A mathematical model for the pricing of derivative financial products: the role of the banking supervision and of the model risk

Abstract

This paper aims to identify an innovative procedure to assess the model risk of a derivative financial product. More precisely, the authors, after briefly discussing the role of banking supervision, present a framework to estimate the model risk at the current time of a bond and they distinguish its systematic component. This model, consisting of two different expressions of the security price, is based on the hypothesis that the underlying interest rate is a swap rate that may be represented also as an AR(1)-GARCH(1,1) process. Moreover, the authors show that their model can be used for forecasting purposes.
In reality, the aim of this study is to investigate the link between pricing models of financial derivatives and model risk and to highlight that this latter depends on the volatility of the security and on the price distribution.

Keywords: Model risk; Pricing model; Reverse Floater; Financial derivatives; Banking supervision
JEL Classification: C02; C60; G17; G20

1. Introduction

The innovations introduced in recent years, promoted by large foreign banks, have certainly created new opportunities for international financial intermediaries, but, at the same time, have opened the way for new problems; among these, the difficulties relating to the pricing of increasingly complex and articulated financial products.
For the evaluation of these different financial products, or rather for the determination of their price, different theoretical models can be applied, the use of which can, however, generate non-homogeneous prices, generating serious problems, especially in relation to the capital endowment that the issuing institutions must meet in order to comply with the regulations introduced with the Basel II agreement.
In other words, due to the difficulties associated with the pricing of increasingly complex and articulated financial products, the use of different theoretical models, each of which can be attributed a different degree of reliability (where reliability means the ability to generate a price compatible with that of the market), can give rise to evaluations that are also very different from each other and, therefore, only after having carefully analyzed the characteristics and limits of the model to be adopted it is possible to establish a realistic price for the product to be evaluated. So, whatever the model chosen for pricing, the risk of incurring in an evaluation error, due to the wrong choice of the pricing model used or to a choice of a model that is not very consistent with the valuations, can be very high; if, on the other hand, one of the models used has a higher reliability margin than the others, the risk of incurring an evaluation error, due to an incorrect choice of the pricing model used, can be significantly reduced and, in some cases, it can be reduced so far as to cancel itself and of being almost certain that the model chosen is the right one for implementing the pricing procedure.
Following these considerations, the problem of quantifying the uncertainty arises, the uncertainty connected with the choice of the model to be adopted for the evaluation, that is, the problem of quantifying the so-called Model Risk. Indeed, in parallel with the creation of more diversified financial products and the development of new markets for these products, pricing models and risk measurement models, used as risk management tools, have also become increasingly complex. However, several major financial institutions have often reported losses deriving from the use of these models and this has drawn attention to the resulting risk, the model risk, which can lead to a capital endowment lower than that necessary to face the risks actually incurred by the bank or could lead to allocative choices not based on a prudent management.
In conclusion, for the evaluation of financial securities are used mathematical models whose use can give rise to the model risk that arises, mainly for two reasons:
a) the model may have been built as intended, but may have fundamental errors in its design and implementation and produce inaccurate results with respect to the design objective and intended use.
b) the model could be used incorrectly or inappropriately
To all this is also added the inability, from the corporate bodies, to understand the limitations underlying each model, even if, it should be noted, there are no explanatory and forecast models suitable for each type of evaluation of a financial product or, more precisely, there are no models that can be determined in

advance, because the adequacy of the model and the tools to be used must be chosen based on the structural characteristics of the product.
Therefore, only after having carefully analyzed the characteristics and the limits of the model to be adopted, it is possible to establish a realistic price for the product to be evaluated, namely it is possible to attribute to one of the models in question a weight, in probabilistic terms, the most as high as possible.
Correctly estimating the model risk associated with a particular portfolio or security is essential for a sound assessment of the operational risk in compliance with the rules imposed by the Basel committee. The authors' contribution, as well specified in Section 2 below, aims to fill a gap existing in the current literature by introducing a new model for calculating the model risk associated with a portfolio of Reverse Floater bonds.

2. Research aims

As already mentioned, the model risk is represented by the possibility that the price of a financial instrument is materially sensitive to the choice of the evaluation methodology. In the case of complex financial instruments, for which there is no standard evaluation methodology on the market, or in particular periods in which new evaluation methodologies are established on the market, it is possible that different methodologies, although they evaluate the financial instruments consistently, provide different price values, especially for derivative instruments. In other words, for the pricing of increasingly complex and articulated financial products, different theoretical models can be used and their use may carry out very different evaluations, and, consequently, the problem of quantifying the phenomenon of the model risk emerges and namely the problem of quantifying the uncertainty connected with the choice of the model to be adopted for the purposes of the assessment.
This paper intends to highlight the critical issues underlying the valuation procedures of a financial product with significant income potential which, however, incorporates high degrees of complexity such as to prevent the identification of a robust and unambiguous calculation process.
In reality, the paper aims to identify an innovative procedure for assessing the model risk of a derivative financial product, or rather, the aim of this study is to investigate the link between pricing models of financial derivatives and model risk.
More precisely, the authors' aim is to draw, on the model risk, some considerations that can be extended to any pricing procedure of financial instruments for which the methods of price determination are not unique. In other words, the paper presents some general considerations on the model risk, considerations according to which the only way to decisively reduce it is a careful analysis of the financial instrument and identification, with the highest possible safety margin of the model that best suits the specific situation.
In the paper, in particular, the authors deal with a specific type of bonds, the Reverse Floater that fall within the broader class of so-called structured bonds; their purpose is to offer a model risk estimation and forecasting procedure for these particular financial instruments using two different expressions of the price built on the assumption that the underlying interest rate is a swap rate and that this swap rate is an AR (1) - GARCH (1,1) process. The topic of this manuscript is important because in the literature there are few contributions (many of them outdated) in which the pricing models of the Reverse Floaters are presented and in which the model risk associated with these particular security is studied.
The paper is organized as follows. Section I offers a brief introduction to the concept of model risk. In section 2 are exposed the research aims, the method and the results of this manuscript. Section 3 discusses the importance of quantifying the model risk for the purpose of complying with the capital requirements established by supervisory regulations. Section 4 offers an examination of the pricing models of derivative instruments, with particular reference to Reverse Floater bonds. In paragraph 5, in order to estimate and predict the model risk of a Reverse Floater, two different expressions of the price of the latter are presented.
In the same Section, is also proposed a minimization strategy of the Model risk of a reverse floater portfolio.
Section 6 validates the assumptions on which the two different expressions of the price used to estimate the model risk of the Reverse Floater were built. Finally, section 7 summarizes the results obtained by the authors.

3. The model risk and the role of the banking supervision

The model risk phenomenon is a kind of risk contemplated by the "Supervisory Instructions for Banks" introduced by the Bank of Italy; in fact, consistently with the supervisory guidelines expressed internationally, banking intermediaries are required to comply with capital requirements, aimed at dealing with the losses that may arise from trading on the markets.

In other words, the capital requirements envisaged constitute a prudential prescription: compliance with prudential rules must be accompanied by procedures and control systems that ensure sound and prudent management of this type of risk.

The prudential discipline for banks and banking groups, organically revised following the changes in the international regulations, takes into account the evolution in risk management methods by financial intermediaries. The structure of the prudential regulation is based on "three pillars". The first introduces a capital requirement to face the typical risks of banking and financial activities (credit, counterparty, market and operational); and for this reason, alternative methods for calculating capital requirements are provided for, characterized by different levels of complexity in risks measurement and in organizational and control requirements. The second pillar requires banks to have a strategy and a process for controlling capital adequacy, both current and prospective, leaving the Supervisory Authority with the task of verifying the reliability and consistency of the related results and of adopting, where the situation requires it, the appropriate corrective measures. Finally, the third pillar introduces disclosure obligations to the public regarding capital adequacy, exposure to risks and the general characteristics of the related management and control systems.

This regulatory framework, based on a renewed system of rules and incentives, allows the objectives of prudential regulation to be pursued more effectively. In fact, it ensures an accurate measurement of a wider range of risks and a capital endowment more closely commensurate with the actual degree of risk exposure of each financial intermediary; stimulates banks to improve management practices and risk measurement techniques, also in consideration of possible capital savings; it favors competitive parity, through a greater extension of the activities and techniques subject to harmonization. In other words, in line with the supervisory guidelines expressed internationally, the obligation is imposed for banks to comply with capital requirements, aimed at coping with the losses that may derive from operations on the markets regarding financial instruments, currencies and goods. The capital requirements provided for in these provisions fall within the scope of the so-called "First pillar" (minimum capital requirements) and therefore constitute a minimum prudential requirement: compliance with prudential rules must be accompanied by procedures and control systems that ensure sound and prudent management of this type of risk.

The calculation of the minimum endowment of assets can be carried out using a standard methodology or using a methodology based on internal models, subject to compliance with organizational and quantitative requirements and subject to authorization by the Supervisory Authority.

The standard method to compute the minimum capital requirements is based on the so-called *building block* approach, according to which separate capital requirements are identified for the different types of risk.

As an alternative to the standardized methodology, there is a methodology based on internal models: it is a method that, based on statistical procedures, leads to the calculation of the "value at risk" (VaR). In other words, banks can calculate capital requirements based on their own internal models, provided that these meet certain conditions and are explicitly recognized by the National Supervisory Authority

The number of intermediaries that use internal models to calculate the requirement is limited but the majority of them have made investments in recent years to equip themselves with VaR-type models, used for management purposes and not strictly for supervisory purposes. The use of these models can, however, give rise to model risk phenomenon that could lead to a capital endowment lower than that necessary for the risks actually borne by banks, or induce allocative choices not based on prudent management. In this regard, the Supervisory regulations provide for quantitative and qualitative requirements to deal with this form of risk and in particular the principle according to which mere compliance with statistical requirements is not a sufficient condition to protect the bank from market risks is consolidated.

In this context, therefore, the phenomenon of the model risk takes on particular importance, a form of risk contemplated by the European Banking Authority, deriving from the models that the financial institution uses to evaluate derivatives.

Toshinao and Toshiyasu [1] analyze the model risk separately in pricing models and risk measurement models. In pricing models, the model risk is defined as "the risk arising from the use of a model which cannot accurately evaluate market prices, or which is not a mainstream model in the market." In risk measurement models, the model risk is defined as "the risk of not accurately estimating the probability of future losses."

Derman [2], one of the most authoritative authors in this field of research, states that the origin of the model risk should be sought in the impossibility of determining with certainty the trend of a stock index or of an interest rate, and that there is no procedure able both to cancel the model risk and to provide an exact value for each assessment. He also says that only a deep knowledge of the subject allows a correct implementation of the model to be used and the achievement of results close to reality. In short, after having quantified the model risk of some theoretical pricing models, it is compared with the market price and critical insights are drawn.

Rebonato [3] defines model risk as "the risk that there will be a significant difference between the mark-to-model value of a complex and / or illiquid instrument and the price at which the same instrument is revealed to have been traded on the market ".

4. The pricing of particular derivative financial instruments: the Reverse Floater bonds

In recent years there has been an explosive growth in financial derivatives and, in particular, in structured bonds, financial products that simultaneously present characteristics typical of bonds and standard or exotic financial options and, therefore may be linked to the performance of equities, indices, funds or may be written on interest rates or bonds with and without coupons. Structured bonds are so called because the subscriber, with a single contract, defines his investment by taking on multiple financial instruments.

This paper deals with a specific type of structured bonds: the Reverse Floater, structured bonds with guaranteed capital and variable yield with periodic coupons and very high maturities. Reverse Floaters owe their name to the fact that they are characterized by a pay-off structure inversely linked to the trend of the interest rate taken as a reference; namely they have a particular indexing mechanism through which their market value is determined: as the level of interest rates decreases, their value increases according to a law of inverse proportionality. For this reason, they show a strong commercial attractiveness and become the preferred tools by banks and other financial institutions to raise capital. In fact, in periods in which interest rates are at particularly low values, they show extraordinary levels of growth in transaction volumes.

The Reverse Floaters are characterized by a long maturity (15-20 years) and by the payment of fixed initial coupons which provide return rates significantly higher than the yields offered by the stock market at the time of the issue. During the first years of life of the bond, the holder periodically receives a predetermined coupon which usually stands at levels markedly above the current interest rate (to compensate for the risk associated with variable coupons and to make the loan more attractive to the public); the subsequent coupons, on the other hand, are indexed at a short-term interest rate, generally represented by the Libor o Euribor rate on 6 or 12 month basis. The coupon rate is calculated as the difference between a constant rate fixed at the issue and a variable reference rate calculated at the time of detachment of the coupon.

Therefore, they move from fixed coupons to variable coupons and from the opposite trend with respect to short-term interest rates. The term “reverse” then indicates the inversely proportional ratio of the coupon in the second period, in the sense that the higher the reference rate is, the lower the coupon is (reverse indexation). In summary, the return of the Reverse Floaters is nothing more than the payment of a premium for the assumption of a risk by the purchaser of the bond, a risk constituted by the future trend of market rates and in particular by their increase which, for of the mechanism of the difference between the fixed rate and the variable rate, would lead the subscriber to receive an effective coupon lower than the market one and, at the limit, not to receive any coupon in the event that the variable rate is equal to or higher than the fixed rate initially established. In other words, the subscriber would collect a lower price than that paid initially and, therefore, would be subject to a capital loss.

The evaluation of the price of the Reverse Floaters is particularly complex because the underlying is represented by an interest rate and the pricing is linked to the complexity of the stochastic processes used to describe the dynamics of the reference rate. These processes determine the value of the elementary components that make up the Reverse Floater and consequently affect the determination of the price of the security itself [4].

For the valuation of Reverse Floater, the *Building Block approach* is used, although, in the financial world, another methodology is often used, namely the *Full Evaluation approach*.

The Building Block approach assumes that the financial product in question can be broken down into elementary components for which the calculation of the price determination is simple; the price of the complex financial product is, in fact, obtained as the algebraic sum of the prices of the individual components.

Obviously, the breakdown should be done so that the pay-off of the original financial structure can be replicated through the combination of several elementary financial instruments or replicated by the synthesis of several simple elementary instruments; consequently the replication portfolio and the original structured product must have the same price. Once the structure of the elementary components has been identified, it is possible to proceed with a pricing analysis through the use of standard evaluation formulas.

Ultimately, the Building Block approach presupposes:

a) the breakdown of the structured product into its elementary components;

b) the enhancement of the individual elementary components;

c) the recomposition of the values of the individual elementary components into a single price called the price of structured product.

The main pricing model of financial options that exploits the building block approach is precisely that of Black-Schols [5]. In other words, as a pricing tool, many market players continue to use the Black-Schols model with full knowledge of its limitations. Indeed, the Black-Scholes model [6] [7], which is a standard pricing model for options, assumes that the prices of the underlying assets fluctuate according to a lognormal process, while the fluctuations in actual market prices do not follow necessarily this process.

In the literature, in addition to the previous model, that by Cox, Ingersoll and Ross [8] is also well known.

Scholars have also proposed numerous extensions of the Black-Scholes model generalizing it to a broad class of stochastic processes [9] [10] [11] [12] [13]. Among these, the one-factor stochastic model of Black, Derman and Toy [14] is particularly interesting and it is based on the hypothesis that a single short-term interest rate determines the future trend of all other rates. An extension of the Black, Derman and Toy model is the one-factor Black and Karasinsky model [15]. The importance of the latter is due to the fact that it can be used to price not only options but also zero coupon bonds.

The models by Merton [16], by Merton, Scholes and Gladstein [17] [18], by Health, Jarrow and Morton [19] and of Nelson and Ramaswamy [20] also deserve to be cited by virtue of their relevance.

5. The model risk of the Reverse Floater bonds with two expressions of the price

In this paragraph, to estimate and to predict the model risk of a Reverse Floater, the authors present a theoretical model consisting of two different expressions of the price of the security.

The model in question is written considering that, in general, the model risk at the current time depends on the volatility of the security and on the distribution of the prices.

The authors initially present the expression of the model risk associated with any security, at the current time with J expressions of the price. The model risk at the valuation date m, with $m = 1, \dots T$, given j different pricing models, with $j = 1, \dots, J$, can be calculated as follows:

$$MR_m = \sum_{m=1}^{T}\sum_{j=1}^{J} p_j \, {P_{j,m}}^2 - \sum_{m=1}^{T}\sum_{j=1}^{J}\left(p_j P_{j,m}\right)^2$$

where p_j is the relative probability of the price obtained with the j-th expression of the price of the security and $P_{j,m}$ is the security price calculated with the j-th expression of the price of the security at time m.

The probability p_j can be estimated using the appropriate Kernel techniques (cfr. Appendix Section I).

Subsequently, however, the authors present two expressions of the price of the security used to calculate the model risk associated with the Reverse Floater at current time m.

5.1 First expression of the price of the bond ($P_{1,m}$)

Suppose you want to price a reverse floater bond, with maturity T, that for the first s periods (with $s = 1, \dots, S$) pays a fixed coupon equal to M and in the remaining t periods (with $t = s + 1, \dots, T$) pays a coupon C_t calculated as:

$$C_t = \bar{r} - kr_t$$

where $\overline{r}$ is a fixed rate, k is a multiplier and r_t is swap interest rate. The authors price the Reverse Floater bond with reference to interest rate swaps because this is the approach most commonly used in financial markets. In fact, an issuer of a Reverse Floater bond can easily replicate the cash flows on that bond with a fixed rate bond and an interest rate swap, namely he receives a fixed rate and pays a floating rate. Also assume that for r_t the hypotheses of Engle [21] hold (cfr. Appendix Section II) and that, therefore, it can be represented as an AR (1) - GARCH (1,1) process:

$$\sigma_t^2 = \omega + \alpha_1 r_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

with $\omega > 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$ and

$$\mu_t = \varphi_0 + \varphi_1 r_t$$

with $\varphi_1 \neq 1$.
The price of the reverse floater at time m, with $m = 1, \dots, S, S+1, \dots, T$, under these assumptions is given by the following expression:

$$P_{1,m} = \sum_{s=1}^{S} \frac{M}{(1+i_s)^s} + \sum_{t=S+1}^{T} \frac{\bar{r} - k(\varphi_0 + \varphi_1 r_t + \omega + \alpha_1 r_{t-1}^2 + \beta_1 \sigma_{t-1}^2)}{(1+i_t)^t} \tag{1}$$

where i_s and i_t are the same swap interest rate, respectively, at the maturity s and at maturity t.

5.2 Second expression of the price of the bond ($P_{2,m}$)

Assume, again, that one want to price a reverse floater with maturity T, that for the first s periods (with $s = 1, \dots, S$) pays a fixed coupon equal to M and in the remaining t periods (with $t = s+1, \dots, T$) pays a coupon C_t calculated as:

$$C_t = \bar{r} - k r_t$$

Furthermore, consider a fixed-income security issued at time 1, that has its maturity in T and pays a coupon equal to F in each period. Let $B_m(F,T)$ with $m = 1, \dots, S, S+1, \dots, T$ the price of that fixed income security. According to Filigrana [22], the price of the reverse floater at time m (with $m = 1, \dots, S, S+1, \dots, T$) under the hypotheses considered above is the following:

$$P_{2,m} = B_m(F,T) - \sum_{s=1}^{S} \frac{F-M}{(1+i_s)^s} - \gamma \sum_{t=S+1}^{T} \frac{\bar{r} - k r_t}{(1+i_t)^t} \tag{2}$$

where, similarly to the previous expression of the price, i_s and i_t are the same swap interest rate, respectively at maturity s and at maturity t and γ is a constant.

5.3 The model risk with two expressions of the price of the bond

The authors calculate now the model risk at current time using the two previous expressions of the price of the security.
The risk model at time m with only two expressions of the price is equal to:

$$MR_m = p_1 P_{1,m}^2 + p_2 P_{2,m}^2 - \left(p_1 P_{1,m} + p_2 P_{2,m}\right)^2$$

This formula represents the variance of the Reverse Floater prices at time m and it is calculated by adding the squares of the prices weighted for the respective probabilities and subtracting the square of the expected value of the prices.
Given the first expression of the price $P_{1,m}$ (expressed by (1)) and given the second expression of the price $P_{2,m}$ (expressed by (2)), replacing in the previous equation the (1) and the (2) it follows:

$$MR_m = p_1\left(\sum_{s=1}^{S}\frac{M}{(1+i_s)^s}+\sum_{t=S+1}^{T}\frac{\bar{r}-k(\varphi_0+\varphi_1 r_t+\omega+\alpha_1 r_{t-1}^2+\beta_1\sigma_{t-1}^2)}{(1+i_t)^t}\right)^2 + p_2\left(B_m(F,T)-\sum_{s=1}^{S}\frac{F-M}{(1+i_s)^s}-\gamma\sum_{t=S+1}^{T}\frac{\bar{r}-k[1+r_t]}{(1+i_t)^t}\right)^2 - \left[p_1\left(\sum_{s=1}^{S}\frac{M}{(1+i_s)^s}+\sum_{t=S+1}^{T}\frac{\bar{r}-k(\varphi_0+\varphi_1 r_t+\omega+\alpha_1 r_{t-1}^2+\beta_1\sigma_{t-1}^2)}{(1+i_t)^t}\right) + p_2\left(B_m(F,T)-\sum_{s=1}^{S}\frac{F-M}{(1+i_s)^s}-\gamma\sum_{t=S+1}^{T}\frac{\bar{r}-k[1+r_t]}{(1+i_t)^t}\right)\right]^2 \quad (3)$$

From this expression, follows that the model risk at current time depends on the current value of the underlying interest rate, on the value of the underlying interest rate at the previous time, on the volatility of the underlying interest rate at the previous time and on the current value of the discount rate.
From the latter equation, it is possible to calculate the systemic component of the model risk.
In fact, putting $p_1 = p_2 = 1$, it is obtained:

$$MR_m = 2\left(\sum_{s=1}^{S}\frac{M}{(1+i_s)^s}+\sum_{t=S+1}^{T}\frac{\bar{r}-k(\varphi_0+\varphi_1 r_t+\omega+\alpha_1 r_{t-1}^2+\beta_1\sigma_{t-1}^2)}{(1+i_t)^t}\right)\left(B_m(F,T) - \sum_{s=1}^{S}\frac{F-M}{(1+i_s)^s}-\gamma\sum_{t=S+1}^{T}\frac{\bar{r}-k[1+r_t]}{(1+i_t)^t}\right) \quad (4)$$

that is precisely the value of the model risk at current time when the reverse floater price is exactly equal to the theoretical one given by the two previous expressions of the price.

5.4 The forecast of the model risk

In this paragraph, however, the authors, using the previous expressions of the price, estimate the model risk in a future time.
They show that, given the (1), it is possible to estimate the price of the reverse floater at future time $m+h$ as:

$$P_{1,m+h} = \sum_{s+h=1}^{S}\frac{M}{(1+i_t)\left(1+f_{t,t+h}\right)^h}+\sum_{t+h=S+1}^{T}\frac{\bar{r}-k\left[\hat{\mu}_{T,h}+\hat{\sigma}_{T,h}^2\right]}{(1+i_t)\left(1+f_{t,t+h}\right)^h} \quad (5)$$

where:

$$\hat{\mu}_{T,h} = \varphi^h r_T$$

and:

$$\hat{\sigma}_{T,h}^2 = \omega\sum_{i=1}^{h-1}(\alpha_1+\beta_1)^{i-1}+(\alpha_1+\beta_1)^{h-1}[\omega+\alpha_1 r_T^2+\beta_1\sigma_T^2]$$

are, respectively, the forecast at time h of the average component of r_t and the forecast at future time h of the conditional variance of r_t.
Similarly, given the (2), it is possible to calculate the price of the reverse floater at future time h as follows:

$$P_{2,m+h} = B_m(F,T) - \sum_{s+h=1}^{S}\frac{F-M}{(1+i_t)\left(1+f_{t,t+h}\right)^h}-\gamma\sum_{t+h=S+1}^{T}\frac{\bar{r}-k\left[(1+r_t)\left(1+s_{t,t+h}\right)^h\right]}{(1+i_t)\left(1+f_{t,t+h}\right)^h} \quad (6)$$

where $s_{t,t+h}$ and $f_{t,t+h}$ are, respectively, the forward rate relative to r_t and the forward rate relative to i_t.
Finally, the (5) and the (6) allow to estimate the model risk at future time h as follows:

$$MR_{m+h} = p_1 \left(\sum_{s+h=1}^{S} \frac{M}{(1+i_t)(1+f_{t,t+h})^h} + \sum_{t+h=S+1}^{T} \frac{\bar{r} - k[\hat{\mu}_{T,h} + \hat{\sigma}^2_{T,h}]}{(1+i_t)(1+f_{t,t+h})^h} \right)^2 + p_2 \left(B_m(F,T) - \sum_{s+h=1}^{S} \frac{F-M}{(1+i_t)(1+f_{t,t+h})^h} - \gamma \sum_{t+h=S+1}^{T} \frac{\bar{r} - k\left[(1+r_t)(1+s_{t,t+h})^h\right]}{(1+i_t)(1+f_{t,t+h})^h} \right)^2 - \left[p_1 \left(\sum_{s+h=1}^{S} \frac{M}{(1+i_t)(1+f_{t,t+h})^h} + \sum_{t+h=S+1}^{T} \frac{\bar{r} - k[\hat{\mu}_{T,h} + \hat{\sigma}^2_{T,h}]}{(1+i_t)(1+f_{t,t+h})^h} \right) + p_2 \left(B_m(F,T) - \sum_{s+h=1}^{S} \frac{F-M}{(1+i_t)(1+f_{t,t+h})^h} - \gamma \sum_{t+h=S+1}^{T} \frac{\bar{r} - k\left[(1+r_t)(1+s_{t,t+h})^h\right]}{(1+i_t)(1+f_{t,t+h})^h} \right) \right]^2 \quad (7)$$

The equation (7) indicates that the model risk at future time h depends not only on swap rates but also on forward rates. Furthermore, the dependent variables, unlike the (3), do not include the values at previous time of the underlying interest rate and the conditional variance, but the predicted value at time h of the average component and the conditional variance of the underlying interest rate .

In addition to what has been obtained, it is important to underline that some authors have highlighted that the trend of the model risk depends on the market uncertainty. More precisely, it tends to increase in periods of financial distress and becomes negligible in periods of low uncertainty [23].

This implies that also the model risk of the Reverse Floaters follows a well-determined temporal trend that is characterized by a very irregular trend that alternates maximum peaks with moments in which the value of the model risk is drastically reduced. More specifically, it is lower in the periods in which the price of the Reverse Floater is lower and higher in the years in which the value of the structured product rises, revealing a trend that, in many aspects, recalls the trend in the price of the stock. The reason of this trend can be found in the circumstance that the higher the price of the Reverse Floater, the greater the consequences of an incorrect choice of the valuation model implemented.

In fact, the model risk is calculated as a variance of the prices of the Reverse Floater obtained with the different assessments made, weighted by the weight attributed to the different models and, consequently, its value will be the greater the higher the values that fall into the equation. This obviously implies that higher values of the Reverse Floater price will correspond to higher values of the model risk.

5.5 Model risk minimization

In this subsection, the authors propose a minimization strategy of the Model risk of a reverse floaters portfolio. More precisely, they find the wealth to be invested in the Reverse Floater portfolio that minimizes the model risk borne by the investor.

Let W_m be the total wealth of the investor at time m. The wealth invested in reverse floaters priced using (1) is equal to:

$$W_{1,m} = n_{1,m} P_{1,m}$$

where $n_{1,m}$ is the number of reverse floaters purchased at the price $P_{1,m}$.

Similarly, the wealth invested in reverse floaters priced using (2) is equal to:

$$W_{2,m} = n_{2,m} P_{2,m}$$

where $n_{2,m}$ is the number of reverse floaters purchased at the price $P_{2,m}$.

The model risk associated with the portfolio can be written as:

$$MR_m = p_1 W_{1,m}^2 + p_2 W_{2,m}^2 - \left(p_1 W_{1,m} + p_2 W_{2,m}\right)^2 \quad (8)$$

similarly to what was done at the beginning of paragraph 5.3 with $P_{1,m}$ and $P_{2,m}$.
The share of wealth $W_{1,m}^*$ that should be invested in reverse floaters priced with (1) in order to minimize the Model risk MR_m associated with the portfolio, is (Cfr. Appendix, Section III):

$$W_{1,m}^* = \frac{p_2 W_{2,m}}{(1 - p_1)} \quad (9)$$

The share of wealth $W_{2,m}^*$ that should be invested in reverse floaters priced with (2) in order to minimize the Model risk MR_m associated with the portfolio, is (Cfr. Appendix, Section III):

$$W_{2,m}^* = \frac{p_1 W_{1,m}}{(1 - p_2)} \quad (10)$$

In other words, the (9) and the (10) express necessary and sufficient conditions to minimize the Model risk associated with the reverse floater portfolio. These expressions, in fact, indicate the share of wealth that should be invested in reverse floaters priced with the two pricing models proposed by the authors (equation (1) and equation (2)) in order to minimize the Model risk associated with the portfolio.

6. Verification of the hypothesis of the theoretical model

In this paragraph, the authors show two estimation outputs in order to demonstrate the validity of the hypothesis, formulated in the previous paragraph, that the swap rate r_t can be represented as an AR (1) - GARCH (1,1).
As a reference swap rate, the authors consider the ICE Swap Rates, namely the principal global benchmark for swap rates and spreads for interest rate swaps.
More precisely, the authors considered the time series of the ICE Swap Rates on a daily basis in the sample period 2021/04/06 – 2021/10/29. The data were taken from the dataset of the Federal Reserve of Saint Louis (FRED). Table 1 below shows the AR(1) estimate relating to ICE Swap Rates:

Table 1. AR(1) estimation for ICE Swap Rates.

Model AR(1) for ICE Swap Rates using observations 2021/04/06-2021/10/29					
Coefficients					
φ_0	− 0,182463 *** (3,81322e-09)				
φ_1	0,994274 *** (0,00681065)				
Root 1 Modulus	*Log-likelihood*	*R-squared*	*Adjusted R-squared*	*Mean*	*Standard deviation*
1,0024	395,2935	0,963435	0,963435	−0,267450	0,086713

As can be seen, both estimated coefficients are significantly different from zero, the root associated with φ_1 in modulus is greater than 1.
This result indicates that the assumption that the ICE Swap Rates follows an AR(1) process is correct.
After estimating the AR(1) on the ICE Swap Rates, the authors proceeded by estimating a GARCH (1,1) on the residuals of the same variable. Table 2 below shows the estimated values of the GARCH (1.1) relating to the residuals of AR(1) model of the ICE Swap Rates:

Table 2. AR(1)-GARCH(1,1) estimation for the AR(1) residuals of the ICE Swap Rates.

GARCH(1,1) for the AR(1) residuals of the ICE Swap Rates using observations 2021/04/06-2021/10/29 and with $\varepsilon_t = \frac{r_t}{\sigma_t} \sim N(0,1)$	
Conditional mean equation	
Coefficient	
μ_m	0,0545463 *** (0,000126937)
Conditional variance equation	
Coefficient	
ω	5,80580e-06 *** (8,48505e-09)
α_1	0,108666 ** (0,0474147)
β_1	0,887462 *** (0,0553304)

The mean component μ_m is significantly different from 0. The coefficient ω, although having a very small value, is significantly different from 0 with size $\alpha = 0{,}01$.
The two results just presented, therefore, demonstrate that the assumption according to which r_t ~ AR(1) - GARCH (1,1) is verified in the case of the ICE Swap Rates.
In conclusion, in this paragraph the authors proved that swap interest rates follow an AR(1)-GARCH(1,1) process, as assumed in the previous Section 5. It is important because this is a key assumption of their model.

7.Concluding remarks

The results presented in the previous paragraph allow us to draw some general considerations on the model risk, valid not only in the case of the Reverse Floater, but that may be extended to any pricing procedure of financial instruments for which the methods of price determination are not unique.
The authors have, in fact, shown that it is possible to arrive at a formula to calculate:
a) the model risk at the current time (expression (3)),
b) the model risk in a future time (expression (7)),
of a Reverse Floater using two different expressions of the price (given by expression (1) and expression (2)), under the assumption that the underlying interest rate is a swap rate that can also be represented as an AR (1) - GARCH (1,1) process.
Moreover, the authors proposed a minimization strategy of the Model risk of a reverse floaters portfolio (expressions (9) and (10)) and, finally, they verified that the hypotheses underlying their theoretical model are satisfied for the swap interest reference rate (ICE Swap Rates).
The innovative result of the authors' analysis is that the model risk at current time (including its systemic component) depends on the underlying interest rate at the current time, on the underlying interest rate at the previous time, on the volatility of the underlying interest rate at the previous time and on the discount rate at the current time. On the other hand, the model risk at the future time depends on: the underlying interest rate at the current time, the discount rate at the current time, the forward rate associated with the underlying interest rate, the forward rate associated with the underlying discount rate, the underlying interest rate expected at the future time and the volatility of the underlying interest rate expected at the future time.
From the analysis carried out by the authors, may be deduced that the only feasible way to reduce, in a decisive manner, the model risk is that which makes an accurate analysis of the financial instrument under exam, a detailed and meticulous study that allows to identify, with the highest margin of safety, which model is most suited to the concrete situation, so as to attribute to one of the models considered a weight in probabilistic terms that is as high as possible.
In reality, the use of mathematical models to minimize the exposure to the model risk, is the final step in a predefined corporate process of assumption and managenent of the risk within the company. This process is articulated on two levels; the first concerns the units responsible for defining and implementing the models, while the second directly concerns the corporate bodies responsible of the risk assumption ultimately.

The need to associate a risk process with the use of mathematical models represents, then, the real added value of pricing modeling and also one of the main reasons why the supervisory authorities at the international level welcome the disuse of standard methods in favour of the diffusion of internal models.
In conclusion, the debate on the applicability of the currently available modeling is, in any case, more heated than ever, and still far from definitive considerations; in fact, the awareness of the complexity of reality, together with the reduced forecasting capacity of the human being and its limited rationality, leave wide margin for improvement of the pricing procedures in an attempt to reduce, as far as possible, the model risk.

Conflict of interests

Author A declares that she has no conflict of interest.

Author B declares that he has no conflict of interests.

Ethical approval

This article does not contain any studies with human partecipants performed by any of the author.

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Appendix

Section I

Let $(P_{j,1}, P_{j,2}, \dots, P_{j,T})$ be the historical series relative to the jth expression of the price. The density function for P_j is:

$$\hat{f}_{b_j}(P_j) = \frac{1}{Tb_j}\sum_{m=1}^{T} K_{b_j}\left(\frac{P_j - P_{j,m}}{b_j}\right)$$

where $b_j > 0$ and K_{b_j} are, respectively, the bandwidth and the Kernel function.
Therefore:

$$p_j(P_{j,m} = 1) = \int_{-\infty}^{1} \hat{f}_{b_j}(P_j)\, d_{P_j}$$

Section II

The rate of return at time t is equal to:

$$r_t = \mu_t + \sigma_t z_t$$

where μ_t is the average component of the rate of return at time t and is calculated as the expected value of r_t conditionated to an information set I^{t-1}, that is $\mu_t = E(r_t|I^{t-1})$; σ_t^2 is the conditional variance of the rate of return at time t and it is computed as the conditional variance of r_t with respect to an information set I^{t-1}, namely $\sigma_t^2 = Var(r_t|I^{t-1})$; $z_t \sim D(0,1)$ is an error term.

Section III

Let consider the (8):

$$MR_m = p_1 W_{1,m}^2 + p_2 W_{2,m}^2 - \left(p_1 W_{1,m} + p_2 W_{2,m}\right)^2$$

In order to find the values of $W_{1,m}$ and $W_{2,m}$ that minimize MR_m, the authors calculate:

$$\frac{\partial MR_m}{\partial P_{1,m}} = 2p_1 W_{1,m} - 2p_1\left(p_1 W_{1,m} + p_2 W_{2,m}\right) = 0$$

and:

$$\frac{\partial MR_m}{\partial P_{2,m}} = 2p_2 W_{2,m} - 2p_2\left(p_1 W_{1,m} + p_2 W_{2,m}\right) = 0$$

From the two previous expressions, the coordinates of the only candidate point:

$$W_{1,m}^{*} = \frac{p_2 W_{2,m}}{(1 - p_1)}$$

and:

$$W_{2,m}^{*} = \frac{p_1 W_{1,m}}{(1 - p_2)}$$

It may easily be verified that this point is a minimum point. In fact, given the Hessian matrix:

$$H = \begin{bmatrix} \frac{\partial^2 MR_m}{\partial P_{1,m}^2} & \frac{\partial^2 MR_m}{\partial P_{1,m} \partial P_{2,m}} \\ \frac{\partial^2 MR_m}{\partial P_{2,m} \partial P_{1,m}} & \frac{\partial^2 MR_m}{\partial P_{2,m}^2} \end{bmatrix}$$

it results:

$$|H| = 4p_1(1 - p_1)p_2(1 - p_2) + 4p_1^2 p_2^2 > 0$$

Since $|H| > 0$ e $\frac{\partial^2 MR_m}{\partial P_{1,m}^2} > 0$, the point is a minimum point.