+k)!}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\prod_{j=1}^{\lfloor p\rfloor+1}(-p+j-1)\prod_{j=1}^{k-1}(1-p+\lfloor p\rfloor+j-1)}{(\mathrm{i}u-p+\lfloor p\rfloor+k)(\lfloor p\rfloor+1)!\prod_{j=1}^{k-1}(2+\lfloor p\rfloor+j-1)} \\
&= \frac{(-1)^{\lfloor p\rfloor+1}}{1+\mathrm{i}u-p+\lfloor p\rfloor}\begin{pmatrix} p \\ \lfloor p\rfloor+1\end{pmatrix}\frac{\prod_{j=1}^{k-1}(1-p+\lfloor p\rfloor+j-1)}{\prod_{j=1}^{k-1}(2+\lfloor p\rfloor+j-1)}\frac{1+\mathrm{i}u-p+\lfloor p\rfloor}{1+\mathrm{i}u-p+\lfloor p\rfloor+k-1} \\
&= \frac{(-1)^{\lfloor p\rfloor+1}}{1+\mathrm{i}u-p+\lfloor p\rfloor}\begin{pmatrix} p \\ \lfloor p\rfloor+1\end{pmatrix}\frac{\prod_{j=1}^{k-1}(1-p+\lfloor p\rfloor+j-1)}{\prod_{j=1}^{k-1}(2+\lfloor p\rfloor+j-1)}\frac{\prod_{j=1}^{k-1}(1+\mathrm{i}u-p+\lfloor p\rfloor+j-1)}{\prod_{j=2}^{k}(1+\mathrm{i}u-p+\lfloor p\rfloor+j-1)} \\
&= \frac{(-1)^{\lfloor p\rfloor+1}}{1+\mathrm{i}u-p+\lfloor p\rfloor}\begin{pmatrix} p \\ \lfloor p\rfloor+1\end{pmatrix}\frac{(1)_k(1-p+\lfloor p\rfloor)_{k-1}(1+\mathrm{i}u-p+\lfloor p\rfloor)_{k-1}}{k!(2+\lfloor p\rfloor)_{k-1}(2+\mathrm{i}u-p+\lfloor p\rfloor)_{k-1}}
\end{aligned}
\tag{B.4}
$$

where in the final step the Pochhammer symbol $(\cdot)_\cdot$ has been used for reduction. The second fraction in the last line of (B.4) is yet equivalent to the kth term of a hypergeometric function ${}_3\mathbf{F}_2$, thus giving rise to Υ as in (5.3.8). This completes the proof for the call.

Similarly, for a symmetric power put, let $\mathscr{E}_{2,k} := \mathbb{E}^*\big[e^{-rT}S_T^k\mathbf{1}_{\{S_T<K\}}\big]$. For $k=0$, it is immediate that $\mathscr{E}_{2,0} = e^{-rT}\big(1-\Pi_2\big)$ for Π_2 as in (5.1.4).

For a given $k\geq 1$, by performing inverse Fourier transform again,

$$
\begin{aligned}
\mathscr{E}_{2,k} &= e^{-rT}\int_{\mathbb{R}} e^{kx}\mathbf{1}_{\{x<\ln K\}}f^*_{\ln S|T}(x)\mathrm{d}x \\
&= e^{-rT}\int_{-\infty}^{0} e^{k(y+\ln K)}\frac{1}{\pi}\int_0^\infty \Re\big\{e^{-\mathrm{i}u(y+\ln K)}\psi^*_{\ln S|T}(u)\big\}\mathrm{d}u\mathrm{d}y \\
&= \frac{K^k e^{-rT}}{\pi}\int_0^\infty \Re\left\{e^{-\mathrm{i}u\ln K}\psi^*_{\ln S|T}(u)\int_{-\infty}^0 e^{(k-\mathrm{i}u)y}\mathrm{d}y\right\}\mathrm{d}u \\
&= \frac{K^k e^{-rT}}{\pi}\int_0^\infty \Re\left\{\frac{e^{-\mathrm{i}u\ln K}\psi^*_{\ln S|T}(u)}{k-\mathrm{i}u}\right\}\mathrm{d}u
\end{aligned}
\tag{B.5}
$$

Notably, here Fubini's theorem is applicable for $k>0$. This implies (5.3.11). Combined with (5.3.10), further reduction can be made by

$$
\begin{aligned}
& K^p e^{-rT}\left(\frac{1}{2}+\sum_{k=0}^{\infty}\begin{pmatrix} p \\ k\end{pmatrix}(-1)^k\frac{1}{\pi}\int_0^\infty \Re\left\{\frac{e^{-\mathrm{i}u\ln K}\psi^*_{\ln S|T}(u)}{k-\mathrm{i}u}\right\}\mathrm{d}u\right) \\
&= K^p e^{-rT}\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \Re\left\{e^{-\mathrm{i}u\ln K}\psi^*_{\ln S|T}(u)\sum_{k=0}^{\infty}\begin{pmatrix} p \\ k\end{pmatrix}\frac{(-1)^k}{k-\mathrm{i}u}\right\}\mathrm{d}u\right) \\
&= K^p e^{-rT}\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \Re\left\{e^{-\mathrm{i}u\ln K}\psi^*_{\ln S|T}(u)\int_0^1\sum_{k=0}^{\infty}\begin{pmatrix} p \\ k\end{pmatrix}(-1)^k z^{k-\mathrm{i}u-1}\mathrm{d}z\right\}\mathrm{d}u\right) \\
&= K^p e^{-rT}\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \Re\left\{e^{-\mathrm{i}u\ln K}\psi^*_{\ln S|T}(u)\int_0^1 z^{-\mathrm{i}u-1}(1-z)^p\mathrm{d}z\right\}\mathrm{d}u\right)
\end{aligned}
\tag{B.6}
$$

for which $\int_0^1 z^{-\mathrm{i}u-1}(1-z)^p \mathrm{d}z =: \mathbf{B}(-\mathrm{i}u, 1+p) = \mathbf{B}(1+p, -\mathrm{i}u)$, by the symmetric property of the Beta function, and thus the pricing formula (5.3.12).

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