On a New Differential Operator

By

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Abstract

In this paper, we give a new differential operator for the class of analytic functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

and we obtained a univalent condition for the harmonic function defined by the said differential operator as well as its coefficient bounds.

Keyword: New differential operator, harmonic function, univalent, coefficient bound.

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1. Introduction and Preliminaries

Let A denote the class of all analytic functions $f(z)$ defined in the open unit disk $U = \{z \in C : |z| < 1\}$ and of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

A continuors complex valued function $f = u + iv$ defined in a simply connected complex domain $D \subset C$ is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain, we write $f = h + \bar{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$ in D.

Let H denote the family of functions $f = h + \bar{g}$ that are harmonic univalent and sense preserving in the unit disk $U = \{z \in C : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. The

harmonic function $f = h + \bar{g}$ reduces to an analytic function $f = h$ when $g \equiv 0$.
Many Authors [1,3,4,5] and several others have studied the family of harmonic univalent functions. In 2012, Makinde and Afolabi [2], introduced and studied the subclass $T_H(\alpha, \beta, t)$ of harmonic univalent functions.
In this paper, for $f(z) \in A$, we introduce the differential operator $F^k f(z)$ denoted by

$$F^k f(z) = z + \sum_{n=2}^{\infty} c_{nk} a_n z^n \tag{2}$$

where $c_{nk} = \frac{n!}{|(n-k)|!}$ and

$$F^k f(z) = z^k \left[z^{-(k-1)} + \sum_{n=2}^{\infty} c_{nk} a_n z^{n-k} \right], k \geq 0$$

and

$$F^0 f(z) = f(z),$$
$$F^1 f(z) = z + \sum_{n=2}^{\infty} c_{n1} a_n z^n = z + \sum_{n=2}^{\infty} n a_n z^n$$

Thus, it impies that $F^k f(z)$ is identically the same as $f(z)$ when $k = 0$ and when $k = 1$, we obtain the first differential coefficient of the Salagean differential operator.
For $f = h + \bar{g} \in H$, we express the analytic functions h and g as;

$$h(z) = z + \sum_{n=2}^{\infty} c_{nk} a_n z^n, \tag{3}$$

and

$$g(z) = \sum_{n=1}^{\infty} c_{nk} b_n z^n, \quad |b_1| < 1 \tag{4}$$

We present and prove the main results of this paper in what follows.

2. Main Results

Theorem 1

Let the function $f = h + \bar{g}$ be such that h and g are as given in (3) and (4) respectively and for $z_1 \neq z_2$. If

$$\frac{\sum_{n=1}^{\infty} c_{nk}|b_n|}{1 - \sum_{n=2}^{\infty} c_{nk}|a_n|} < 1, \quad k \geq 0, \quad a_n, b_n \texttt{ are complex numbers}$$

Then f is univalent in U.

Proof

If $z_1, z_2 \in D$, then

$$\left|\frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)}\right| \geq 1 - \left|\frac{g(z_1 - g(z_2)}{h(z_1) - h(z_2)}\right|$$

$$= 1 - \left|\frac{\sum_{n=1}^{\infty} c_{nk} b_n (z_1^n - z_2^n)}{(z_1 - z_2) + \sum_{n=2}^{\infty} c_{nk} a_n (z_1^n - z_2^n)}\right|$$

$$> 1 - \frac{\sum_{n=1}^{\infty} c_{nk}|b_n|}{1 - \sum_{n=2}^{\infty} c_{nk}|a_n|} > 0, \texttt{ by hypothesis}.$$

Hence f is univalent in U

Corollary 1

Let the function $f = h + \bar{g}$ be univalent in U such that h and g are as given in in (3) and (4) respectively . Then,

$$|b_n| < \frac{1}{c_{nk}} - \sum_{n=2}^{\infty} |a_n|$$

Corollary 2

Let the function $f = h + \bar{g}$ be univalent in U such that h and g are as given in in (3) and (4) respectively . Then,

$$|a_n| < \frac{1}{c_{nk}} - \sum_{n=1}^{\infty} |b_n|$$

Theorem 2

Let the function $f = h + \bar{g}$ be univalent in U such that h and g are as given in in (3) and

(4) respectively . If

$$1-\sum_{n=2}^{\infty} nc_{nk}\,|a_n| > \sum_{n=1}^{\infty} nc_{nk}\,|b_n|$$

Then f is sense preserving and locally univalent in U.

Proof

Let

$$h(z) = z + \sum_{n=2}^{\infty} c_{nk} a_n z^n$$

Then

$$|h'(z)| = \left|1 + \sum_{n=2}^{\infty} nc_{nk} a_n z^{n-1}\right|$$

$$\geq 1 - \sum_{n=2}^{\infty} nc_{nk}|a_n|$$

$$\geq \sum_{n=1}^{\infty} nc_{nk}|b_n| = |g'(z)|$$

Hence f is sense preserving and locally univalent in U.

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