On Extension of Euler's Beta Function

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Abstract

Generalization and extension of Beta function is presented. In addition, a new generalization of the beta function with three and four parameters k, m, n and r are introduced. This new beta function gives upto 4! derivative of new function including Euler's beta function and generalized beta function. The objective of this generalization and extension of beta function is to extend the domain of intersection of beta and gamma function.

Keywords: Special function; Beta function; Gamma function; Generalized; Extended.

1 Introduction

A gamma function $\Gamma(k)$ for $\mathrm{Re}\,(k) > 0$ is used to represent a beta function $B(k, m)$ in integral form with two parameters $\mathrm{Re}(k) > 0$ and $\mathrm{Re}(m) > 0$. A beta function is defined as a ratio of the product of gamma function with each parameter to the gamma function of the sum of the parameters. It was first studied by Euler and Legendre, and the name beta function was given by Jacques Binet [1-7].

A beta integral is defined as,

$$B(k, m) = \int_0^1 x^{k-1}(1-x)^{m-1}dx; \quad \mathrm{Re}(k) > 0,\ \mathrm{Re}(m) > 0 \tag{1}$$

Equation (1) is called as First kind of Eulerian integral. And, interms of gamma function $\Gamma(k)$ may be written as

$$\frac{\Gamma(k)\ \Gamma(m)}{\Gamma(k+m)} = B(k, m) \tag{2}$$

In addition, equation (2) may also be written in the form of factorial as,

$$B(k,m) = \frac{(k-1)!(m-1)!}{(k+m-1)!} \tag{3}$$

Theorem 1 *The solution of equation (2) is the beta function $B(k,m)$ with parameters k and m.*

Proof. Using the definition of gamma function, $\Gamma(k) = \int\limits_0^\infty x^{k-1}\, e^{-x}\, dx,\ \mathrm{Re}\,(k) > 0$. The product of gamma function of two parameters k and m is represented by,

$$\Gamma(k)\ \Gamma(m) = \int\limits_0^\infty x^{k-1}\ e^{-x}\ dx \int\limits_0^\infty y^{m-1}\ e^{-y}\ dy$$

On substituting $x = u^2$ and $y = v^2$, it is obtained

$$\begin{aligned}
\Gamma(k)\ \Gamma(m) &= \int\limits_0^\infty (u^2)^{k-1}\ e^{-u^2}\ 2\ u\ du \int\limits_0^\infty (v^2)^{m-1}\ e^{-v^2}\ 2\ v\ dv \\
&= 4\int\limits_0^\infty u^{2k-1}\ e^{-u^2}\ du \int\limits_0^\infty v^{2m-1}\ e^{-v^2}\ dv \\
&= 4\int\limits_0^\infty\int\limits_0^\infty e^{-(u^2+v^2)}\ u^{2k-1}\ v^{2m-1}\ du\ dv \\
&= \int\limits_{-\infty}^\infty\int\limits_{-\infty}^\infty e^{-(u^2+v^2)}\ |u|^{2k-1}\ |v|^{2m-1}\ du\ dv
\end{aligned}$$

Transforming to polar coordinates with $u = r\cos\theta$, and $v = r\sin\theta$ then

$$\begin{aligned}
\Gamma(k)\ \Gamma(m) &= \int\limits_0^{2\pi}\int\limits_0^\infty e^{-r^2}\, |r\cos\theta|^{2k-1}\ |r\sin\theta|^{2m-1}\ r\ dr\ d\theta \\
&= \int\limits_0^\infty e^{-r^2} r^{2k+2m-2}\ r\ dr \int\limits_0^{2\pi} \left|(\cos\theta)^{2k-1}\,(\sin\theta)^{2m-1}\right|\ d\theta \\
&= \Gamma(k+m)\ B(k,m)
\end{aligned}$$

$$\Gamma(k)\ \Gamma(m) = \Gamma(k+m)\ B(k,m)$$

$$\therefore B(k,m) = \frac{\Gamma(k)\ \Gamma(m)}{\Gamma(k+m)}$$

This completes the proof. ■

Theorem 2 *For* $\mathrm{Re}(k) > 0$ *and* $\mathrm{Re}(m) > 0$*, the symmetry property of beta function is*

$$B(k,m) = B(m,k) \tag{4}$$

Proof. From equation (1),

$$B(k,m) = \int_0^1 x^{k-1}(1-x)^{m-1}dx$$

Substituting $t = 1 - x$ then,

$$B(k,m) = \int_1^0 (1-t)^{k-1}\ t^{m-1}\ (-dt) = \int_0^1 t^{m-1}\ (1-t)^{k-1}\ dt = B(m,k)$$

This completes the proof. ■

Theorem 3 *For* $\mathrm{Re}(k) > 0$ *and* $\mathrm{Re}(m) > 0$*, the semi-recurrence formula is given as*

$$B(k,m+1) = \frac{m}{k+m}\ B(k,m) \tag{5}$$

Proof. From equation (1),

$$B(k,m) = \int_0^1 x^{k-1}(1-x)^{m-1}dx$$

Now, let us evaluate $B(k, m+1)$,

$$B(k,m+1) = \int_0^1 x^{k-1}(1-x)^{m+1-1}dx = \int_0^1 x^{k-1}(1-x)^{m}dx$$

Integrating by parts by assuming $u = (1-x)^m$and $dv = x^{k-1}dx$,

$$B(k,m+1) = \frac{x^k}{k}(1-x)^m\Big|_0^1 + \frac{m}{k}\int_0^1 x^k\ (1-x)^{m-1}\ dx$$

The first term equates to zero. In order to solve, the second term, substituting $x^k = x^{k-1} - x^{k-1}(1-x)$, then

$$
\begin{aligned}
B(k,m+1) &= \frac{m}{k}\int_0^1 \left[x^{k-1} - x^{k-1}(1-x)\right](1-x)^{m-1}\,dx \\
&= \frac{m}{k}\int_0^1 x^{k-1}(1-x)^{m-1}\,dx - \int_0^1 x^{k-1}(1-x)(1-x)^{m-1}\,dx \\
&= \frac{m}{k}\left[B(k,m) - \int_0^1 x^{k-1}(1-x)^m dx\right] \\
&= \frac{m}{k}\left[B(k,m) - B(k,m+1)\right] \qquad (6)
\end{aligned}
$$

Now, solving equation (6) for $B(k,m+1)$,

$$
\begin{aligned}
B(k,m+1) &= \frac{m}{k}\left[B(k,m) - B(k,m+1)\right] \\
B(k,m+1)\left[1+\frac{m}{k}\right] &= \frac{m}{k}B(k,m) \\
B(k,m+1) &= \frac{m}{k}\left(\frac{k}{k+m}\right)B(k,m) \\
B(k,m+1) &= \frac{m}{k+m}B(k,m)
\end{aligned}
$$

This completes the proof. ■

Hence using equation (5), one may obtain the following properties,

Property 1 $B(k+1,m) = \frac{k}{k+m}B(k,m)$

Property 2 $B(k,m) = \frac{k-1}{k-1+m}B(k-1,m)$

Property 3 $B(k,m) = \frac{m-1}{k+m-1}B(k,m-1)$

Property 4 $B(k,m) = B(k+1,m) + B(k,m+1)$

Theorem 4 *There exist an integral form of beta function as*

$$
B(k,m) = \int_0^\infty x^{k-1}\,(1+x)^{-(k+m)}\,dx \qquad (7)
$$

Proof. The proof of this is simple and straightforward. ■

Theorem 5 *From equation (7) and using the symmetry property equation (4),*

$$B(m,k) = \int_0^\infty x^{m-1}\ (1+x)^{-(k+m)}\ dx$$

Proof. The proof of this is simple and straightforward. ■

Theorem 6 *The beta function in trigonometric function is given as*

$$B(k,m) = 2\int_0^{\frac{\pi}{2}} \sin^{2k-1}(\theta)\cos^{2m-1}(\theta)\ d\theta \tag{8}$$

Proof. The proof of this is simple and straightforward. ■

2 Generalization of Beta Function

In 1997, generalization & extension of beta function was first attemped by Chaudhry et al. [8]. This generalization arose by introducing an exponential factor, $\exp\left[\frac{-\sigma}{x(1-x)}\right]$ where $\mathrm{Re}(\sigma) > 0$, in the integrand of equation (1), which is given as

$$B(k,m,\sigma) = \int_0^1 x^{k-1}(1-x)^{m-1}\exp\left[\frac{-\sigma}{x(1-x)}\right] dx; \tag{9}$$
$$\mathrm{Re}(k) > 0,\ \ \mathrm{Re}(m) > 0,\ \ \mathrm{Re}(\sigma) > 0$$

When $\sigma = 0$, then $B(k,m,0) = B(k,m)$ for $\mathrm{Re}(k) > 0$ and $\mathrm{Re}(m) > 0$. i.e., when σ vanishes, the generalized function (equation 9) reduces to the original beta function. The logic behind introducing this function is given in Chaudhry et al. [8].

3 Extended Generalization of Beta Function

Using equations (1 & 7), the authors propose a new extended family of beta function with three parameters as $B_r(k,m) = \int_0^\infty x^{k-1}\left[(1-x)^{m-1}\right]^{-r} dx$. The derivation this beta function is as follows:

$$
\begin{aligned}
B(k,m) &= \int\limits_0^1 x^{k-1}(1-x)^{m-1}dx \\
&= \int\limits_0^\infty x^{k-1}(1-x)^{-(k+m)}dx \\
&= \int\limits_0^\infty x^{k-1}\left[(1-x)^{m-1}\right]^{-\frac{(k+m)}{m-1}} dx \\
&= \int\limits_0^\infty x^{k-1}\left[(1-x)^{m-1}\right]^{-\left[1-\frac{k+1}{m-1}\right]} dx
\end{aligned} \tag{10}
$$

Assuming $r = \left[1 - \frac{k+1}{m-1}\right]$ then equation (10) becomes

$$
B_r(k,m) = \int\limits_0^\infty x^{k-1}\left[(1-x)^{m-1}\right]^{-r} dx; \quad \mathrm{Re}(k) > 0,\ \mathrm{Re}(m) > 0,\ r \in \mathbb{R} \tag{11}
$$

When $r = -1$, equation (11) becomes the original beta function.

By introducing an additional parameter $n \in \mathbb{Z}$, equation (11) may be further extended to four parameters as,

$$
B_r(k,m,n) = \int\limits_0^\infty x^{k-1}\left[n + (1-x)^{m-1}\right]^{-r} dx; \tag{12}
$$
$$
\mathrm{Re}(k) > 0,\ \mathrm{Re}(m) > 0,\ \mathrm{Re}(n) > 0,\ r \in \mathbb{R}
$$

When n vanishes, and $r = -1$, equation (12) becomes the original beta function,

$$
B_{-1}(k,m,0) = \int\limits_0^\infty x^{k-1}\left[0 + (1-x)^{m-1}\right]^{-(-1)} dx = B(k,m)
$$

Theorem 7 *For $r \in \mathbb{R}$, equation (12) may be written in terms of eqution (11) as,*

$$B_r(k,m,n) = \begin{cases} \displaystyle\sum_{i=0}^{r} \frac{n^{-(r-i)}}{\binom{r}{i}} B_i(k,m); & r \in \mathbb{N} \\ \displaystyle\sum_{i=0}^{-r} \frac{\binom{-r}{i}}{n^{(r+i)}} B_i(k,m); & -r \in \mathbb{N} \end{cases}$$

Proof. For $r \in \mathbb{N}$,

$$\begin{aligned} B_r(k,m,n) &= \int_0^\infty x^{k-1} \left[n + (1-x)^{m-1}\right]^{-r} dx \\ &= \int_0^\infty \frac{x^{k-1}}{\left[n + (1-x)^{m-1}\right]^{r}} \, dx \\ &= \int_0^\infty \frac{x^{k-1}}{\sum_{i=0}^{r} \binom{r}{i} n^{(r-i)} (1-x)^{(m-1)i}} \, dx \\ &= \sum_{i=0}^{r} \frac{n^{-(r-i)}}{\binom{r}{i}} \int_0^\infty x^{k-1} (1-x)^{-(m-1)i} \, dx \\ &= \sum_{i=0}^{r} \frac{n^{-(r-i)}}{\binom{r}{i}} \, B_i(k,m); \quad \forall \ r \in \mathbb{N} \end{aligned} \tag{13}$$

For $-r \in \mathbb{N}$,

$$\begin{aligned} B_r(k,m,n) &= \int_0^\infty x^{k-1} \left[n + (1-x)^{m-1}\right]^{-r} dx \\ &= \int_0^\infty x^{k-1} \sum_{i=0}^{-r} \binom{-r}{i} \, n^{-(r+i)} (1-x)^{(m-1)i} \, dx \\ &= \sum_{i=0}^{-r} \binom{-r}{i} \, n^{-(r+i)} \int_0^\infty x^{k-1} (1-x)^{(m-1)i} \, dx \\ &= \sum_{i=0}^{-r} \binom{-r}{i} \, n^{-(r+i)} \, B_i(k,m); \quad \forall \ -r \in \mathbb{N} \end{aligned} \tag{14}$$

This completes the proof. ■

It is clear from equations (13 & 14), that a beta function of four paramters may be represented as a beta function of three parameter with weighted factors.

Theorem 8 *Convergence of the integral:* $B_r(k,m,n) = \int\limits_0^\infty x^{k-1}\left[n+(1-x)^{m-1}\right]^{-r} dx;$ $\mathrm{Re}(k)>0$, $\mathrm{Re}(m)>0$, $\mathrm{Re}(n)>0$, $r\in\mathbb{R}$.

Proof. For $r \geq 1$ and $\mathrm{Re}(n) > 0$,

$$x^{k-1}\left[n+(1-x)^{m-1}\right]^{-r} < x^{k-1}\left[(1-x)^{m-1}\right]^{-r},$$

then

$$\begin{aligned} 0 &\leq \int_0^\infty x^{k-1}\left[n+(1-x)^{m-1}\right]^{-r} dx \\ &\leq \int_0^\infty x^{k-1}\left[(1-x)^{m-1}\right]^{-r} dx \leq B_r(k,m) < \infty \end{aligned}$$

For $0 < r < 1$ and $k > 0$,

$$\begin{aligned} \int_0^\infty x^{k-1}\left[n+(1-x)^{m-1}\right]^{-r} dx &\leq \int_0^\infty x^{k-1}\left[(1-x)^{m-1}\right]^{0} dx \\ &\leq \int_0^\infty x^{k-1} dx < \infty. \end{aligned}$$

For $-r > 0$, $(-r)-1 \leq -r \leq (-r)+1$ and $(-r)\in\mathbb{N}$, then

$$\begin{aligned} \int_0^\infty x^{k-1}\left[n+(1-x)^{m-1}\right]^{(-r)-1} dx &\leq \int_0^\infty x^{k-1}\left[n+(1-x)^{m-1}\right]^{-r} dx \\ &\leq \int_0^\infty x^{k-1}\left[n+(1-x)^{m-1}\right]^{(-r)+1} dx \end{aligned}$$

and

$$\begin{aligned} \sum_{i=0}^{(-r)-1}\begin{pmatrix} (-r)-1 \\ i \end{pmatrix} n^{-((r)+1+i)}\, B_i(k,m) &\leq \int_0^\infty x^{k-1}\left[n+(1-x)^{m-1}\right]^{-r} dx \\ &\leq \sum_{i=0}^{(-r)+1}\begin{pmatrix} (-r)+1 \\ i \end{pmatrix} n^{-((r)-1+i)}\, B_i(k,m). \end{aligned}$$

This completes the proof. ■

4 Conclusions

Beta function, its generalization and extension upto four parameters is studied. In addition to portray the beta function, its generalization & extension; a new generalization of the beta function with three and four parameters k, m, n and r are introduced. This new beta function gives upto 4! derivative of new function including Euler's beta function and generalized beta function. The objective of this generalization and extension of beta function is to extend the domain of intersection of beta and gamma function. The authors are looking forward to work in further generatization of beta function upto seven parameters. This new generalized functions helps in representing mixture random phenomena in distribution theory, and also opens a new direction of reasearch for statistical quality control.

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