How Many Days Equal A Year? A Note on the Mean-Variance Model

1. Introduction

The mean-variance model and the related efficient frontier provided the basis of the development of modern portfolio theory.   It offered the first systematic analysis of the risk-return trade-off faced by each investor and it still constitutes a fundamental theory for portfolio formation.   Markowitzs' ground-breaking theoretical model and associated numerical solutions of portfolio selection under uncertainty supported the foundation for many important developments in financial economics like the Capital Asset Pricing Model (CAPM), derived by Sharpe (1964), Lintner (1965), and Mossin (1966)) and the now classical distinction between systematic and idiosyncratic risks. Although the importance of his work remains undeniable, mean-variance portfolio analysis generated more questions than answers and, consequently, caused a large body of research.

In this paper, we address the importance of an issue which is very practical, but often ignored, on estimating the means and variances of stock returns. In practice, mean and variance on a stock return are often estimated on a 1-day horizon, and then they are converted to longer horizons as needed, by scaling. However, the conversion method used is not always trivial, as the time scale used for this conversion may not be the same as the calendar days and depends on the stock price movement.

First, in the literature, Fama (1965), French (1980), Roll (1984), French and Roll (1986) and many other researchers find that volatility is caused by trading activities. This means that only trading days should be counted to determine the time scale. For

example, the 1-year variance of return on a stock traded on the New York Stock Exchange (NYSE) should be equal to 1-day variance multiplied by 252, the number of trading days in a year on NYSE, rather than by 365, as shown in Hull (2012). Although some researchers, including Smithsonite and Minton (1996 a,b), J.P. Morgan (1996), Diebold et al (1997), among others, show different perspectives, this approach has been widely accepted by industry and regulators such as the Basel committee.

However, little attention has been paid to the issue of converting a 1-day mean to a longer time horizon. While the volatility is caused by trading activities, the mean of a series of returns depends on a variety of factors among which time and the time value of money consideration. This implies that non-trading days should also be counted toward calculating the time scale. At the same time, since only trading days contribute to volatility, or market price risk, the expected returns on non-trading days cannot be same as those on trading days. Thus, for instance, the simple rule of multiplying by 252 may not be correct to convert a 1-day mean to 1-year mean.

For example, when estimating average daily returns in practice, a weekend return, i.e., Friday close to Monday close return, is often treated as one observation with no difference from other trading day returns. Thus, the number of observations in a period of a year in the sample is equal to the number of trading days in the year. Thus, statistically, it is reasonable to convert such an estimated average daily return to the average annual return by multiplying the number of trading days in the year. However, the average daily return estimated in this way ignores the returns on non-

trading days. As a result, it is neither the average return on a trading day nor the average return on a calendar day.

In this paper, we present an approach to convert a 1-day mean to a longer horizon, based on the Capital Asset Pricing Model (CAPM), and show how this conversion may affect the decisions of capital allocation and on the portfolio performance evaluation.

2. Mathematical Framework

Let $E(r_x)$ and σ_x be the mean and standard deviation of the daily return on stock X, i.e., 1-day mean and 1-day standard deviation, respectively. If the time horizon of our investment is a period of N calendar days, in order to use the mean-variance model to make optimal investment decisions, we need to adjust these parameters to fit our time horizon.

We denote $E^*(r_x)$ and σ_x^* to be the mean and standard deviation of the return on stock X for the time period of N calendar days, and n the number of trading days in the period. Then, since only trading days contribute to volatility, we have:

$$\sigma_X^* = \sigma_X \sqrt{n} \tag{1}$$

However, when looking at calculating the mean, we note that non-trading days have an impact on this mean, not only the number of trading days as in the case of variance.

Thus, the mean for an N-day period could be seen as consisting of two components: the total mean of n trading days and the total mean of (N-n) non-trading days. The first component is the 1-day mean multiplied by the number of trading days, i.e., $nE(r_x)$. Regarding the second component, in the light of the CAPM, the expected return consists of the risk free rate of return and a risk premium, representing compensation for investment time and for the risk assumed, respectively. During a non-trading day, an investment will require compensation for time value of money only. However, since risk or volatility is mainly caused by trading activities, the risk on a non-trading day can be ignored, and the risk premium required should be close to zero. Thus, the second component is equal to $(N-n)R_F$, where R_F is the daily risk free rate of interest. Therefore, in summary, we can calculate the mean for an N-day period as follows:

$$E^*(r_x) = n\,E(r_x) + (N-n)R_F \tag{2}$$

In some financial markets, such as foreign currency exchange market, trading is carried out seven days a week, and therefore N, the number of calendar days, is the same as n, the number of trading days. Since both the mean and standard deviation are adjusted by the same time scale, the investment horizon should have no impact on the portfolio decision.

However, in other financial markets, such as stock markets around the globe, trading is interrupted during weekends and stock market holidays, and therefore, number of trading days is less than the number of calendar days. For instance, there are 252

trading days on the New York Stock Exchange in one year, that is n=252 while there are N=365 calendar days. Thus, formula to convert 1-day mean and standard deviation to 1-year ones are:

$$E^*(r_x) = 252E(r_x) + 113R_F \tag{3}$$

$$\sigma_X^* = \sigma_X \sqrt{252} \tag{4}$$

Since the mean and standard deviation are adjusted by different time scales, the optimal portfolio selection may differ with different investment horizons. In next section, we will show some examples on this effect.

3. Effects of Time Horizon Changes

In this section, we show the effects of time horizon changes with several examples.

3.1.   Effects on Capital Allocation

For the purpose of illustration, we present a numerical example as follows:

From the historical daily data from December 1, 2009, to December 1, 2012, we obtained the estimates of the mean and standard deviation of the daily return on the S&P 500 as below:

$E(r_{S\&P500}) = 0.000407$

$\sigma_{S\&P500} = 0.011648$

For simplicity assume the risk free rate is R_f *=3% per annum, or R_f=0.000082 per day.[1]

Following Bodie et al (2012), we assume an investor has a utility function as follows:

$$U = E(r) - (A/2)\,\sigma^2$$

with a degree of risk aversion, A, equal to 4,

First, assuming that the investment horizon is one day, following Bodie et al. (2012), the optimal capital allocation between the risk free asset and the equity for the investor is:

$$y_d = [E(r_{S\&P500}) - R_{F]}/(A\,\sigma_{S\&P500}{}^2) = (0.000407 - 0.000082)/(4 \times 0.011648^2) = 60\%$$

This suggests investing 60% in the equity portfolio and the remaining 40% in T-Bills. Now, assuming that the investment horizon is one year and using Eq. (3) and (4), the annual mean and standard deviations are calculated as follows:

---

[1] 0.03/365=0.000082

$E*(r_{S\&P500}) = (252x0.000407)+(113x0.000018)=0.1046$

$\sigma*_{S\&P500}= 0.011648x252^{1/2}=0.1849$

Similarly, by following Bodie et al.(2012) the optimal allocation between the risk free asset and the equity is:

$y_a = (E*(r_{S\&P500})-R*_f)/(A \sigma*_{S\&P500}^2) = (0.1046- 0.03)/(4x0.1849^2) = 55\%$

This suggests investing 55% in the equity portfolio and the remaining 45% in T-Bills. The above simple example shows that the investment horizon could affect capital allocation decision considerably.

3.2.   Effect on Performance Measurement

On a daily basis, the Sharpe ratio of an asset is given by:

$S_x = (E(r_X)-R_F) / \sigma_X$

and on an annual basis, the Sharpe ratio is

$S^*_x = (E^*(r_X) - R_f^*) / \sigma^*_X$

Since

$E^*(r_X) - R^*_F = nE(r_x) + (N-n)R_F - NR_F = n(E(r_X)-R_F)$

the annual Sharpe ratio becomes:

$S^*_x = n(E(r_X) - R_F) / \sigma_X\, n^{1/2} = n^{1/2}\, S_x$

The above equation indicates that, though the value of the Sharpe ratio is dependent on the investment horizon, the ranking among securities traded in the same market will remain the same. For example, for two portfolios of NYSE traded stocks, if one is better than the other based on the daily Sharpe ratio, the same ranking will be maintained if the annual Sharpe ratio is used. However, when we compare the performances of securities traded in different markets with different number of trading days, the investment horizon may affect their ranking.

For the purpose of illustration, we can look at the following numerical example:

Consider a stock X traded on NYSE and a foreign currency Y, traded on the FX market. Assume that the means and standard deviations of the annual rate of returns on X and Y are given as follows:

Stock X $\quad\quad E^*(r_X) = 20\%, \quad \sigma^*_X = 30\%$

Currency Y: $\quad E^*(r_Y) = 12\%, \quad \sigma^*_Y = 8\%$

The risk free interest rate is $R^*_F = 4\%$ per annum.

The number of calendar days is N=365, while the number of trading days for X is n_x=252 and that for Y is n_y=365. Then on an annual basis, the Sharpe ratios of X and Y are as follows:

$$S_X^* = (E^*(r_X) - R_f^*) / \sigma^*_X = (0.2 - 0.04)/0.3 = 0.53$$

$$S_Y^* = (E^*(r_Y) - R^*_F) / \sigma^*_Y = (0.12 - 0.04)/0.14 = 0.57$$

Since $S_X^* < S_Y^*$, we conclude that currency Y has better performance than stock X. If we perform the same analysis on a daily basis, using equations (2) and (3), the means and standard deviations of the daily rate of returns on X and Y are calculated as follows:

Stock X: $\quad\quad E(r_X) = 0.000745, \quad \sigma_X = 0.018898$

Currency Y: $\quad E(r_Y) = 0.000329, \quad \sigma_Y = 0.007328$

The risk free interest rate is $R_F = 0.000110$ per day.

Then the Sharpe ratios of X and Y are as follows:

$S_X = (0.000745-0.000110)/0.018898=0.0336$

$S_Y = (0.000329-0.000110)/0.007328=0.0299$

Since $S_X > S_Y$, we conclude that stock X has better performance than currency Y.

The above example indicates that the difference in investment horizon may affect the performance ranking of the assets.

One note worth mentioning in this context is that since beta is independent of time horizon, investment horizon should not affect the performance ranking given by the beta- based performance measures, such as Treynor ratio.

4. Conclusions and Suggestions

When using the mean-variance framework in portfolio management, it is common to estimate the 1-day means and variances of stocks and other assets, and then convert them to ones with longer horizons, such as annual means and variance. While the conversion of variances has been widely addressed and there is an approach popularly accepted, the conversion of means has been rarely addressed.

In this paper, we present a formula to convert 1-day means to 1-year means or means with other horizons, based on the CAPM model. Then, with this formula, we

investigate the impact of time horizon changes to the optimal capital allocation decision and to the performance evaluation, with numerical examples.

We show that the optimal capital allocation decision can be different for a particular investor if he or she would change his or her investment horizon. In addition, when the Sharpe ratio is used to evaluate the performance of portfolios, the choice of the time horizon may significantly affect the evaluation.

While we present a simple approach for the conversion of means with different time horizons, the issue is much more complex. As we are working in the mean-variance framework, the expected return consists of two distinguishable components: the risk-free rate and market risk premium. Thus, since market risk is mainly caused by trading activities, risk premium on non-trading days can be safely ignored. Consequently, the mean on a non-trading day should include only the risk-free interest rate. However, many recent studies have shown that there are some other factors which also contribute to mean levels. One of the most significant developments in this direction of researches is the Liquidity-based Asset Pricing Model (LAPM) developed by Amihud and Mendelson (1986), Acharya and Pedersen (2005), and Liu (2006), among others, which indicate that illiquidity is priced as a security characteristic and/or as a risk factor and such, the illiquidity premium should be added to the expected returns. In the light of the LAPM model, non-trading days should have higher illiquidity premium than trading days, although they have lower market risk premiums. How to incorporate these issues into the conversion formula for means will be a direction of our future research.

References

[1] V. Acharya and L. Pedersen, "Asset pricing with liquidity risk," *Journal of Financial Economics*, vol. 77, 2005, pp. 375–410.
[2] Y. Amihud and H. Mendelson, "Asset pricing and the bid–ask spread," Journal of Financial Economics, vol.17, 1986, pp.223–249.
[3] Z. Bodie, A. Kane and A. Markus, "Investments," 9th ed, McGraw-Hill Ryerson Press, 2011.
[4] F. Diebold, A. Hickman, A. Inoue and T. Schuermann, "Converting 1-day volatility to h-day volatility: Scaling by h is worse than you think," *University of Pennsylvania working paper,* 1997.
[5] Fama, E., "The behavior of stock-market prices," *Journal of Business*, vol.38, no.1, 1965, pp. 34-105.
[6]K. French, "Stock returns and the weekend effect," *Journal of Financial Economics*, vol. 8, 1980, pp. 55-69.
[7]K. French and R. Roll, "Stock return variances: The arrival of information and the reaction of traders," *Journal of Financial Economics*, vol. 17, 1986, pp. 5-26.
[8]J. Hull, "Options, futures, and other derivatives," 8th ed., *Prentice Hall,* 2012.
[9] J.P. Morgan, "RiskMetrics—Technical Document," 4th ed., New York, 1996.
[10]J. Lintner, "The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets," *Review of Economics and Statistics*, vol.47, 1965, pp.13-37.
[11]W. Liu, "A liquidity-augmented capital asset pricing model," *Journal of Financial Economics*, vol. 82, 2006, pp. 631–671.
[12]J. Mossin, "Equilibrium in a capital asset market," *Econometrica*, vol. 35, 1966, pp.768-783.
[13]R. Roll, "Orange juice and weather," *American Economic Review*, vol.74, no.5, 1984, pp.861-880.
[14]W. Sharpe, "Capital asset prices: A theory of market equilibrium under conditions of risk," *Journal of Finance*, vol.19, 1964, pp. 425-442.
[15]C. Smithson and L. Minton. "Value-at-risk: The first of two articles on value-of-risk, outlining the different ways of calculating this popular risk measure." *RISK-LONDON-RISK MAGAZINE LIMITED-* 9, 1996, pp. 25-28.