Bounded Solutions to the Differential Equation of Planetary Motion under General Relativity

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Introduction. In this note, a straightforward account will be given of the well-known nonlinear differential equation for planetary motion under general relativity. For the derivation of the relativistic model see [1, pp. 270-276] while for a thorough discussion of the Newtonian classical model of planetary motion see [2, pp. 471-496]. Planetary motion is an integral part of celestial mechanics. For an excellent introduction to this subject see [3].

Main result. The relativistic equation is given by

(1) $u''(\theta) + u(\theta) - c_1 u(\theta)^2 = c_2$ (where $u=1/r$, r being the radius from the given object to a foci and c_1 and c_2 are positive constants).

First, multiply (1) by 2u' and integrate for 0 to θ obtaining

(2) $$u'(\theta)^2 + u(\theta)^2 - 2c_1u(\theta)^3/3 = 2c_2\, u(\theta) - 2c_2\, u(0) + u'(0)^2 + u(0)^2 - 2c_1u(0)^3/3.$$

Next, using the fact that $u=1/r$ and $u' = -r'/r^2$ and then multiplying equation (2) by r^4 transforms equation (2) into

(3) $$r'(\theta)^2 + r(\theta)^2 - 2c_1r(\theta)/3 = 2c_2\, r(\theta)^3 - 2c_2r(\theta)^4u(0) + kr(\theta)^4$$

where $k = u'(0)^2 + u(0)^2 - 2c_1u(0)^3/3$. If $k-2c_2u(0) < 0$, then should $r \rightarrow \infty$ the LHS of (3) approaches ∞ while the RHS approaches $-\infty$ which is impossible. In other words, the solutions must remain bounded as $t\rightarrow\infty$ given these conditions. Should $k-2c_2u(0) \geq 0$, then the solutions may be unbounded.

We could study boundedness in another way by looking at the phase space (r,r'). We start by finding the equilibrium points of equation (3), i.e., the points (r,r') where $r(\theta) \geq 0$ and $r'(\theta)=0$. When $r'(\theta) = 0$, we have from (3) after rearranging terms

(4) $2u(0)c_2r^4 - kr^4 - 2c_2r^3 + r^2 - 2c_1r/3 = 0.$

Equation (4) now can be rewritten as

(5) $f(r) = rg(r) = r(2u(0)c_2 - k)r^3 - 2c_2r^2 + r - 2c_1/3] = 0.$

In other words, equation (3) may be transformed into

(6) $r'(\theta)^2 + f(r(\theta)) = r(\theta)g(r(\theta)) = 0.$

Next, we need to discuss the zeros of quartic polynomial $f(r)$ in (6) in the phase space (r,r'). These points correspond to the equilibrium points of (6) when $r'(\theta)$ equals zero. The zeros are $r_0 = 0$ and the zeros of the cubic polynomial $g(r) = (2u(0)c_2 - k)r^3 - 2c_2r^2 + r - 2c_1/3$. When $(2u(0)c_2 - k) > 0$, we can invoke Descartes rule signs (see [4, p. 211]) to conclude that $g(r)$ has at least one positive real root r_1 and possibly two more positive real roots r_2 and r_3 (with the possibility of double roots should $r_2 = r_3$ or $r_1 = r_2$) because there are three sign changes occurring in the cubic polynomial $g(r)$. Note, too, since the signs alternate in (5), $f(r)$ has no negative roots. Furthermore, equation (6) implies that the bounded solutions can only occur when $f(r) \leq 0$ which occurs when the cubic polynomial $g(r) \leq 0$. Should there be only one positive root r_1, then the bounded solutions must exist over the interval $[0,r_1]$. When r_2 and r_3 are two additional distinct zeroes of g(r) then the bounded solutions also exist over the interval $[r_2,r_3]$. However, since g(r) is positive over the interval (r_1,r_2), no bounded solutions can exist there because equation (6) would be positive which is impossible. When $r_2 = r_3$ we have a local minimum at the equilibrium point $(r_2,0)$ so in addition to the equilibrium point $(r_2,0)$, bounded solutions exist only on $[0,r_1]$ since $f(r) > 0$ for $r > r_1$ except when $r_2=r_3$. On the other hand, should $r_1 = r_2$, then bounded solutions exist over the entire interval $[0,r_3]$. In this case, we have a local maximum for $r = r_1$ so $f(r)\leq 0$ on the entire interval $[0,r_3]$. Furthermore, since $f(r) > 0$ for $r > r_3$, no bounded solutions can exist from our previous remarks.

Remark. In the case of a double root, one can easily calculate its value since it is a critical point of both g(r) and f(r) as well. Consequently, we have $g'(r) = 0$. Therefore, we have

$$(7) \quad g'(r) = 3(2u(0)c_2 - k)r^2 - 4c_2 r + 1 = 0.$$

Solving for r yields

$$(8) \quad r = \frac{4c_2 \pm ((-4c_2)^2 - 12(2u(0)c_2 - k))^{1/2}}{12u(0)c_2 - 6k}.$$

The correct root can be chosen by inspection. The sign of the second derivative of (7), i.e.

$$(9) \quad g''(r) = (12u(0)c_2 - 6k)r - 4c_2$$

determines whether r is a local maximum or minimum of g(r).

Conclusion. By using standard analytical methods from differential equations the above approach clearly gives a straightforward and qualitative analysis when the initial conditions yield orbits of the planets, an essential element of celestial mechanics.

References

[1] L. Brand, *Differential and Difference Equations*, John Wiley, New York, 1966.

[2] M. Tenenbaum and H. Pollard, *Ordinary Differential Equations*, Dover, New York, 1985.

[3] V.I. Arnold, *Mathematical Methods of Celestial Mechanics*, Springer Verlag, New York-Heidelberg-Berlin, 1989.

[4] L. Holder, *College Algebra* (3rd edition), Wadsworth Publishing, Belmont, California, 1984.