Analysing the optimal conversion boundary of convertible bonds close to maturity

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Abstract

This paper studies the short-time behavior of the optimal conversion boundary of convertible bonds using singular perturbation technique. The fundamental result is the analytic prediction of the optimal conversion price close to maturity. Even though the asymptotic expansion is valid for a short time interval, it complements the conventional approaches to evaluate this financial instrument at times that are not close to expiry. The analysis presented here is applicable to a wide range of nonlinear derivatives pricing problems.

AMS subject classification: 91G20, 91G30, 91G80

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1 Introduction

Convertible bonds (CBs) are hybrid financial instruments that can be converted into the bond issuing firm's common stocks with a preset conversion ratio, or hold the bond till maturity to receive coupons and the face value prescribed in the contract agreement. The investors may decide to convert the CBs into equity based on the price path followed by the underlying stock. A higher dividend per bond if converted often lead to a higher possibility to trigger conversion that dilutes existing equity holder's value [7]. There were also theoretical and empirical arguments for early conversion when investors faced short-sale costs, transaction costs, or funding costs [3, 11]. CBs have a financial structure that combines the characteristics of stocks and bonds. The dominant determinant of value depends on the prevailing market conditions. From the perspective of a borrower, CBs have the benefit of lower interest rate cost than the straight bond, and it offers a relatively cheap way for many companies to raise capital. However, there is a drawback that the issuer faces capital structure uncertainty. On the other hand, in return for a declined yield faced by investors, there is an upside participation in the performance of the equities of the issuing firm [3].

The pricing of CBs has long been acknowledged as a very challenging problem in quantitative finance because of the associated moving boundary, along with singular behaviour at expiry date. It is numerically challenging for most conventional pricing methods to effectively track the dynamic of CBs in the surrounding neighborhood of maturity date. For instance, when using the finite difference method [10, 17] and finite element method [8, 12], a fine discretization of the time domain is needed near expiry to obtain a reasonably accurate result [13]. This practice is not only computationally expensive but has limited accuracy. Financially, the exact location of critical conversion prices is crucial to the investors for financial decisions and hedging [1]. Therefore, it is desirable to further study the dynamics of this financial security near maturity date to complement the traditional pricing approaches.

The literature analyzing singular behavior of early exercise price of financial derivatives are limited. Among the works trying to study this phenomenon is Chen & Zhu [6]. Their work focused on American put options with no dividend payment modeled on the Black-Scholes framework. While this appears a source of inspiration, it is never optimal to convert CBs without dividend yield on the underlying before expiry. Financially, when both dividend and coupons payment are zero, the investors have no incentive to give up their conversion right early, then an American-style CB resembles its European counterpart. In a related study, Alobaidi & Mallier [2] analyzed zero-coupon American converts based on Vasicek and Cox-Ingersoll-Ross models. A matched asymptotic solution is obtained within the transitional region between the two co-existing states.

This paper aims to match solutions that are asymptotically valid on different regimes to derive a solution that is uniformly valid close to maturity. We consider standard CBs under the Black-Scholes model with zero coupons, which can be exchanged for one unit of stock at any time at or before the maturity date. The bonds pay an amount K, the face value, at maturity if the option to redeem is not exercised. The choice of our model allows the evaluation of our results within a framework that permits objective comparison with the existing works.

The remainder of this paper proceeds as follows. Section 2 states the underlying assumptions of the Black-Scholes model, and present the free boundary problem of CBs with the embedded early conversion right. In Section 3, we derive the asymptotic behavior of CBs near the expiration time. Section 4 discusses our explicit analytical results and concludes the paper.

2 Free boundary problem of CBs

Following the Grundy and Verwijmeren [9] empirical results that CBs pricing models assumed a perfect market, like Brennan and Schwartz [4], we adopt this perfect market setting. Under this setting, there is no transaction cost, both equity and bond holders have equal access to market information and trade continuously without arbitrage opportunities. We assume further that there is no senior or junior debt issued, only block conversion is allowed, possession of all convertibles is diffused, and default risk is neglected [1].

Let $V(S,t)$ denotes the value of a CB, which is a measurable function of the underlying stock price, S and time, t. Under the risk neutral measure, the stock price follows lognormal diffusion process

$$\frac{dS}{S} = \mu dt + \sigma dZ, \tag{2.1}$$

where the drift μ volatility σ are constants, and dZ denotes the increment of a standard Brownian motion process. In light of no-arbitrage arguments, the following free boundary problem modelled the pricing dynamics of the standard CBs

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV = 0, \quad 0 < S < S_f, \quad t > 0 \\
&V(S,T) = \max(nS, Z) \\
&V(S_f, t) = nS_f \\
&\frac{\partial V}{\partial S}(S_f, t) = n \\
&\lim_{S \to 0} V(S,t) = Ze^{-r(T-t)},
\end{aligned} \tag{2.2}$$

where r is the risk-free interest rate, K is the strike price, and δ is the continuous dividend payment. For simplicity, we assume $r - \delta > 0$. Problem (2.2) is defined on $S \in [0, S_f(t)]$, $t \in [0,T]$. For each $t \in [0,T]$, there exists a stock price S for which conversion before the final time T is optimal. This value defines a continuous curve S_f; a moving interface separating the region where it is advantageous to hold the bond from where exercise is optimal. The location of $S_f(t)$ at final time,T is given as $S_f(T) = \frac{K}{n}$. Intuitively, problem (2.2) is defined in a domain part of whose boundary is moving as time passes, and its behavior is similar to the so called Stefan problem [15, 14].

3 Matched asymptotic expansion of the optimal conversion boundary

In this section, we derive the matched asymptotic expansion of $S_f(t)$ from differential system (2.2) using the singular perturbation technique [5]. First, we transform (2.2) into an equivalent dimensionless form by introducing new variables:

$$x = \ln\left(\frac{S}{K}\right),\ x_f(\tau) = \ln\left(\frac{S_f(t)}{K}\right),\ Kv(x,\tau) = (V(S,t) - S)e^{k_1\tau},\ \text{and}\ \tau = \frac{\sigma^2(T-t)}{2}$$

With the new variable and conversion ratio $n = 1$, one can easily check that the free boundary problem (2.2) is transformed into the following problem:

$$\begin{aligned}
&\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k_1 - k_2 - 1)\frac{\partial v}{\partial x} - k_2 e^x e^{k_1\tau} \\
&p(x,0) = \max(1 - e^x, 0) \\
&\lim_{x\to-\infty} v(x,\tau) = 1 - e^x e^{k_1\tau} \\
&p(x_f(\tau),\tau) = 0 \\
&\frac{\partial v}{\partial x}(x_f,\tau) = 0,
\end{aligned} \tag{3.1}$$

where k_1 and k_1 are defined as $k_1 = \frac{2r}{\sigma^2}$, $k_2 = \frac{2\delta}{\sigma^2}$ respectively. Since $S_f(T) = K$, $x_f(0) = 0$. It should be remarked that the continuity of state variables, $V(S,t)$ and its first derivative with respect to S on $S_f(t)$ corresponds to the continuity of $v(x,\tau)$ and its first derivative on $x_f(\tau)$.

According to the essence of the so-called singular perturbation approach, we set $\tau = \epsilon T'$, where $T' = O(1)$ and ϵ is a small positive artificial parameter $0 < \epsilon \ll 1$. Then,we obtain

$$\begin{aligned}
&\frac{\partial v}{\partial T'} = \epsilon\frac{\partial^2 v}{\partial x^2} + \epsilon(k_1 - k_2 - 1)\frac{\partial p}{\partial x} - \epsilon k_2 e^{x+k_1\epsilon T'} \\
&v(x,0) = max(1-e^x, 0) \\
&\lim_{x\to-\infty} v(x,T') = 1 - e^x e^{\epsilon k_1 T'} \\
&v(x_f, T') = 0 \\
&\frac{\partial v}{\partial x}(x_f, T') = 0.
\end{aligned} \tag{3.2}$$

To construct the matched asymptotic solution, we naively treat Equation (3.2) as a regular perturbation problem. Then, we produce the outer approximation by assuming that the solution $v(x,T')$ of PDE system (3.2) can be expanded in powers of ϵ. Thus, we obtain an outer expansion which valid for $x < 0$:

$$v(x,T') = 1 - e^x(1 + k_1\epsilon T') + O(\epsilon^2) \tag{3.3}$$

As it is shown, the outer expansion breaks down as $x \to 0^+$, i.e, the outer solution is discontinuous in this regime. Because we cannot impose boundary conditions on the leading-order outer solution, we require inner boundary layer to satisfy these boundary conditions. To achieve this, we represent $v(x,T')$ by inner expansion with corner layer $x - x_{f_1} = O(\epsilon^p)$ and introduce a stretched or interior layer variable

$$y = \frac{x - x_f}{\epsilon^p},$$

where $y = O(1)$. Substituting y into the governing PDE contained in Equation (3.2) and balance the leading-order terms, we obtain $p = 1$. The problem $v(x, T') = v(y, T')$ becomes

$$\begin{aligned} &\epsilon \frac{\partial v}{\partial T'} - \frac{dx_f}{d\tau}\frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial y^2} + \epsilon(k_1 - k_2 - 1)\frac{\partial v}{\partial y} - k_2\epsilon^2[1 + y\epsilon + x_f + k_1\epsilon T'] \\ &v(0, T') = 0, \frac{\partial v}{\partial y}(y = 0, T') = 0 \end{aligned} \tag{3.4}$$

The technical justification for the inner expansion require the scaling $x = \epsilon^p X$ where $X = O(1)$. Adopting x into Equation (3.1) and equate the higher terms produces $p = 1/2$. The scaling $x = \epsilon^{1/2}X$ bridges between the outer region and the inner region near the moving boundary x_f.// To continue with the asymptotic analysis, we explore the solution of (3.4) in a series form:

$$v(x, T') = \epsilon^{1/2}v_0 + \epsilon v_1 + O(\epsilon^{3/2}), \tag{3.5}$$

where $v(x, T') = O(\epsilon^{3/2})$. Substituting $v(X, T')$ in Equation (3.4), with $x = \epsilon^{1/2}X$ and $x = y\epsilon + x_f$, we obtain sequence of leading-order PDE systems

$$\begin{aligned} &\frac{\partial v_0}{\partial T'} = \frac{\partial^2 v_0}{\partial X^2} \\ &v_0(X, 0) = \max(-X, 0) \\ &\lim_{X \to -\infty} v_0(X, T) = -X \\ &\lim_{X \to \infty} \frac{\partial v_0(X, T)}{\partial T'} = 0 \end{aligned} \tag{3.6}$$

$$
\begin{aligned}
&\frac{\partial v_1}{\partial T'} = \frac{\partial^2 v_1}{\partial X^2} + (k_1 - k_2 - 1)\frac{\partial v_0}{\partial X} - k_2 \\
&v_1(X,0) = max(\frac{-X^2}{2!}, 0) \\
&\lim_{X \to -\infty} v_1(X,T') = \frac{-X^2}{2!} - k_1 T' \\
&\lim_{X \to \infty} \frac{\partial v_1(X,T')}{\partial X}(X,T') = 0
\end{aligned}
\tag{3.7}
$$

For mathematical justification, the boundary conditions as $X \to -\infty$ are obtained by matching with the outer expansion, whereas the ones as $X \to \infty$ are needed to complete the PDE systems.

In order to obtain analytical solutions of the PDE systems (3.6) and (3.7, we adopt the similarity solution techniques [16]. Following this method, we consider a solution structure $v_0(X,T') = T'^{1/2} g_0(\gamma)$ and $v_1(X,T') = T g_1(\gamma)$ for (3.6) and (3.7) respectively, where $\gamma = \frac{X}{2\sqrt{T'}}$. Thus, it is not difficult to show that the above systems transformed respectively to ordinary differential systems

$$
\begin{aligned}
&g_0''(\gamma) + 2\gamma g_0'(\gamma) - 2g_0(\gamma) = 0 \\
&\lim_{\gamma \to -\infty} g_0(\gamma) = -2\gamma, \quad \lim_{\gamma \to \infty} g_0'(\gamma) = 0
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
&g_1''(\gamma) + 2\gamma g_1'(\gamma) - 4g_0(\gamma) = 2(k_1 - k_2 - 1)\text{erfc}(\gamma) + 4k_2 \\
&\lim_{\gamma \to -\infty} g_1(\gamma) = -2\gamma^2 + k_1(\alpha - 1), \quad \lim_{\gamma \to \infty} g_1'(\gamma) = 0.
\end{aligned}
\tag{3.9}
$$

The general solution of $g_0(\gamma)$ is obtained as

$$
g_0(\gamma) = C_0 \gamma C_1 [e^{-\gamma^2} + \gamma\sqrt{\pi}\text{erfc}(\gamma)] \tag{3.10}
$$

After imposing the limit condition $\gamma \to -\infty$ and the boundary conditions $g_0(\gamma) = g_0'(\gamma) = 0$ as $\gamma \to \infty$, it is straightforward to obtain the analytical solution as

$$g_0(\gamma) = \frac{e^{-\gamma^2}}{\sqrt{\pi}} - \gamma \text{erfc}(\gamma), \tag{3.11}$$

where

$$\text{erfc}(\gamma) = 1 - \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \left(\frac{(-1)^n \gamma^{2n+1}}{n!(2n+1)} \right).$$

Following the same procedures for system (3.9), we assume for the moment the general solution

$$g_1(\gamma) = g_{1_a}(\gamma) e^{-\gamma^2} + g_{1_b}(\gamma) \text{erfc}(\gamma) + g_{1_c}(\gamma), \tag{3.12}$$

which, after performing some mathematical manipulations results in

$$g_1(\gamma) = \left(k_2 - k_1 + 1 + \frac{\gamma e^{-\gamma^2}}{2\sqrt{\pi}} - \frac{(1 + 2\gamma^2)\text{erfc}(\gamma)}{4} \right) \text{erfc}(\gamma) - k_2. \tag{3.13}$$

By exploring the asymptotic expansion of erfc(γ), the solutions $g_0(\gamma)$ and $g_1(\gamma)$ as $\gamma \to \infty$, can be written respectively as

$$g_0(\gamma) = \frac{e^{-\gamma^2}}{2\gamma^2\sqrt{\pi}} + O\left(\frac{e^{-\gamma^2}}{\gamma^2} \right) \tag{3.14}$$

$$g_1(\gamma) = -k_2. \tag{3.15}$$

Finally, the transformed unknown boundary $x_f(\tau)$ is determined by matching the solution $v(x, \tau)$ in the two different regions. We require that the expansions agree asymptotically in these regimes, where $X \to \infty$ and $\gamma \to \infty$ as $\epsilon \to 0$. Hence, by taking the limit values as $\gamma \to \infty$ corresponding to $x \to x_f$, and using the leading-order terms, we obtain the transcendental equation:

$$\frac{e^{-x_f^2/4\tau}}{2\sqrt{\pi\tau}} - k_2 = 0, \tag{3.16}$$

with the solution

$$x_f(\tau) = 2\sqrt{\tau}\Big(- \ln(2k_2\sqrt{\pi\tau})\Big)^{1/2} \tag{3.17}$$

Reverting to the state variables with $x_f(\tau) = \ln\left(\frac{S_f(t)}{K}\right)$, we obtain

$$S_f(t) = K\exp\left[\sigma\sqrt{2(T-t)}\Big(- \ln\frac{2\delta\sqrt{2\pi(T-t)}}{\sigma}\Big)^{1/2}\right] \tag{3.18}$$

The matched asymptotic expansion for the optimal conversion boundary of CBs is now complete. The new-found Equation (3.18) is important in CBs trading. Specifically, once $S_f(t)$ is known, the nonlinear pricing problem (3.2) becomes a linear one, and it is straightforward to predict the CBs dynamics. It should be remarked that as $\delta \to 0$, $S_f(t) \to \infty$. This implies that the conversion option should never be exercised when the bond pays no dividend and investors are better off holding the bond to maturity. Higher dividend per bond when converted results in a higher possibility to trigger conversion that dilutes existing share holder's value [7].

4 Discussion and conclusion

The specific result of this paper is Equation (3.17), the optimal conversion price of CBs valid close to maturity. This result can be used to validate numerical solutions designed for more complicated cases where no analytical solutions exist.It worth mentioning that provided the dividend yield on underlying stock is non negative, x_f is of the form $\sqrt{\tau}\big(-\ln\sqrt{\tau}\big)$. This result follows the existing works where related issues were addressed [2].

To investigate the singular behaviour of optimal conversion boundary close to expiry, we estimate the limiting of $S_f(t)$ and its first derivative as $t \to T$:

$$\lim_{t\to T} S_f(t)) = K, \text{ and } \lim_{t\to T} \frac{dS_f}{dt} = \infty$$

The limiting value of $S_f(t)$, K as $t \to T$ aligns with both theoretical and empirical reasonings that if a zero-coupons CBs held to due date, it could be traded for a cash amount of the ratio of principal and the conversion ratio (here conversion ratio, n=1). Furthermore, the infinite slope at $t = T$ should be expected because the optimal asset price changes drastically in the surrounding neighborhood of maturity time. The future work would extend the singular perturbation technique to multiple stopping problems, including convertible bonds with embedded call and put options.

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