On Complex Vectors in $\mathbb{C}^3$ with Real Valued Scalar Product

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Abstract

We consider a space S of complex vectors in $\mathbb{C}^3$ with physically relevant constraints and the corresponding representation of the group $\mathrm{SO}(3,\mathbb{C})$ acting on S. The constraints are introduced to provide real-valued and hyperbolically calculated vector magnitudes. Additionally, in order to acquire the benefits that real numbers provide, we introduce a real-valued scalar product in S using scalar product definition with relaxed conditions. This, in turn, leads to consider a specific $\mathrm{SO}(3,\mathbb{C})$ representation and restricted $\mathrm{SO}(3,\mathbb{C})$ action on S in order to keep the scalar product invariant.

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1 Introduction

It is well known that the restricted Lorentz group $\mathrm{SO}(1,3)^+$ is isomorphic to the complex special orthogonal group $\mathrm{SO}(3,\mathbb{C})$. So, $\mathrm{SO}(3,\mathbb{C})$, which naturally acts on $\mathbb{C}^3$, offers an alternative way to represent elements of physical theories commonly expressed in terms of $\mathrm{SO}(1,3)^+$. Bringing elements of physical theories in $\mathbb{C}^3$ opens some useful prospects. The vector product, which is not applicable in the four-dimensional real space, in three-dimensional complex space becomes an important tool. The increased number of vector components can be used to include additional physical quantities. Furthermore, the presence of real and imaginary vector parts enables a corresponding separation of distinct physical concepts.

We introduce a space S of complex vectors in $\mathbb{C}^3$ with a real-valued scalar product and physically relevant constraints that include fixed vector magnitudes and orthogonality between real and imaginary vector parts. The orthogonality constraint is already used elsewhere (see [1] as a newer example), and the space S and the corresponding $\mathrm{SO}(3,\mathbb{C})$ representation is analyzed by means of a complex-valued scalar product [2]. Here, we

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examine the real-valued scalar product in S and the restricted $\mathrm{SO}(3,\mathbb{C})$ action that keeps the scalar product invariant. Firstly, we introduce representation G of the group $\mathrm{SO}(3,\mathbb{C})$ according to the constraints imposed on the complex vectors in S. The representation G is given in details through the polar decomposition of the $\mathrm{SO}(3,\mathbb{C})$ matrices on (real orthogonal)/(positive definite Hermitian). However, while G preserves the vector magnitudes, in general case it does not preserve the introduced real-valued scalar product. Thus, we consider restricted G action that leaves the scalar product invariant. This action corresponds to the action of the group SO(1,2), which is known to have applications in various branches of physics (see for example [3], Ch. 9).

2 Representation of Vectors in the Space *S*

Let $u = \vec{x} + i\vec{a}$ be a complex vector in $\mathbb{C}^3$, where $\vec{x}, \vec{a} \in \mathbb{R}^3$. Our intention is to define a space of complex vectors with fixed, real-valued magnitude. Additionally, the magnitude calculation should exhibit a hyperbolic property in order to satisfy some physical requirements. Indeed, the "usual" conjugate scalar product in $\mathbb{C}^3$ gives real-valued vector magnitudes, but they are not hyperbolically calculated, $u \cdot \bar{u} = \vec{x}^2 + \vec{a}^2$. On the other hand, the non-conjugate scalar product in its real part gives hyperbolically calculated vector magnitudes, but they are complex-valued, $u \cdot u = \vec{x}^2 - \vec{a}^2 + 2i\vec{x}\vec{a}$. To achieve our intention, it seems less demanding to adjust the non-conjugate scalar product by introducing a constraint on complex vectors in the form $\vec{x}\vec{a} = 0$ (orthogonality constraint). Taking in mind physical applications where the orthogonality between 3-vectors is commonly used, it seems it is a good trade-off between obtaining real, hyperbolically calculated vector magnitudes and the complications that the orthogonality constraint triggers.

Throughout the paper we will use $\cdot$ to denote the non-conjugate scalar product in $\mathbb{C}^3$.

Definition 1. *The space $S \subset \mathbb{C}^3$ defined by $S = \{u = \vec{x} + i\vec{a} \mid u \cdot u = \lambda^2,\ \lambda \in \mathbb{R},\ \vec{x}\vec{a} = 0\}$ is the space of s-vectors in $\mathbb{C}^3$.*

The scalar product $\cdot$ applied on two vectors in S is, indeed, complex-valued scalar product that gives real vector magnitudes. However, some physical applications strongly benefit from real-valued scalar product where the differences in vectors directions will be real-valued. A natural way to provide a real-valued scalar product in S is to rotate one of the s-vectors until the directions of their real parts coincide.

Definition 2. *For given two s-vectors $u = \vec{x} + i\vec{a}$ and $v = \vec{y} + i\vec{b}$ in S, the real-valued scalar product $\hat{\cdot}$ is defined by*

$$u \,\hat{\cdot}\, v = u \cdot Rot_{\hat{y}\to\hat{x}} v, \tag{1}$$

where $\hat{x} = \frac{\vec{x}}{\|\vec{x}\|}$, $\hat{y} = \frac{\vec{y}}{\|\vec{y}\|}$ and $Rot_{\hat{y}\to\hat{x}}$ is a rotation that carries $\hat{y}$ to $\hat{x}$.

Since the choice of a rotation axis does not affect the further considerations, we can choose it to be orthogonal to both $\vec{x}$ and $\vec{y}$ and so, the rotation will be in $\vec{x} \wedge \vec{y}$ plane. Now, $Rot_{\hat{y}\to\hat{x}}$ can be easily represented by the Rodrigues' rotation formula, which adapted to this case, takes the form

$$Rot_{\hat{y}\to\hat{x}} = (\hat{y}\hat{x})I + \hat{x}\otimes\hat{y} - \hat{y}\otimes\hat{x} + \frac{1}{1+\hat{y}\hat{x}}(\hat{y}\times\hat{x})\otimes(\hat{y}\times\hat{x}), \tag{2}$$

where I is the identity matrix.

Observe that the scalar product (1) is in agreement with the already introduced magnitude of s-vectors in Definition 1, since $u \hat{\cdot}\, u = u \cdot u$. Actually, the scalar product $\hat{\cdot}$ can be considered as a restriction of the scalar product $\cdot$ in the sense that it can be seen as a specific application of $\cdot$ on s-vectors with different real part directions. Indeed, on s-vectors u and v with parallel real parts ($\vec{x} \parallel \vec{y}$), the scalar products $\hat{\cdot}$ and $\cdot$ coincide.

The standard addition and multiplication by a scalar are not closed operations in S, and also, the zero vector $\vec{0} + i\vec{0}$ is not in S, so S is not a vector subspace of $\mathbb{C}^3$. This implies that the scalar product $\hat{\cdot}$ is defined with relaxed conditions and it remains to show the symmetry, i.e. commutativity of $\hat{\cdot}$.

Proposition 1. *The scalar product $\hat{\cdot}$ is commutative, i.e $u \hat{\cdot}\, v = v \hat{\cdot}\, u$.*

Proof. The proof can be obtained by direct calculations. However, since $Rot_{\hat{y}\to\hat{x}}$ is an orthogonal matrix with respect to the scalar product $\cdot$ it follows that

$$u \hat{\cdot}\, v = u \cdot Rot_{\hat{y}\to\hat{x}} v = Rot^{-1}_{\hat{y}\to\hat{x}} u \cdot v = v \cdot Rot^{-1}_{\hat{y}\to\hat{x}} u = v \cdot Rot_{\hat{x}\to\hat{y}} u = v \hat{\cdot}\, u. \quad \square$$

It is easy to show that an arbitrary vector $\vec{p}$ (not necessarily orthogonal to $\vec{a}$ or $\vec{b}$) can be also used to calculate the scalar product $u \hat{\cdot}\, v$.

Consequence 1. *The following equality holds, $u \hat{\cdot}\, v = Rot_{\hat{x}\to\hat{p}} u \cdot Rot_{\hat{y}\to\hat{p}} v$.*

Proof. It follows from

$$Rot_{\hat{x}\to\hat{p}} u \cdot Rot_{\hat{y}\to\hat{p}} v = Rot_{\hat{x}\to\hat{p}} u \cdot Rot_{\hat{x}\to\hat{p}}\left(Rot_{\hat{y}\to\hat{x}} v\right) = u \cdot Rot_{\hat{y}\to\hat{x}} v.$$

The last equality holds since the rotation of $\hat{x}$ to $\hat{p}$ preserves the angle between the vectors $\vec{a}$ and $Rot_{\hat{y}\to\hat{x}}\vec{b}$. $\square$

Consequence 2. *The scalar product $\hat{\cdot}$ is always positive, i.e. $u \hat{\cdot}\, v > 0$.*

Proof. From the definition of s-vectors, it follows that $\vec{x}^2 > \vec{a}^2$ and $\vec{y}^2 > \vec{b}^2$. Since the rotations preserve the magnitudes of the real and the imaginary parts of s-vectors, we have

$$u \hat{\cdot}\, v = (\vec{x} + i\vec{a}) Rot_{\hat{y}\to\hat{x}}\left(\vec{y} + i\vec{b}\right) = \|\vec{x}\|\|\vec{y}\| - \cos\sphericalangle\left(\vec{a}, Rot_{\hat{y}\to\hat{x}}\vec{b}\right)\|\vec{a}\|\|\vec{b}\|$$

$$> \|\vec{a}\|\|\vec{b}\|\left[1 - \cos\sphericalangle\left(\vec{a}, Rot_{\hat{y}\to\hat{x}}\vec{b}\right)\right] > 0. \quad \square$$

Although the space S does not possess a suitable vector addition to become a vector space, we can introduce the vector addition inherited from $\mathbb{C}^3$ with the requirement the real part of the second vector to be parallel to the real part of the first one. Thus, analogously to the scalar product $\hat{\cdot}$ we can define s-vector addition $\hat{+}$ by

$$u \,\hat{+}\, v = u + Rot_{\hat{y}\to\hat{x}} v, \tag{3}$$

which obviously is not a closed operation in S. The addition $\hat{+}$ is not commutative, since

$$u \,\hat{+}\, v = u + Rot_{\hat{y}\to\hat{x}} v = Rot_{\hat{y}\to\hat{x}}\big(Rot_{\hat{x}\to\hat{y}} u + v\big) = Rot_{\hat{y}\to\hat{x}}(v \,\hat{+}\, u),$$

which is similar to the gyrocommutative law [4]. However, unlike the gyroassociative law, $\hat{+}$ is an associative operation. Namely, for $w = \vec{z} + i\vec{c}$ we have

$$u \,\hat{+}\, (v \,\hat{+}\, w) = u \,\hat{+}\, \big(v + Rot_{\hat{z}\to\hat{y}} w\big) = u + Rot_{\hat{y}\to\hat{x}}\big(v + Rot_{\hat{z}\to\hat{y}} w\big)$$

$$= \big(u + Rot_{\hat{y}\to\hat{x}} v\big) + Rot_{\hat{y}\to\hat{x}} Rot_{\hat{z}\to\hat{y}} w = (u \,\hat{+}\, v) + Rot_{\hat{z}\to\hat{x}} w = (u \,\hat{+}\, v) \,\hat{+}\, w.$$

Corollary. *The scalar product* $\hat{\cdot}$ *is distributive with respect to* $\hat{+}$.

Proof. We have

$$(u \,\hat{+}\, v) \,\hat{\cdot}\, w$$

$$= \big(u + Rot_{\hat{y}\to\hat{x}} v\big) \cdot Rot_{\hat{z}\to\hat{x}} w = u \cdot Rot_{\hat{z}\to\hat{x}} w + Rot_{\hat{y}\to\hat{x}} v \cdot Rot_{\hat{z}\to\hat{x}} w = u \,\hat{\cdot}\, w + v \,\hat{\cdot}\, w. \ \square$$

The real dimension of the space S is four, since one dimension is lost by the ortogonality constraint $\vec{x}\vec{a} = 0$ and another dimension is lost by the constraint of fixed vector magnitude $u^2 = \lambda^2$.

3 Action of SO(3, ℂ) on *S*

Let us consider the spaces $(S, \cdot)$ and $(S, \hat{\cdot})$. Recall that S is not a vector space and the scalar products $\cdot$ and $\hat{\cdot}$ are with relaxed conditions. Both of them are commutative with strictly positive magnitudes, while the linearity condition can be inherited from $\mathbb{C}^3$. Actually, $\cdot$ is linear with respect to the ordinary +, while $\hat{\cdot}$ is linear with respect to $\hat{+}$ in $\mathbb{C}^3$. Since the calculations of vector magnitudes give the same results in both spaces, we can consider the complex orthogonal group $\mathrm{SO}(3, \mathbb{C})$ acting on each of them and preserving vector magnitudes. So, for any $g \in \mathrm{SO}(3, \mathbb{C})$,

$$gu \cdot gu = u \cdot u = u \,\hat{\cdot}\, u = gu \,\hat{\cdot}\, gu,$$

for all $u \in S$. In fact, we are interested in representation G of SO(3, ℂ) acting on S.

According to the polar decomposition of complex orthogonal matrices, a matrix $M \in \mathrm{SO}(3, \mathbb{C})$ can be represented as $M = Re^{iA}$, where R is a real orthogonal matrix (a rotation matrix) and A is a real antisymmetric matrix. The latter implies that e^{iA} is a positive definite Hermitian coninvolutory matrix [5] (p.487). So, every G-matrix is a product of an SO(3) matrix representing a rotation and a positive definite Hermitian matrix, which by analogy of $SO(1,3)^{+}$, can be called hyperbolic rotation (h-rotation). The h-rotations deserve special attention, since unlike the rotations, they change magnitudes of the real and the imaginary parts of s-vectors.

Definition 3. *The matrix H_u, parameterized by the s-vector $u = \vec{x} + i\vec{a}$ and given by*

$$H_u = I + \frac{k-1}{\lambda^2 k^2}\vec{x} \otimes \vec{x} + \frac{1}{\lambda^2(1+k)}\vec{a} \otimes \vec{a} + i\frac{1}{\lambda^2 k}(\vec{a} \otimes \vec{x} - \vec{x} \otimes \vec{a}), \textit{ where } k = \sqrt{1 + \frac{\vec{a}^2}{\lambda^2}} \tag{4}$$

is an h-rotation acting on S.

Now, we will show that the matrix H_u is in agreement with the polar decomposition of complex orthogonal matrices.

Proposition 2. *The matrix H_u given by (4) is an orthogonal coninvolutory and positive definite Hermitian matrix.*

Proof. The straightforward calculation gives

$$\mathcal{Re}(H_u)\mathcal{Im}(H_u) = \mathcal{Im}(H_u)\mathcal{Re}(H_u)$$

and then,

$$H_u\overline{H}_u = H_u H_{\overline{u}} = \left[I + \frac{k-1}{\lambda^2 k^2}\vec{x} \otimes \vec{x} + \frac{1}{\lambda^2(1+k)}\vec{a} \otimes \vec{a}\right]^2 + \left[\frac{1}{\lambda^2 k}(\vec{a} \otimes \vec{x} - \vec{x} \otimes \vec{a})\right]^2 = I = \overline{H}_u H_u\,.$$

Thus, $\overline{H}_u = {H_u}^{-1}$ and since $\overline{H}_u = {H_u}^{\mathrm{T}}$, it follows that H_u is orthogonal. Obviously, it follows then that ${\overline{H}_u}^{\mathrm{T}} = H_u$ and with $H_u\overline{H}_u = I$ it means that H_u is a Hermitian coninvolutory matrix.

From

$$H_u v = \left[\vec{y} + \left(\frac{k-1}{\lambda^2 k^2}\vec{x}\vec{y} + \frac{1}{\lambda^2 k}\vec{a}\vec{b}\right)\vec{x} + \left(\frac{1}{\lambda^2(1+k)}\vec{y}\vec{a} - \frac{1}{\lambda^2 k}\vec{b}\vec{x}\right)\vec{a}\right]$$

$$+i\left[\vec{v} + \left(\frac{k-1}{\lambda^2 k^2}\vec{b}\vec{x} - \frac{1}{\lambda^2 k}\vec{y}\vec{a}\right)\vec{x} + \left(\frac{1}{\lambda^2(1+k)}\vec{a}\vec{b} - \frac{1}{\lambda^2 k}\vec{x}\vec{y}\right)\vec{a}\right], \tag{5}$$

by using simple algebraic operations, one obtains $\bar{v} \cdot (H_u v) > 0$. Observe that in case of $\hat{\cdot}$, we also have $\bar{v}\,\hat{\cdot}\,(H_u v) > 0$, as it follows from the Consequence 2 of Proposition 1. □

Notice that the matrix H_u applied to the corresponding "zero" vector $\lambda\hat{x}$ in S gives u, i.e. $H_u\lambda\hat{x} = u$. One can easily check that, in general, two h-rotations do not commute, even when they are generated from s-vectors with a common real part. It means that, generally, a product of two h-rotations is not a Hermitian matrix and so the resulting transformation is not an h-rotation.

Although both scalar products give the same vector magnitudes in S, it is important to underline that $\cdot$ is complex-valued, while $\hat{\cdot}$ is real-valued. Definition 3 and Proposition 2 show that h-rotation H_u is defined with respect to the scalar product $\cdot$. Thus, as one can expect, the scalar product $\hat{\cdot}$ is not in accordance with h-rotations, i.e. in general, $H_w u \,\hat{\cdot}\, H_w v \neq u \,\hat{\cdot}\, v$. One can directly verify that

$$H_w u \,\hat{\cdot}\, H_w v = H_w u \cdot Rot_{\hat{q}\to\hat{p}} H_w v \neq u \cdot Rot_{\hat{y}\to\hat{x}} v = u \,\hat{\cdot}\, v,$$

where $\vec{p} = \mathcal{R}\mathrm{e}(H_w u)$, $\vec{q} = \mathcal{R}\mathrm{e}(H_w v)$ and for $H_w u$ and $H_w v$ see (5). Thus, the convenience to have a real-valued scalar product $\hat{\cdot}$ is paid by breaking the scalar product invariance. However, $H_w u \,\hat{\cdot}\, H_w v = u \,\hat{\cdot}\, v$ when $\vec{x} \parallel \vec{y}$, since then, the involved rotation in (1) vanishes and the scalar product $\hat{\cdot}$ coincides with $\cdot$. This situation is a motivation for the next section.

4 Restricted Action of SO(3, $\mathbb{C}$) on S

As one can see from (5), applying an h-rotation H_u on a given s-vector v results in an s-vector whose real part is a linear combination of $\vec{x}, \vec{y}$ and $\vec{a}$, which is indeed, different from the real parts of both, u and v. We introduce a restricted h-rotation action which does not change the real parts of s-vectors, in order to obtain invariance of the scalar product $\hat{\cdot}$. The change of the real parts of s-vectors is left to the rotations.

Definition 4. *A restricted action $\widehat{H}_u$ of the h-rotation H_u on vectors in S is given by*

$$\widehat{H}_u = Rot^{-1}_{\hat{\star}\to\hat{x}} H_u Rot_{\hat{\star}\to\hat{x}}\,, \tag{6}$$

where $\vec{\star}$ is the real part of the s-vector on which the h-rotation $\widehat{H}_u$ is applied.

The matrix $\widehat{H}_u$ obviously remains orthogonal and positive definite on S (see Consequence 2, of Proposition 1). It is also Hermitian, since H_u is Hermitian and $Rot_{\hat{\star}\to\hat{x}}$ and its inverse are unitary matrices. Thus, $\widehat{H}_u$ is also an h-rotation which obviously keeps the real parts of s-vectors in place. Actually, $\widehat{H}_u$ should be considered as a "locally implemented" h-rotation in G with respect to the scalar product $\hat{\cdot}$. Notice that H_u and $\widehat{H}_u$ coincide when they are applied to an s-vector with real part $\vec{\star}$ parallel to $\vec{x}$ (the labels are as in (6)), as was the case for the scalar products $\cdot$ and $\hat{\cdot}$.

Let us write (6) in the form $\widehat{H}_u Rot_{\hat{x}\to\hat{\star}} = Rot_{\hat{x}\to\hat{\star}} H_u$. This equality represents left and right polar decomposition of an orthogonal matrix in G. The unitary matrices in the left and the right polar decomposition, represented by rotation $Rot_{\hat{x}\to\hat{\star}}$ are indeed equal. The h-rotation matrices are connected by the equality

$$\widehat{H}_u = H_{Rot_{\hat{x}\to\hat{\star}} u}$$

since this restricted h-rotation $\widehat{H}_u$ relates to the corresponding action in the group SO(1,2) $\subset$ SO(3, $\mathbb{C}$). To explain how $Rot^{-1}_{\hat{\star}\to\hat{x}} H_u Rot_{\hat{\star}\to\hat{x}}$ and $B_{Rot_{\hat{x}\to\hat{\star}} u}$ work, let us apply them on a given s-vector, say $v = \vec{y} + i\vec{b}$. In the case of $\widehat{H}_u v = Rot^{-1}_{\hat{y}\to\hat{x}} H_u Rot_{\hat{y}\to\hat{x}}$, the real part of v (i.e. $\vec{y}$) is rotated toward the real part of the s-vector which parameterizes the h-rotation (i.e. $\vec{x}$) and after the h-rotating, the direction of the real part of the resulting s-vector is returned to the direction of $\vec{y}$. In the case $\widehat{H}_u v = H_{Rot_{\hat{x}\to\hat{y}} u} v$, the real part of the s-vector which parameterizes the h-rotation (i.e. $\vec{x}$) is rotated toward direction of $\vec{y}$. A resembling equality appears in action of SO(1,3)$^+$ on upper-half hyperboloid in $\mathbb{R}^4$ (see [6] (p.140)).

Proposition 4. *The restricted h-rotation preserves the scalar product* $\hat{\cdot}$ *in S, i.e.*

$$\widehat{H}_w u \,\hat{\cdot}\, \widehat{H}_w v = u \,\hat{\cdot}\, v$$

for any s-vectors $u = \vec{x} + i\vec{a}$, $v = \vec{y} + i\vec{b}$ *and* $w = \vec{z} + i\vec{c}$.

Proof. Using (6) one can directly obtain

$$\widehat{H}_w u \,\hat{\cdot}\, \widehat{H}_w v = H_{Rot_{\hat{z}\to\hat{x}} w} u \; Rot_{\hat{y}\to\hat{x}} H_{Rot_{\hat{z}\to\hat{y}} w} \left(Rot_{\hat{x}\to\hat{y}} Rot_{\hat{y}\to\hat{x}}\right) v$$

$$= H_{Rot_{\hat{z}\to\hat{x}} w} u \; \left(Rot_{\hat{y}\to\hat{x}} H_{Rot_{\hat{z}\to\hat{y}} w} Rot_{\hat{x}\to\hat{y}}\right) Rot_{\hat{y}\to\hat{x}} v$$

$$= H_{Rot_{\hat{z}\to\hat{x}} w} u \; H_{Rot_{\hat{y}\to\hat{x}}(Rot_{\hat{z}\to\hat{y}} w)} Rot_{\hat{y}\to\hat{x}} v$$

$$= H_{Rot_{\hat{z}\to\hat{x}} w} u \; H_{Rot_{\hat{z}\to\hat{x}} w} \; Rot_{\hat{y}\to\hat{x}} v$$

$$= u Rot_{\hat{y}\to\hat{x}} v = u \,\hat{\cdot}\, v.$$

The second from last equality holds since it is an equality of preserving the scalar product $\cdot$ by an h-rotation in G. □

The restriction of h-rotations actually makes the action of the group G on S to be partial, since the function G$\times$ $S \to S$ becomes a partial function. The concept of restriction of h-rotations corresponds to the concept of partial action of groups [7], or even more, to the slightly broader concept of local transformation groups [8] (p.20) usually related to local actions of Lie groups in geometry.

5 Disscusion

Vectors in the space S have two constraints, fixed magnitude and orthogonal real and imaginary parts. Both of the constraints have important mathematical and physical implications.

From geometrical point of view, it is important the group G acting on S to be orthogonal in order to preserve vector lengths and angles. Additionally, it is desirable the action of G to be transitive, that is, for any two elements $u, v \in S$, there should be an element $g \in$ G such that $gu = v$. Then, the fixed vector magnitudes are required to obtain transitive action of G, i.e. to make the space S homogeneous. Actually, in the case of $(S, \cdot)$ one can directly show that the representation G of SO(3, $\mathbb{C}$) acts transitively. Taking in mind that for $u, v \in S$, we have $(u+v) \cdot u = \lambda^2 + u \cdot v$ and $(u+v)^2 = 2(\lambda^2 + u \cdot v)$, one can write the transformation $H_{u \to v}$ that carry u to v in the following way

$$H_{u\to v} = I - \tfrac{2}{(u+v)^2}(u+v)\otimes(u+v) + \tfrac{2}{v^2} v\otimes u, \quad (u+v)^2 \neq 0.$$

In the case of $(S, \hat{\cdot})$, we have $(u \hat{+} v)^2 = 2(\lambda^2 + u \,\hat{\cdot}\, v) = 2(u \hat{+} v) \cdot u$ and the corresponding transformation is

$$T_{u\to v} = Rot_{\hat{x}\to\hat{y}} \widehat{H}_{u\to Rot_{\hat{y}\to\hat{x}} v}\ \text{, where}$$

$$\widehat{H}_{u\to Rot_{\hat{y}\to\hat{x}} v} = I - \frac{2}{(u \hat{+} v)^2}(u \hat{+} v)\otimes(u \hat{+} v) + \frac{2Rot_{\hat{y}\to\hat{x}} v\otimes u}{v^2}.$$

The orthogonality of $H_{u\to v}$,$\widehat{H}_{u\to Rot_{\hat{y}\to\hat{x}} v}$ and the equalities $H_{u\to v} u = v$,$T_{u\to v} u = v$ can be straightforwardly checked. Observe that $\widehat{H}_{u\to Rot_{\hat{y}\to\hat{x}} v}$ is not given in form (6), as the vector parameterizing $H_{u\to Rot_{\hat{y}\to\hat{x}} v}$ is rather complicated. However, $\widehat{H}_{u\to Rot_{\hat{y}\to\hat{x}} v}$ is a restricted h-rotation action adapted to the vector u, since it keeps the real part of u in place.

From physical point of view, the fixed vector magnitudes indicate that the real and imaginary parts of vectors describe dependent physical quantities with magnitudes that hyperbolically complement each other.

The orthogonality condition $\vec{x}\vec{a} = 0$ on $u = \vec{x} + i\vec{a}$ seems more interesting constraint. Mathematically, it is necessary condition to achieve both, real vector magnitudes and their hyperbolic calculations. From physical point of view, as we already mentioned in the introduction, separation of complex vectors on real and imaginary parts enables a corresponding separation of distinct physical concepts. However, the complex vector as a whole should be related to some physical system, and the relationship between real and imaginary part is necessary to reflect an important characteristic of the system. Beside fixed vector magnitudes, the orthogonality constraint provides an additional relation between the real and imaginary vector parts, i.e. between 3-vectors, which is useful concept in physical applications.

6 Conclusions

We considered a space S of complex vectors in $\mathbb{C}^3$ with constraints , called s-vectors. Taking in mind the physical applications, we introduced the real-valued scalar product $\hat{\cdot}$ which is always calculated from the point of view of a fixed direction of the s-vector real parts. So, the latter can be naturally associated to a physical coordinate system, where the fixed 3-vector can be related to an observer. Then, we introduced a representation G of the group SO(3, ℂ) acting on *S* through the polar decomposition of the SO(3, ℂ) matrices on (real orthogonal) / (positive definite Hermitian), i.e. rotation / h-rotation matrices. We found that the scalar product $\hat{\cdot}$ is not in agreement with the action of G since the h-rotations break the scalar product invariance. So, we introduced a restricted action of G that locally preserves the scalar product $\hat{\cdot}$ and corresponds to the action of the group SO(1,2), which is known to have applications in various branches of physics, including classical, relativistic and particle mechanics.

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