Moment-matching technique and General mean model in pricing Lookback option

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Abstract

In option pricing one of the main problems to solve is how to determine the fair price of an option when no-arbitrage opportunity is considered. To solve this problem many models have been developed but most of them there is no closed form solutions. In this paper, general mean model is used to price Lookback option since it can entervene in determination of minimum and maximum of underlying asset price under some conditions. The study shows the Construction of lattice using moment-matching which provide a system of linear equations where real world probabilities are unknown. To solve this system, Vandermonde matrix is preferred as one of the easiest way to use. Since it is not allowed to price with real world probabilities and as this paper deals with incomplete market which has more than one martingale measure, it is needed to choose the best one to use in pricing. Therefore, the relative entropy method is introduced to find the minimum entropy martingale measure which is the neutral probability in other words. Finally, the results from pricing Binomial floating lookback option is compared to well known Black-Scholes model.

Keywords: Lookback option, Incomplete market, moment-matching, general mean, relative entropy martingale measure, Vandermonde matrix, Binomial model, Black-Scholes model

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1 Introduction

Option trading has a long history even before Christ. Option is one of types of derivatives that give the holder the rights but not obligation to buy or to sell an underlying asset at a fixed price on the expiry date. Lookback option is one of Exotic options which is not new to the financial market. It came into existence many years ago before the birth of the first organised option exchange in the world named "Chicago Board of Option Exchange" in 1973 (Zhang, 1995). This is the largest option exchange because it can provide more than one million contracts per day (Boyle & Ananthanarayanan, 1977). In 1973, Myron Scholes and Fisher Black introduced famous option pricing model named Black-Scholes model which deals with continuous time under some assumptions. Since that time many research has been done in option pricing in both continuous and discrete time and noticed that standard models in continuous time are not doing well in discrete counterpart that it why other methods like lattice, Monte Carlo, numerical, statistical methods,... were created to solve this problem. Since Exotic options can play a special role in which standard options cannot do without difficulities, Exotic options are the best to use with discrete time methods.

Lookback options are path-dependent exotic options whose payoffs depend on the maximum and minimum of the underlying asset price attained throughout the optin lifetime. Standard Lookback options was first introduced by (Goldman, Sosin & Gatto, 1979). Lookback option as one of Exotic options allows the holders of the option to know the historical path of the underlying asset and when to exercise. Holders can choose the most beneficial price of the underlying asset which is occurred in that time. Lookback option provide numerous advantages for option traders since always end up in money due to its floating strike price. The payoff for a call option is provided by the asset price at maturity minus the minimum price observed during the option lifetime. For put option the payoff is given by the maximum price observed during the option lifetime minus the asset price at maturity time.

General mean function was used by (Zhang, 1998) to study the difference between arithmetic mean and geometric mean in order to approximate mathematically, the arithmetic Asian options and geometric Asian options. Since Lookback option payoffs depend on minimum and maximum of underlying asset, in this study general mean model is used to find minimum and maximum of the underlying asset when the path of lattice is considered.

(Ogutu, Lundengård, Silvestrov & Weke, 2014) described how to construct lattice using moment-matching technique to get a system of equations which contain jump probabilities as unknown. To solve that system, a Vandemonde matrix was used with some condition on jump size denoted as α which stands for the distance between two outcomes of stock when stock is considered as an exponential Le$^{'}$vy process. This paper is dealing with moment-matching and general mean in pricing Lookback options and It has the following structure: First section is introduction, Second section is moment matching technique in binomial model, section three is minimum relative entropy martingale measure, section four is general mean model, and the last is to price lookback option and compare the result to Black-Scholes model.

2 Moment-matching technique in binomial model

Consider the stochastic distribution of the price of paying non-dividend stock price in a risk-neutral economy. Let stock price Y_t be a stochastic random variable at time t in a period $[t, T]$ such that $Y_t = Y_{t-1}Z$ where Z is a discrete random variable defined as follows:

$$Z = \begin{cases} \lambda_1 & \text{with probability} \quad p_1 \\ \lambda_2 & \text{with probability} \quad p_2 \end{cases} \tag{1}$$

Such that $\lambda_1 > \lambda_2$ implies $\lambda_1 \neq \lambda_2$

Matching the moments of a random variable X with a discrete random variable D where $E(X) = m_1$ as given below

$$D_t = m_1 + Y_t \tag{2}$$

where $t = 1, 2, 3, ..., T$

Considering an incomplete market, the probabilities cannot be the same at each period.

$Y_1 = y_0 Z$ where Z is expressed in equation (1) then at $t = 1$ the equation (2) will be

$$D_1 = m_1 + Y_1$$

By applying moment matching technique yields

$$\begin{cases} E(Y_1^0) = p_1 + p_2 = \mu_0 \\ E(Y_1) = E(y_0 Z) = y_0\lambda_1 p_1 + y_0\lambda_2 p_2 = \mu_1 \end{cases}$$

In matrix form, it can be written as

$$\begin{pmatrix} \mu_0 \\ \mu_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ y_0\lambda_1 & y_0\lambda_2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \tag{3}$$

Let $V = \begin{pmatrix} 1 & 1 \\ y_0\lambda_1 & y_0\lambda_2 \end{pmatrix}$ represents the Vandermonde matrix obtained in equation (3) then jump probability can be determined as

$$\overrightarrow{p} = V^{-1}\overrightarrow{\mu} \tag{4}$$

where $\overrightarrow{p}$ and $\overrightarrow{\mu}$ are vectors containing the probabilities and moments respectively. The probabilities p on each period is unique as it is possible to determine the inverse of Vandermonde matrix since it has been confirmed by (Macon & Spitzbart, 1958).

Definition 2.1 *Vandermonde matrix is investigated by Alexandre-Théophile Vandermonde, It is a matrix with the terms of a geometric progression in each row. (Some authors use the* *transpose of the matrix). It has the following form*

$$V_N = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \delta_1 & \delta_2 & \cdots & \delta_N \\ \delta_1^2 & \delta_2^2 & \cdots & \delta_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ \delta_1^{N-1} & \delta_2^{N-1} & \cdots & \delta_N^{N-1} \end{pmatrix} \tag{5}$$

The determinant has been proven by (Gourdon, 1994) and it is written as

$$det(V_N) = \prod_{2 \leq i < j \leq N} (\delta_j - \delta_i)$$

If all δ_i are distinct and different from zero then, the matrix is also guaranteed to be invertible. Consider

$$\delta_i = y_0 \lambda_i \qquad \text{where} \quad 1 \leq i \leq N \qquad \text{with} \quad N \in \aleph \tag{6}$$

Will give the general lattice matrix with the final row missing.

Theorem 2.1 *For a Vandermonde matrix V_N with elements defined by (6), the elements of inverse are given by*

$$(V_N^{-1})_{ij} = \frac{(-1)^{j-i} \sigma_{N-j,i}}{\prod_{k=1,k \neq i}^{N} y_0(\lambda_k - \lambda_i)} \tag{7}$$

where $\sigma_{N-j,i}$ is a cofactor matrix

Matching the lattice to the first $N-1$ moments gives the equation (4) Using formulas (7) and (4) gives

$$p_i = \sum_{j=1}^{N} (V^{-1})_{ij} \mu_{j-1} = \sum_{j=1}^{N} \frac{(-1)^{j-i} \sigma_{N-j,i}}{\prod_{k=1,k \neq i}^{N} y_0(\lambda_k - \lambda_i)} \mu_{j-1} \tag{8}$$

(Ogutu et al., 2014)

2.1 Determination of transition probabilities in binomial lattice

The expression of probabilities when N is even is given in equation (8). For binomial lattice $N = 2$, then by replacing the value of i anf j yields p_1 and p_2 respectively.

$$p_1 = \sum_{j=1}^{2} \frac{(-1)^{j-1}\sigma_{2-j,1}}{\prod_{k=1,k\neq i}^{2} y_0(\lambda_k - \lambda_1)}\mu_{j-1} \qquad = \frac{\sigma_{1,1}\mu_0}{y_0(\lambda_2 - \lambda_1)} - \frac{\sigma_{0,1}\mu_1}{y_0(\lambda_2 - \lambda_1)}$$

$$p_2 = \sum_{j=1}^{2} \frac{(-1)^{j-2}\sigma_{2-j,2}}{\prod_{k=1,k\neq i}^{2} y_0(\lambda_k - \lambda_2)}\mu_{j-1} \qquad = \frac{-\sigma_{1,2}\mu_0}{y_0(\lambda_1 - \lambda_2)} + \frac{\sigma_{0,2}\mu_1}{y_0(\lambda_1 - \lambda_2)}$$

In matrix form we have

$$\overrightarrow{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \frac{\sigma_{1,1}}{y_0(\lambda_2-\lambda_1)} & -\frac{\sigma_{0,1}}{y_0(\lambda_2-\lambda_1)} \\ \frac{-\sigma_{1,2}}{y_0(\lambda_2-\lambda_1)} & \frac{\sigma_{0,2}}{y_0(\lambda_2-\lambda_1)} \end{pmatrix} \begin{pmatrix} \mu_0 \\ \mu_1 \end{pmatrix} \tag{9}$$

From equation (3) the Vandermonde matrix of order two is constructed and its inverse should be compared to the inverse of Vandermonde matrix in equation (9) to get the adjacent matrix. The deteminant of Vandermonde matrix V of order two defined in equation (3) is given by

$$det(V) = y_0\lambda_2 - y_0\lambda_1 = y_0(\lambda_2 - \lambda_1)$$

and the inverse is

$$V^{-1} = \frac{1}{y_0(\lambda_2 - \lambda_1)} \begin{pmatrix} y_0\lambda_2 & -1 \\ -y_0\lambda_1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{y_0\lambda_2}{y_0(\lambda_2-\lambda_1)} & \frac{-1}{y_0(\lambda_2-\lambda_1)} \\ \frac{-y_0\lambda_1}{y_0(\lambda_2-\lambda_1)} & \frac{1}{y_0(\lambda_2-\lambda_1)} \end{pmatrix} \tag{10}$$

From Algabra, two square matrices are equal if and only if the element located in the same position are the same. Then by comparing inverse of Vandermonde matrix in equation (9) and (10) yield

$$\frac{\sigma_{1,1}}{y_0(\lambda_2 - \lambda_1)} = \frac{y_0\lambda_2}{y_0(\lambda_2 - \lambda_1)}; -\frac{\sigma_{0,1}}{y_0(\lambda_2 - \lambda_1)} = \frac{-1}{y_0(\lambda_2 - \lambda_1)}$$

$$-\frac{\sigma_{1,2}}{y_0(\lambda_2-\lambda_1)} = \frac{-y_0\lambda_1}{y_0(\lambda_2-\lambda_1)}; \frac{\sigma_{0,2}}{y_0(\lambda_2-\lambda_1)} = \frac{1}{y_0(\lambda_2-\lambda_1)}$$

From the above equations we have

$$\sigma_{1,1} = y_0\lambda_2; \sigma_{0,1} = 1; \sigma_{1,2} = y_0\lambda_1; \sigma_{0,2} = 1 \tag{11}$$

By replacing (11) in equation (9) then the binomial probabilities would be

$$p_1 = \frac{\lambda_2\mu_0}{(\lambda_2-\lambda_1)} - \frac{\mu_1}{y_0(\lambda_2-\lambda_1)} \tag{12}$$

$$p_2 = -\frac{\lambda_1\mu_0}{(\lambda_2-\lambda_1)} + \frac{\mu_1}{y_0(\lambda_2-\lambda_1)} \tag{13}$$

Since $p_1 + p_2 = \mu_0 = 1$, then $p_1 > 0$ if $\mu_1 > \lambda_2 y_0$ while $p_2 > 0$ when $\mu_1 < \lambda_1 y_0$ which means that both p_1 and p_2 are positive if and only if $\lambda_2 y_0 < \mu_1 < \lambda_1 y_0$ or $\lambda_2 < \frac{\mu_1}{y_0} < \lambda_1$. The above assumptions indicate that all moments are positive which imply the positivity of probabilities. In this study, the case $y_0\lambda_i \geq 1$ is considered where $i = 1, 2$.

3 Minimal Relative Entropy Martingale Measure

Many authors have discused the minimal Entropy Martingale measure in different ways. Some of them say (Ssebugenyi, Mwaniki & Konlack, 2013) described how to use minimal entropy martingale measure to price American and European options in multinomial lattices which take into cumulants information. (Frittelli, 2000) gave the sufficient conditions for the existence and uniqueness of equivalent martingale measure which minimizes the relative entropy with respect to the real world probabilities and many others. In this paper which deals with incomplete market where there is more that one martingale measure, a good method is needed to choose a suitable martingale measure that is why relative entropy were preferred.

Definition 3.1 *Given two probability measure $Q = (q_1, q_2)$ and $P = (p_1, p_2) > 0$; then* *relative entropy of Q with respect to P given by $R(Q||P)$ is defined as*

$$R(Q||P) = \sum_{i=1}^{2} q_i ln(\frac{q_i}{p_i}) \tag{14}$$

Consider binomial one-period model. Suppose λ_i has two possible values, denoted by λ_1 and λ_2 with corresponding probabilities from p_1 to p_2. They must be a positive probability that the stock will go down, similarly going up. We impose a probability distribution q on the set of stock prices $y_0\lambda_1, y_0\lambda_2$ such that the following two conditions are satisfied. If q is a probability measure, then can be expressed as

$$\sum_{i=1}^{2} q_i = 1$$

Another condition is that q has risk neutal implies that the expected value of y_1 under q has to be equal to y_0, it can be written as

$$\sum_{i=1}^{2} q_i\lambda_i = y_0$$

The it is needed to solve the minimization problem of relative entropy between q and the real world probability p subject to these two contraints. Before to do so, let show that the relative entropy is a convex function of q. Consider the function

$$F : V : \Re^n \longrightarrow \Re \qquad \text{and} \qquad q \rightharpoondown F(q) = \sum_{i=1}^{2} q_i ln(\frac{q_i}{p_i}) \qquad \text{with}$$

$$V = V^* = \Re^n, Y = Y^* = \Re^m \qquad \text{and} \qquad q \in V : q = (q_1, q_2), \gamma \in Y : \gamma = (\gamma_1, \gamma_2)$$

Let the set of equivalent martingale measure be defined as

$$M_e = \left\{ q \in V : \sum_{i=1}^{2} q_i = 1, \sum_{i=1}^{2} q_i\lambda_i = y_0, q > 0 \right\}$$

then the convexity of relative entropy in (14) should be determined from

$$i(q, p) = R(Q||P) = \sum_{i=1}^{2} q_i ln(\frac{q_i}{p_i})$$

Let q_1,p_1 and q_2,p_2 be the probability distribution, define q and p as

$$q = \alpha q_1 + (1-\alpha)q_2 \qquad \text{and} \qquad p = \alpha p_1 + (1-\alpha)p_2 \qquad \text{with} \qquad \alpha \in [0,1] \qquad \text{then}$$

$$i(q,p) = i(\alpha q_1 + (1-\alpha)q_2, \alpha p_1 + (1-\alpha)p_2) = \sum_{i=1}^{2}(\alpha q_1 + (1-\alpha)q_2)ln(\frac{\alpha q_1 + (1-\alpha)q_2}{\alpha p_1 + (1-\alpha)p_2})$$

$$\leq \alpha \sum_{i=1}^{2} q_1 ln(\frac{q_1}{p_1}) + (1-\alpha)\sum_{i=1}^{2} q_2 ln(\frac{q_2}{p_2}) = \alpha i(q_1,p_1) + (1-\alpha)i(q_2,p_2)$$

Hence relative entropy in equation (14) is convex. Then, the problem can be solved using the Lagrange multipliers method by formulating the augmented cost function using the constraints that has indicated in condition one and two respectively

$$\begin{cases} L(q,\gamma) = F(q) + \sum_{i=1}^{m} \gamma_i B_i \\ s.t \sum_{i=1}^{2} q_i = 1, \sum_{i=1}^{2} q_i \lambda_i = y_0 \end{cases}$$

Where

$$B_1 = \sum_{i=1}^{2} q_i - 1 \qquad \text{and} \qquad B_2 = \sum_{i=1}^{2} q_i \lambda_i - y_0$$

Lagrange equation becomes

$$L(q,\gamma_1,\gamma_2) = \sum_{i=1}^{2} q_i ln(\frac{q_i}{p_i}) + \gamma_1 B_1 + \gamma_2 B_2 = \sum_{i=1}^{2} q_i ln(\frac{q_i}{p_i}) + \gamma_1(\sum_{i=1}^{2} q_i - 1) + \gamma_2(\sum_{i=1}^{2} q_i \lambda_i - y_0)$$

where $\gamma_1$and γ_2 are Lagrange multipliers. By minimizing L with respect to q , set the partial derivative $\frac{\partial L}{\partial q_i}$ equal to zero for all $i \in \mathbb{N}$. This leads to

$$ln(\frac{q_i}{p_i}) + 1 + \gamma_1 + \gamma_2 \lambda_i = 0$$

by arranging yields

$$q_i = \frac{p_i exp(\eta \lambda_i)}{\sum_{i=1}^{2} p_i exp(\eta \lambda_i)} = \frac{p_i exp(\eta \lambda_i)}{\mathbb{E}[exp(\eta \lambda_i), p]} \tag{15}$$

and

$$f(\eta) = r = \frac{\mathbb{E}[Zexp(\eta Z), p]}{\mathbb{E}[exp(\eta Z), p]} or \frac{\sum_{i=1}^{2} \lambda_i p_i exp(\eta \lambda_i)}{\mathbb{E}[exp(\eta \lambda_i), p]} \tag{16}$$

To determine the values of neutral probability q_i we need to analyse the function of η and find η^* such that $f(\eta^*) = y_0$ this can be determined by trial and error. By studying the limit of the function in (16) yields

$$\lim_{\eta \longrightarrow -\infty} f(\eta) = \lambda_2; \lim_{\eta \longrightarrow +\infty} f(\eta) = \lambda_1$$

Proof 3.1 *$f(\eta)$ could be written as follows*

$$f(\eta) = \frac{\lambda_1 p_1 exp(\eta \lambda_1) + \lambda_2 p_2 exp(\eta \lambda_2)}{p_1 exp(\eta \lambda_1) + p_2 exp(\eta \lambda_2)}$$

Let consider $\alpha_0 = \lambda_1$; $\alpha_1 = \lambda_1 p_1$,$\beta_0 = \lambda_2$,$\beta_1 = \lambda_2 p_2$,$\alpha = p_1$,$\beta = p_2$ then $f(\eta)$ becomes

$$f(\eta) = \frac{\alpha_1 e^{\eta\alpha_0} + \beta_1 e^{\eta\beta_0}}{\alpha e^{\eta\alpha_0} + \beta e^{\eta\beta_0}} = \frac{\frac{\alpha_1 e^{\eta\alpha_0} + \beta_1 e^{\eta\beta_0}}{\alpha_1 e^{\eta\alpha_0}}}{\frac{\alpha e^{\eta\alpha_0} + \beta e^{\eta\beta_0}}{\alpha_1 e^{\eta\alpha_0}}}$$

$$= \frac{1 + \alpha_1^{-1}\beta_1 e^{\eta(\beta_0 - \alpha_0)}}{\alpha_1^{-1}\alpha + \alpha_1^{-1}\beta e^{\eta(\beta_0 - \alpha_0)}} = \alpha_1 \alpha \left[\frac{1 + \alpha_1^{-1}\beta_1 e^{\eta(\beta_0 - \alpha_0)}}{1 + \alpha_1^{-1}\beta e^{\eta(\beta_0 - \alpha_0)}} \right]$$

Set $x = e^{\eta(\beta_0 - \alpha_0)} = e^{\eta(\lambda_2 - \lambda_1)}$ since $\lambda_1 > \lambda_2$ then $\eta \longrightarrow -\infty$ implies $x \longrightarrow +\infty$ then

$$f(x) = \alpha_1 \alpha^{-1} \left[\frac{1 + \alpha_1^{-1}\beta_1 x}{1 + \alpha^{-1}\beta x} \right]$$

$$\lim_{-\infty} f(\eta) = \lim_{+\infty} f(x) = \alpha_1 \alpha^{-1} \frac{\alpha_1^{-1}\beta_1}{\alpha^{-1}\beta} = \frac{\beta_1}{\beta} = \frac{\lambda_2 p_2}{p_2} = \lambda_2$$

If $\eta \longrightarrow +\infty$ then $x \longrightarrow 0$ which means that

$$\lim_{+\infty} f(\eta) = \lim_{0} f(x) = \alpha_1 \alpha^{-1} = \frac{\alpha_1}{\alpha} = \frac{\lambda_1 p_1}{p_1} = \lambda_1$$

It is clear that in binomial case limit should be

$$\lim_{\eta \longrightarrow -\infty} f(\eta) = \lambda_2; \lim_{\eta \longrightarrow +\infty} f(\eta) = \lambda_1$$

By studying equation (16) yields

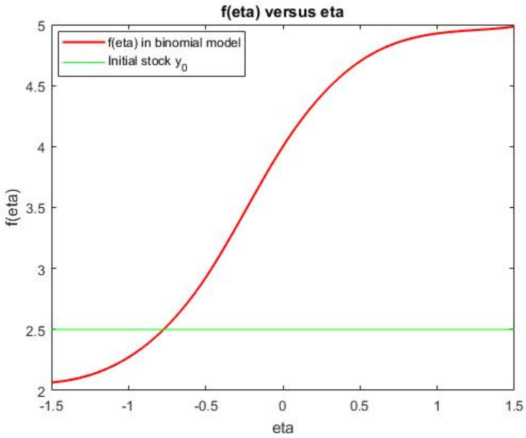

Figure 1: $f(\eta)$ versus η in Binomial

Example 3.1 (Binomial case) *Consider $y_0 = 2.5$, $\lambda_1 = 5$, $\lambda_2 = 2$, $\mu_0 = 1$ from assumption of moments in binomial, one can get $5 < \mu_1 < 12.5$, let in this case consider $\mu_1 = 10$ then by replacing back in equation (12) and (13) yields $p_1 = 0.6667$ and $p_2 = 0.3333$. From these information $f(\eta)$ in (16) is determined in Figure 1 where on should find η^* by trial and error. Consider $\eta^* = -0.8$ and use it to determine neutral probabilities q_i. From the equation (15) we have*

$$q_1 = \frac{p_1 exp(\eta^* \lambda_1)}{p_1 exp(\eta^* \lambda_1) + p_2 exp(\eta^* \lambda_2)} = \frac{0.6667 exp(-0.8 * 5)}{0.6667 exp(-0.8 * 5) + 0.3333 exp(-0.8 * 2)} = 0.154$$

$$q_2 = \frac{p_2 exp(\eta^* \lambda_2)}{p_1 exp(\eta^* \lambda_1) + p_2 exp(\eta^* \lambda_2)} = \frac{0.3333 exp(-0.8 * 2)}{0.6667 exp(-0.8 * 5) + 0.3333 exp(-0.8 * 2)} = 0.846$$

Therefore, after getting this neutral probabilities q_1 and q_2, it is possible to price.

4 General Mean Function

Definition 4.1 *If x is a non-zero real number and $S_1, S_2, ..., S_n$ are positive real numbers which represent the stock, then general mean with exponential x of these positive real numbers is*

$$M(S|x) = f(x) = \left(\frac{1}{N} \sum_{i=1}^{N} S_i^x \right)^{\frac{1}{x}}$$

Assume that for all $i \in 1, 2, ..., N$ and $S_i > 0$ then in exponential form yields

$$f(x) = e^{\frac{1}{x}ln[\frac{1}{N}\sum_{i=1}^{N}S_i^x]} = \left(\frac{1}{N}\sum_{i=1}^{N}S_i^x\right)^{\frac{1}{x}} \tag{17}$$

where N is number of observation, S is stock and x is a parameter which drive the behavior of stock.

1. **Domain of definition**

$$Df = \left\{x \in \Re : x \neq 0, \sum_{i=1}^{N} S_i^x > 0\right\} = \Re - \{0\} \cap \Re = \Re - \{0\} = (-\infty, 0) \cup (0, \infty)$$

2. **Parity**

$$\text{For all} \quad x \in Df, -x \in Df, \qquad \text{such that} \begin{cases} f(x) \neq f(-x) \\ f(-x) \neq -f(x) \end{cases}$$

Therefore, function f is neither even nor odd. Which means geometrically, that the function adimits no symmetry with ordinate (y-axis) and no symmetry with the origine. Other word, if $f(-x) = f(x)$ means that there is symmetry with y since $-x$ and x are symmetry.

3. **Limits on the boundaries**

In order to prove geometric mean given by

$$\lim_{x \longrightarrow 0} M_x = M_0$$

We can rewrite the definition of M_x using the exponential function as it is in equation (17). Then if the limit $x \longrightarrow 0$, we can apply l'Hôspital's rule to the argument of the exponential function. Differentiating the numerator and denominator with respect to x, we have:

$$\lim_{x\longrightarrow 0}\frac{\sum_{i=1}^{n}\omega_i S_i^x}{x}=\lim_{x\longrightarrow 0}\frac{\frac{\sum_{i=1}^{n}\omega_i S_i^x\ln S_i}{\sum_{i=1}^{n}\omega_i S_i^x}}{1}$$

Let

$$y=\sum_{i=1}^{n}\omega_i S_i^x \qquad \text{where} \qquad ln(y)=ln(\sum_{i=1}^{n}\omega_i S_i^x)=x\sum_{i=1}^{n}\omega_i lnS_i$$

$$(ln(y))'=\frac{y'}{y}=\sum_{i=1}^{n}\omega_i lnS_i$$

$$y'=y\sum_{i=1}^{n}\omega_i lnS_i=\sum_{i=1}^{n}\omega_i S_i^x.\sum_{i=1}^{n}\omega_i lnS_i=\sum_{i=1}^{n}\omega_i(S_i^x lnS_i)=\sum_{i=1}^{n}\omega_i S_i^x lnS_i$$

Then

$$\frac{(ln\sum_{i=1}^{n}\omega_i S_i^x)'}{x'}=\frac{\sum_{i=1}^{n}\omega_i S_i^x lnS_i}{\sum_{i=1}^{n}\omega_i S_i^x}$$

$$=\lim_{x\longrightarrow 0}\frac{\sum_{i=1}^{n}\omega_i S_i^x\ln S_i}{\sum_{i=1}^{n}\omega_i S_i^x}=\sum_{i=1}^{n}\omega_i\ln S_i=\ln(\prod_{i=1}^{n}S_i^{\omega_i})$$

By the continuity of the exponential function, we can substitute back into the above relation to obtain

$$\lim_{x\longrightarrow 0}M_x(S_1,...,S_n)=exp(ln(\prod_{i=1}^{n}S_i^{\omega_i}))=\prod_{i=1}^{n}S_i^{\omega_i}=\prod_{i=1}^{n}S_i^{\frac{1}{n}}=\sqrt[n]{\prod_{i=1}^{n}S_i}=M_0(S_1,...,S_n). \tag{18}$$

for other boundary we have

$$\lim_{x\longrightarrow +\infty}M(x)=\lim_{x\longrightarrow +\infty}\left(\frac{1}{N}\sum_{i=1}^{N}S_i^x\right)^{\frac{1}{x}}=\max\left(S_1,S_2,...,S_N\right)=S_1 \tag{19}$$

Where $S_1>S_2>...>S_N>0$

Proof 4.1 *consider*

$$M(S|x)=\left(\frac{1}{N}\sum_{i=1}^{N}S_i^x\right)^{\frac{1}{x}}=\left(\frac{S_1^x+S_2^x+...+S_N^x}{N}\right)^{\frac{1}{x}}$$

$$= exp\left\{\frac{1}{x}ln\left(\frac{S_1^x + S_2^x + ... + S_N^x}{N}\right)\right\} = exp\left\{\frac{1}{x}ln\left[\frac{S_1^x(1+(\frac{S_2}{S_1})^x + (\frac{S_3}{S_1})^x + ... + (\frac{S_N}{S_1})^x)}{N}\right]\right\}$$

$$= exp\left\{\frac{1}{x}ln(S_1^x) + \frac{1}{x}ln(\frac{1}{N}) + \frac{1}{x}ln\left[1+(\frac{S_2}{S_1})^x + ... + (\frac{S_N}{S_1})^x\right]\right\}$$

$$= exp\left\{\ln(S_1) + \frac{1}{x}ln(\frac{1}{N}) + \frac{1}{x}ln(1+Y_x)\right\}$$

Where

$$Y_x = \sum_{i=2}^{N}(\frac{Si}{S_1})^x \tag{20}$$

If $x \longrightarrow +\infty$ *then* $Y_x \longrightarrow 0$ *Since* $\frac{S_i}{S_1} < 1$*, for all* i *belong to* $\{2, 3, ..., N\}$*.*

Then $ln(1 + Y_x)$ *is equivalent to equation (20) at zero.*

Therefore,

$$M(S|x) = exp\left\{\ln(S_1) + \frac{1}{x}ln(\frac{1}{N}) + \sum_{i=2}^{N}(\frac{S_i}{S_1})^x\right\} = exp\left\{\ln(S_1)\right\}\exp\left\{\frac{1}{x}ln(\frac{1}{N}) + \sum_{i=2}^{N}(\frac{S_i}{S_1})^x\right\}$$

Since $\frac{1}{x}ln(\frac{1}{N}) \longrightarrow 0$ *as* $x \longrightarrow +\infty$ *and* $\sum_{i=1}^{N}\left(\frac{S_i}{S_1}\right)^x \longrightarrow 0$ *when* $x \longrightarrow +\infty$ *with* $\frac{S_i}{S_1} < 1$ *for* i *belong to* $\{$*2,3,4,...,N*$\}$ *with finite number of observation* N*.*

Therefore,

$$\lim_{x \longrightarrow +\infty} M(S|x) = S_1 = max\{S_1, S_2, ..., S_N\} \tag{21}$$

Let show that for other boundary

$$\lim_{x \longrightarrow -\infty} M(S|x) = \lim_{x \longrightarrow -\infty}\left(\frac{1}{N}\sum_{i=1}^{N}S_i^x\right)^{\frac{1}{x}} = min\left(S_1, S_2, ..., S_N\right) = S_N \tag{22}$$

Where $S_1 > S_2 > ... > S_N > 0$

Proof 4.2

$$M(S|x) = \left(\frac{1}{N}\sum_{i=1}^{N} S_i^x\right)^{\frac{1}{x}} = \left(\frac{S_1^x + S_2^x + ... + S_N^x}{N}\right)^{\frac{1}{x}} = exp\left\{\frac{1}{x}ln\left(\frac{S_1^x + S_2^x + ... + S_N^x}{N}\right)\right\}$$

$$= exp\left\{\frac{1}{x}ln\left[\frac{S_N^x((\frac{S_1}{S_N})^x + (\frac{S_2}{S_N})^x + ... + (\frac{S_{N-1}}{S_N})^x + 1)}{N}\right]\right\}$$

$$= exp\left\{\frac{1}{x}ln(S_N)^x + \frac{1}{x}ln\left[\frac{(\frac{S_1}{S_N})^x + (\frac{S_2}{S_N})^x + ... + (\frac{S_{N-1}}{S_N})^x + 1}{N}\right]\right\}$$

$$= exp\left\{\ln(S_N) + \frac{1}{x}ln(\frac{1}{N}) + \frac{1}{x}ln(1+K_x)\right\}$$

Let consider

$$K_x = \sum_{i=1}^{N-1}\left(\frac{S_i}{S_N}\right)^x = \left(\frac{S_1}{S_N}\right)^x + \left(\frac{S_2}{S_N}\right)^x + ... + \left(\frac{S_{N-1}}{S_N}\right)^x$$

If $x \longrightarrow -\infty$ *then* $K_x \longrightarrow 0$ *Since* $\frac{S_i}{S_N} > 1$*, for all* i *belong to* $\{1, 2, 3, ..., N-1\}$.

Then $ln(1+K_x)$ *is equivalent to* K_x *at zero.*

Therefore,

$$M(S|x) = exp\left\{\ln(S_N) + \frac{1}{x}ln(\frac{1}{N}) + \sum_{i=1}^{N-1}(\frac{S_i}{S_N})^x\right\} = exp\left\{\ln(S_N)\right\}\exp\left\{\frac{1}{x}ln(\frac{1}{N}) + \sum_{i=1}^{N-1}(\frac{S_i}{S_N})^x\right\}$$

Since $\lim\limits_{x\longrightarrow -\infty} \frac{1}{x}ln(\frac{1}{N}) \longrightarrow 0$ *and* $\lim\limits_{x\longrightarrow -\infty} \sum_{i=1}^{N}(\frac{S_i}{S_N})^x = 0$ *as number of observation* N *is finite. Then*

$$\lim_{x\longrightarrow -\infty} M(S|x) = S_N = min\{S_1, S_2, ..., S_N\}$$

Example 4.1 *Let find* $min(y_0, y_0\lambda_2, y_0\lambda_2^2)$= $min(2.5, 5, 10)$*. Using general mean, given that* $N = 3$*,* $y_0 = 2.5$*,* $y_0\lambda_2 = 5$ *and* $y_0\lambda_2^2 = 10$ *By using general mean equation in (19) yields*

$$\lim_{+\infty} M(S|x) = 2.5 * e^0 = 2.5 = min(2.5, 5, 10)$$

Example 4.2 *In the same example let determine the* $max(y_0, y_0\lambda_2, y_0\lambda_2^2)$= $max(2.5, 5, 10)$*. Using general mean, given that* $N = 3$*,* $y_0 = 2.5$*,* $y_0\lambda_2 = 5$ *and* $y_0\lambda_2^2 = 10$ *By applying* *general mean in (21) yields*

$$\lim_{+\infty} M(S|x) = 10 * e^0 = 10 = max(2.5, 5, 10)$$

4. **Asymptotes**

   This function doesn't admit vertical and oblic asymptotes. Horizontal asymptote as it has been proved in part of limits on the boundaries. For all $x \in (-\infty, 0) \cup (0, \infty)$

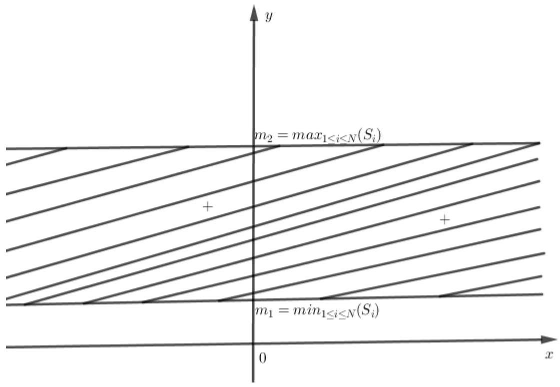

Figure 2: Domain of general mean solutions

5. **Local extrema**

$$f' : Df \longrightarrow \Re$$

f(x) is continuous function on Domain $x \in Df$

Recall:

$$f(x) = exp\left\{\frac{1}{x}ln\left\{\frac{1}{N}\sum_{i=1}^{N} exp\left(xln(S_i)\right)\right\}\right\}$$

$$f(x) = exp\left\{\frac{1}{x}G(x)\right\} \qquad \text{where} \quad G(x) = ln\left\{\frac{1}{N}\sum_{i=1}^{N} exp\left(xln(S_i)\right)\right\}$$

$$G'(x) = \frac{\sum_{i=1}^{N} ln(S_i)exp(xln(S_i))}{\sum_{i=1}^{N} exp(xln(S_i))}$$

Since $f(x) = exp\left\{\frac{G(x)}{x}\right\}$

For all $x \in Df$, f' could be determined as:

$$f'(x) = \left\{ \frac{G(x)}{x} \right\}' exp \left\{ \frac{G(x)}{x} \right\} \tag{23}$$

Since $\left(e^{u(x)}\right)' = u'(x)e^{u(x)}$

$$\left(\frac{G(x)}{x} \right)' = \frac{G(x) - xG'(x)}{x^2}$$

The sign of f' depends on the sign of

$$G(x) - xG'(x) \tag{24}$$

From equation (23) the expression

$exp \left\{ \frac{G(x)}{x} \right\}$ is always positive which means that $\left(\frac{G(x)}{x} \right)'$ is the term which can make $f'(x)$ zero.

Therefore,

$$G(x) - xG'(x) = 0 \qquad \text{implies that} \quad G(x) = xG'(x)$$

Whether $S_i > 1$ or $0 < S_i < 1$ there is no problem because at $x = 0$, $xG'(x) = 0$

Then for all $x \in (0, +\infty)$, $G(x) > 0$ and $G'(x) > 0$

The solution of $f(x)$ where $x \longrightarrow 0$ exist since

$$lim_{x \longrightarrow 0} = lim_{x \longrightarrow 0^-} = lim_{x \longrightarrow 0^+} = \prod_{i=1}^{N} S_i^{\frac{1}{N}} \in \Re_+^*$$

If $x = 0$, $G(0) = ln(\frac{1}{N} \sum_{i=1}^{N} 1)$=$ln(\frac{1}{N}.N) = 0$

$$G(0) - 0G'(0) = 0$$

$f'(x) = 0$ admits one solution at $x = 0$

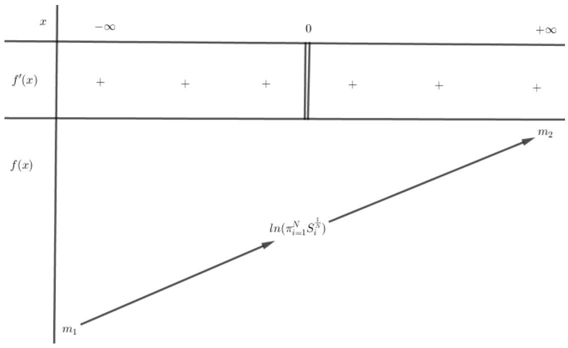

Figure 3: The continuity of general mean function

6. **Concavity**

   From (23) we set

$$H(x) = \frac{G(x) - xG'(x)}{x^2} \qquad \text{such that}$$

$$H'(x) = -\left(\frac{x^2G''(x) - 2xG'(x) + 2G(x)}{x^3}\right)$$

   $H'(x) > 0$ if and only if

$$\frac{x^2G''(x) - 2xG'(x) + 2G(x)}{x^3} < 0 \qquad \text{Then}$$

$$2G(x) > 2xG'(x) - x^2G''(x)$$

   Therefore, we have $H'(x) > 0$ with $x \in (-\infty, 0)$. Since $(H(x))^2$ and $exp\left\{\frac{G(x)}{x}\right\}$ are positive. With $H'(x) > 0$ implies that $f''(x) > 0$ which means that $f(x)$ is convex in the interval of the domain $(-\infty, 0)$ and concave in the interval of domain $(0, \infty)$.

7. **Graph**

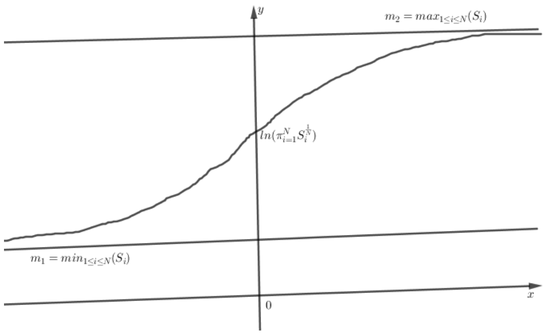

Figure 4: Graph representation of general mean function

4.1 General mean model

By referring to the concept of call and put option where $X_c = max(S_T - K, 0)$ and $X_p = max(K - S_T, 0)$ with T maturity time, K strike price and S the stock. We are considering general mean function as in equation (17) to be the strike price depends on x and denoted by K_x where in (17) S is stock, N is number of observation and x is a parameter. The function in (17) is well defined since $S_i > 0$ and $x \in (-\infty, 0) \cup (0, +\infty)$. Then by valuing the value of x yields different strike price of exotic options as follows

$$
\begin{cases}
\lim\limits_{x \longrightarrow -\infty} f(x) = min(S_i) = K_{-\infty} \\
\dots \\
f(-1) = \dfrac{N}{\frac{1}{x_1} + ... + \frac{1}{x_N}} = K_{-1} \\
\dots \\
\lim\limits_{x \longrightarrow 0} f(x) = \prod_{i=1}^{N} S_i^{\frac{1}{N}} = K_0 \\
\dots \\
f(1) = \frac{1}{N} \sum_{i=1}^{N} S_i = K_1 \\
f(2) = \sqrt{\frac{1}{N} \sum_{i=1}^{N} S_i^2} = K_2 \\
\dots \\
\lim\limits_{x \longrightarrow +\infty} f(x) = max(S_i) = K_{+\infty}
\end{cases}
\tag{25}
$$

The function in (17) is just a generalization of strike price for options between Asian and Lookback option as it is clear in (25). Therefore, the general mean model should be :

$$X_c = max\left(S_T - f(x), 0\right)$$

$$X_p = max\left(f(x) - S_T, 0\right)$$

Generally we call this function $f(x)$ the general mean which is denoted by $M(x)$. It is clear that with x tends to $-\infty$ or $+\infty$ general mean function express the same strike price as the one used in standard lookback option as it is shown in (22) and (19) respectively.

5 Pricing Lookback Option

Let Y_n denote the stock price at time $t = t_n$ with $n = 0, 1, 2, ..., n-1$. Suppose that $\lambda_i \in \Re$ that satisfies $0 < \lambda_2 < 1 + r < \lambda_1$. Then the binomial lattice diagram will be

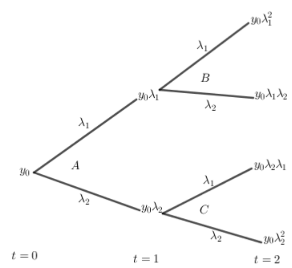

Figure 5: Binomial tree in pricing Lookback option

Determining the stock price at A,B and C nodes. It is needed to consider payoffs and use the backward to find the initial stock price A. Therefore, to determine the price at each node yields

$$B = \frac{1}{1+r}\left[q_1\left(y_0\lambda_1^2 - K_{-\infty}^1\right)^+ + q_2\left(y_0\lambda_1\lambda_2 - K_{-\infty}^2\right)^+\right]$$

$$C = \frac{1}{1+r}\left[q_1\left(y_0\lambda_2\lambda_1 - K_{-\infty}^3\right)^+ + q_2\left(y_0\lambda_2^2 - K_{-\infty}^4\right)^+\right]$$

Where $K_{-\infty}^1 = \min\{y_0, y_0\lambda_1, y_0\lambda_1^2\}$, $K_{-\infty}^2 = \min\{y_0, y_0\lambda_1, y_0\lambda_1\lambda_2\}$, $K_{-\infty}^3 = \min\{y_0, y_0\lambda_2, y_0\lambda_2\lambda_1\}$,

$K_{-\infty}^4 = \min\{y_0, y_0\lambda_2, y_0\lambda_2^2\}$. With indixes $1, 2, 3, 4$ to indicate number of paths in lattice. The initial stock price will be

$$A = \frac{1}{1+r}(q_1 B + q_2 C)$$

From Black-Scholes world, the following equations are used

$$u = \lambda_1 = e^{\sigma\sqrt{T}}, \qquad d = \lambda_2 = e^{-\sigma\sqrt{T}} \qquad \text{and} \qquad q = \frac{e^{rT} - d}{u - d} = \frac{e^{rT} - \lambda_2}{\lambda_1 - \lambda_2}$$

The Black Scholes formula for call option is given by

$$C = S_t N(d_1) - K e^{-rT} N(d_2)$$

Where

$$d_1 = \frac{ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \qquad \text{and} \qquad d_2 = \frac{ln(\frac{S_t}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

6 pricing lookback option via general mean model

Let consider $\lambda_1 = 5$, $\lambda_2 = 2$, $y_0 = 2.5$, $r = 1.5$, $K = 3$, $T = 1$ and $\sigma = \frac{ln(\lambda_1)}{\sqrt{T}} = 1.6$. For p_1 and p_2 to be positive this condition $y_0\lambda_1 > \mu_1 > y_0\lambda_2$ should hold. one can choose any value in that interval. Let choose $\mu_1 = 10$ for example. As it has been done early $q_1 = 0.154$ and $q_2 = 0.846$. From the following figure

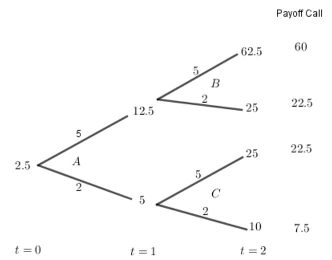

Figure 6: Pricing floating lookback via general mean(Binomial case)

Referring to the binomial formulas in section five, we can find the stock price at the nodes B ,C and A respectively. The stock price should be $B = 11.31$,$C = 3.924$ and $A = 2.025 \approx 2.03$ In Black-Scholes way, the results will be $d_1 = 1.624$,$d_2 = 0.024$ and $C = 2.029 \approx 2.03$

7 Conclusion

Lattice method and Black-Scholes model are famous in financial world in pricing discrete and continuous time respectively. By comparing Binomial model and Black-Scholes model in this study, the out put shows that both models end up with approximately equal results considering call option. It is clear that general mean model in this work helped in determining the payoff of Lookback option can be also away of observing other option which is hidden between Asian option and Lookback option. This work also make clear that even the minimum of the whole lattice is considered in pricing the result will make sense. From the results, one can say that comparing Binomial model and Black-Scholes model there is no significant difference.

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