On Chromaticity of Ladder-Type Graphs

Abdul Rauf Nizami, Mobeen Munir and Amjad Shahbaz

Abstract. We give general formulas of the chromatic polynomial of some interesting families of ladder-type graphs, and conclude that, except two, neither two of them are chromatically equivalent. Moreover, some of them are not chromatically unique.

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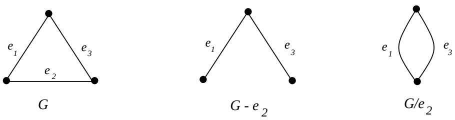
Keywords. Chromatic polynomial, Chromatic equivalence, Chromatic uniqueness, λ-coloring, Ladder-type graph.

1. Introduction

The chromatic polynomial was introduced by G. D. Birkhoff in 1912 as a function that counts the number of graph colorings for planar graphs to solve the four color problem [1]. In 1932 H. Whitney generalized it from the planar graphs to arbitrary graphs [7]. The chromatic polynomial, because of its theoretical and applied importance, has generated a large body of work. Chia [4] provides an extensive bibliography on the chromatic polynomial, and Dong, Koh, and Teo [6] give a comprehensive treatment.

The following two operations are essential to understand the chromatic polynomial definition for a graph G. These are: *edge deletion* denoted by $G' = G - e$, and *edge contraction* $G'' = G/e$.



The deletion/contraction operations

Definition 1.1. The *chromatic polynomial* is a function P from the set of all graphs to the set $\mathbb{Z}[\lambda]$, a ring of polynomials, such that

$$P(G) = \begin{cases} 0 & \textit{if there is a loop in } G \\ \lambda^n & \textit{if } G \textit{ consists of only } n \textit{ isolated vertices} \\ P(G-e) - P(G/e) & \textit{otherwise} \end{cases}$$

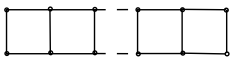
Two graphs are *chromatically equivalent* if they have the same chromatic polynomial; a graph G is chromatically unique if $P(G) = P(G')$ implies $G \cong G'$.

For a positive integer λ, a λ-*coloring* of a graph G is a mapping of $V(G)$ into the set $\{1, 2, 3, \ldots, \lambda\}$ of λ colors. Thus, there are exactly λ^n colorings for a graph on n vertices. If φ is a λ-coloring such that $\varphi(u) \neq \varphi(v)$ for all $uv \in E$, then φ is called a *proper* (or *admissible*) coloring. The *chromatic number* of a graph G, denoted by $\gamma(G)$, is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color.

Remark 1.2. Every evaluation of chromatic polynomial at some number λ actually gives the λ-coloring of the graph.

Since we are interested mainly in ladder-type graphs, we define them here. First, the two closely related definitions:

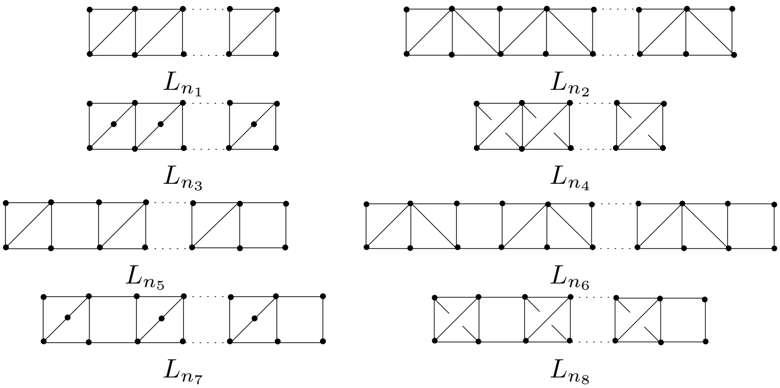
Definition 1.3. A *ladder graph* L_n is the Cartesian product of path graphs p_n and p_2:



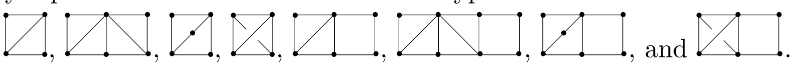
$L_n = p_n \times p_2$

We define a *ladder-type graph* a ladder graph with addition of some edges and vertices, in some pattern, keeping the main structure of L_n intact.

The ladder-type graphs we are concerned with are:



The subscripts n, n_1, n_2, n_3, n_4, n_5, n_6, n_7, and , n_8 in these graphs respectively represent number of 'unit' boxes of types



The following is the chromatic polynomial of the ladder graph L_n, which already exists in the literature.

Proposition 1.4. *The chromatic polynomial of the graph L_n is*

$$P(L_n) \quad = \quad \lambda(\lambda-1)(\lambda^2-3\lambda+3)^n.$$

First we give the chromatic polynomials of four 'basic' ladder-type graphs:

Theorem 1.5. *The chromatic polynomials of $L_{n_1}, L_{n_2}, L_{n_3}$, and L_{n_4} are*

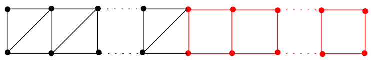
a. $P(L_{n_1}) = \lambda(\lambda-1)(\lambda-2)^{2n_1}$,
b. $P(L_{n_2}) = \lambda(\lambda-1)(\lambda-2)^{4n_2}$,
c. $P(L_{n_3}) = \lambda(\lambda-1)(\lambda^3-5\lambda^2+10\lambda-7)^{n_3}$, *and*
d. $P(L_{n_4}) = \lambda(\lambda-1)(\lambda-2)^{n_4}(\lambda-3)^{n_4}$.

Then we have the proposition:

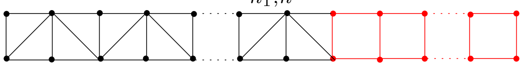
Proposition 1.6. *The chromatic polynomials of $L_{n_5}, L_{n_6}, L_{n_7}$, and L_{n_8} are*

a. $P(L_{n_5}) = \lambda(\lambda-1)(\lambda-2)^{2n_5}(\lambda^2-3\lambda+3)^{n_5}$,
b. $P(L_{n_6}) = \lambda(\lambda-1)(\lambda-2)^{2n_6}(\lambda^2-3\lambda+3)^{n_6}$,
c. $P(L_{n_7}) = \lambda(\lambda-1)(\lambda^2-3\lambda+3)^{n_7}(\lambda^3-5\lambda^2+10\lambda-7)^{n_7}$, *and*
d. $P(L_{n_8}) = \lambda(\lambda-1)(\lambda-2)^{n_8}(\lambda-3)^{n_8}(\lambda^2-3\lambda+3)^{n_8}$.

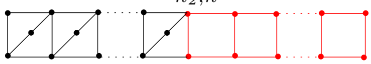
Besides the above graphs, the following are special types of ladder-types graphs. These are actually obtained by appending the ladder graph L_n to the graphs L_{n_1}, L_{n_2}, L_{n_3}, and L_{n_4}.



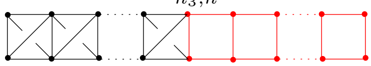
$L_{n_1,n}$



$L_{n_2,n}$



$L_{n_3,n}$



$L_{n_4,n}$

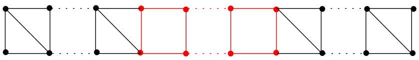
We shall give the chromatic polynomials of these graphs as a corollary of the general result:

Theorem 1.7. *If a graph G is obtained by appending L_n to a graph G_1 such that they share nothing except just one edge, then*

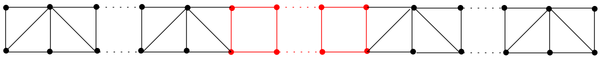
$$P(G) = (\lambda^2-3\lambda+3)^n P(G_1).$$

Corollary 1.8. **a.** $P(L_{n_1,n}) = \lambda(\lambda-1)(\lambda-2)^{2n_1}\left[\lambda^2-3\lambda+3\right]^n$.
b. $P(L_{n_2,n}) = \lambda(\lambda-1)(\lambda-2)^{4n_2}\left[\lambda^2-3\lambda+3\right]^n$.
c. $P(L_{n_3,n}) = \lambda(\lambda-1)(\lambda^3-5\lambda^2+10\lambda-7)^{n_3}\left[\lambda^2-3\lambda+3\right]^n$.
d. $P(L_{n_4,n}) = \lambda(\lambda-1)(\lambda-2)^{n_4}(\lambda-3)^{n_4}\left[\lambda^2-3\lambda+3\right]^n$.

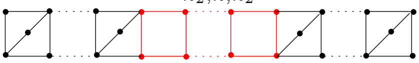
If L_n is sandwiched between a ladder-type graph $L_{n_i}, 1 \leq i \leq 4$, then then we shall denote the resultant ladder-type graph by L_{n_i,n,n_i}. The chromatic polynomials of the following graphs are given in a lemma:



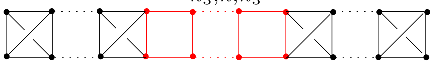
L_{n_1,n,n_1}



L_{n_2,n,n_2}



L_{n_3,n,n_3}



L_{n_4,n,n_4}

Lemma 1.9. **a.** $P(L_{n_1,n,n_1}) = (\lambda - 2)^{2n_1} P(L_{n_1,n})$.
b. $P(L_{n_2,n,n_2}) = (\lambda - 2)^{4n_2} P(L_{n_2,n})$.
c. $P(L_{n_3,n,n_3}) = (\lambda^2 - 5\lambda + 10\lambda - 7)^{n_3} P(L_{n_3,n})$.
d. $P(L_{n_4,n,n_4}) = (\lambda - 2)^{n_4} (\lambda - 3)^{n_4} P(L_{n_4,n})$.

The more general ladder-type graphs appear when L_n is sandwiched k times in $L_{n_i}, 1 \leq i \leq 4$. We denote these graphs by $L_{n_1,n,n_1,\ldots,n_1,n,n_1}$, $L_{n_2,n,n_2,\ldots,n_2,n,n_2}$, $L_{n_3,n,n_3,\ldots,n_3,n,n_3}$, and $L_{n_4,n,n_4,\ldots,n_4,n,n_4}$, and present their chromatic polynomials in the theorem:

Theorem 1.10. **a.** $P(L_{n_1,n,n_1,\cdots,n_1,n,n_1}) = \lambda(\lambda-1)(\lambda-2)^{2(k+1)n_1}(\lambda^2-3\lambda+3)^{kn}$.
b. $P(L_{n_2,n,n_2,\cdots,n_2,n,n_2}) = \lambda(\lambda - 1)(\lambda - 2)^{4(k+1)n_2}(\lambda^2 - 3\lambda + 3)^{kn}$.
c. $P(L_{n_3,n,n_3,\cdots,n_3,n,n_3}) = \lambda(\lambda-1)(\lambda^2-5\lambda+10\lambda-7)^{(k+1)n_3}(\lambda^2-3\lambda+3)^{kn}$.
d. $P(L_{n_4,n,n_4,\cdots,n_4,n,n_4}) = \lambda(\lambda - 1)(\lambda - 2)^{(k+1)n_4}(\lambda - 3)^{(k+1)n_4}(\lambda^2 - 3\lambda + 3)^{kn}$.

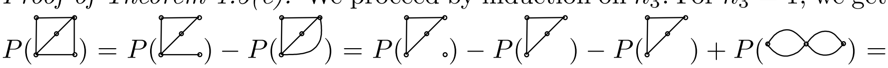
The chromatic equivalence and chromatic uniqueness of these graphs are reflected in the theorem:

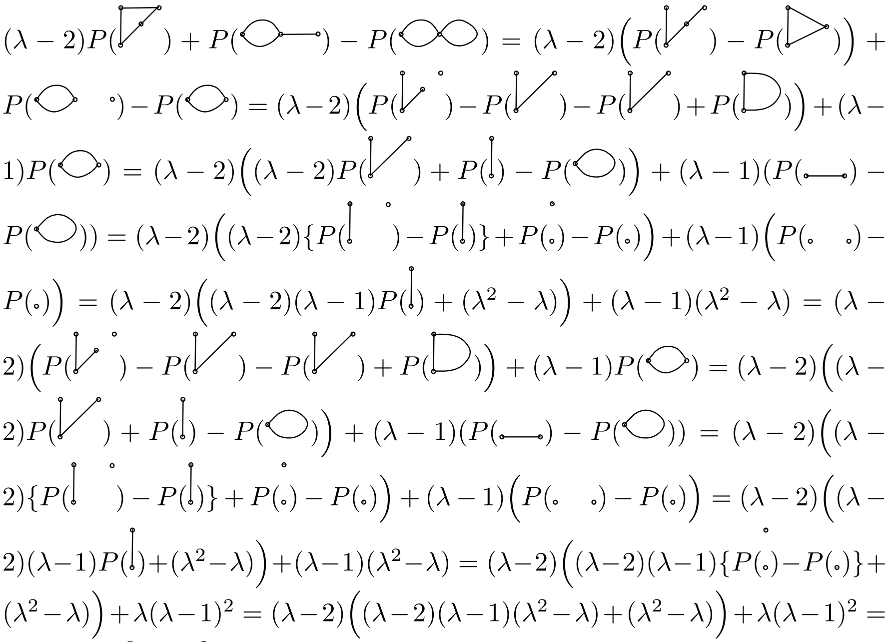
Theorem 1.11. **a.** *Neither two of L_n, L_{n_1}, L_{n_3}, and L_{n_4} are chromatically equivalent.*
b. *L_{n_1} and L_{n_2} are chromatically equivalent if $n_1 = 2n_2$, but are not chromatically unique.*
c. *$L_{n_1,n}, L_{n_3,n}$, and $L_{n_4,n}$ are not chromatically unique.*

2. Proofs

This section contains the proofs of the results we got.

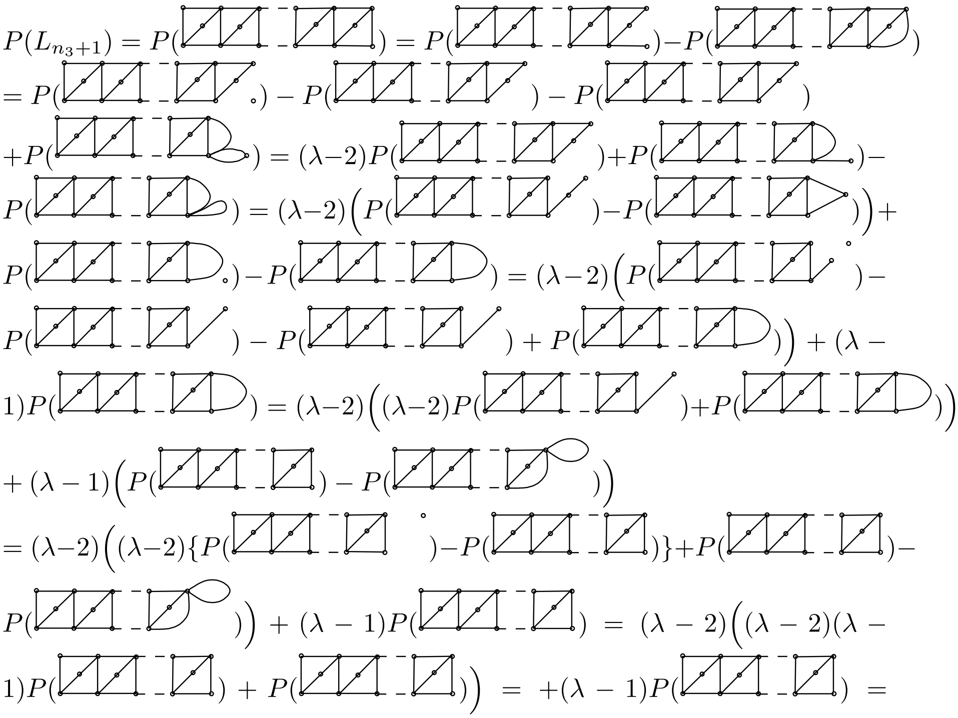
Proof of Theorem 1.5(c). We proceed by induction on n_3. For $n_3 = 1$, we get

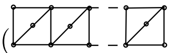




$\lambda(\lambda-1)(\lambda^3-5\lambda^2+10\lambda-7)$.

Now with the assumption that the result holds for an arbitrary n_3, we have

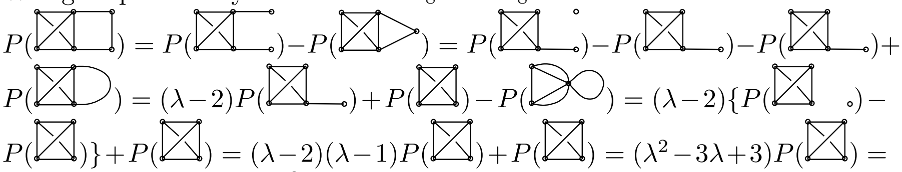


$\Big((\lambda-2)(\lambda^2-3\lambda+3)+(\lambda-1)\Big)P($$) = (\lambda^3-3\lambda^2+3\lambda-2\lambda^2+6\lambda-6+\lambda-1)P(L_{n_3}) = (\lambda^3-5\lambda^2+10\lambda-7)\Big(\lambda(\lambda-1)(\lambda^3-5\lambda^2+10\lambda-7)^{n_3}\Big) = \lambda(\lambda-1)(\lambda^3-5\lambda^2+10\lambda-7)^{n_3+1}$,
as was required.

Proofs of the parts (a), (b), and (d) are similar. □

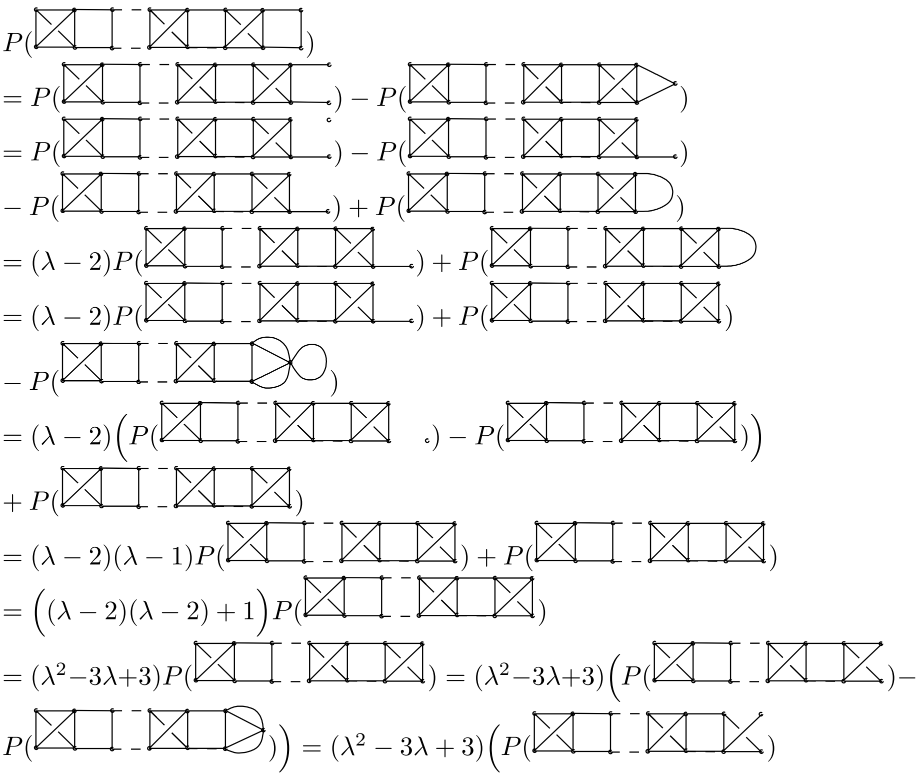
Proof of Proposition 1.6. Here we give only the proof of part (d); other parts cab be proved similarly.

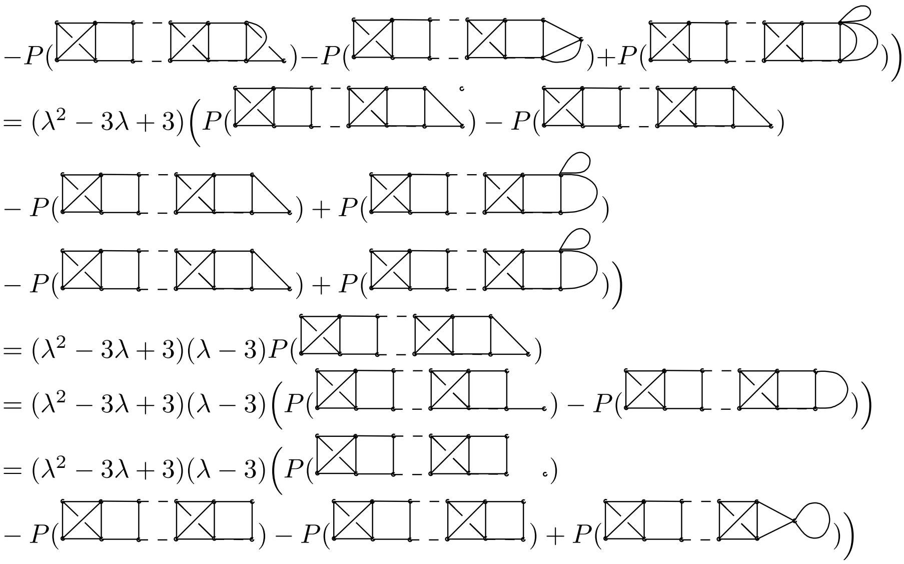
We again prove it by induction on n_8. For $n_8 = 1$ we have

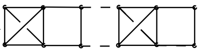


$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda^2-3\lambda+3)$

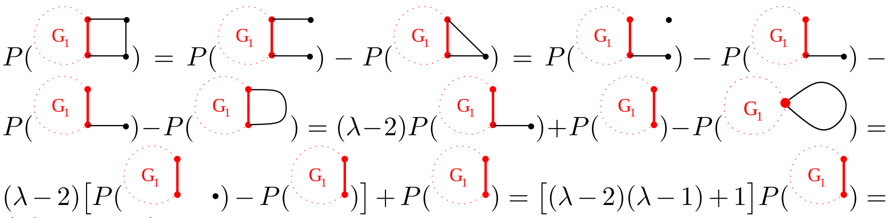
Now with the assumption that the result holds for an arbitrary n_8, we have





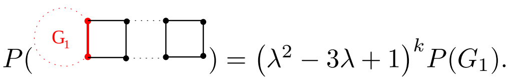
$= (\lambda^2-3\lambda+3)(\lambda-3)(\lambda-2)P($$) = (\lambda^2-3\lambda+3)(\lambda-3)(\lambda-2)\Big(\lambda(\lambda-1)(\lambda-2)^{n_8}(\lambda-3)^{n_8}(\lambda^2-3\lambda+3)^{n_8}\Big) = \lambda(\lambda-1)(\lambda-2)^{n_8+1}(\lambda-3)^{n_8+1}(\lambda^2-3\lambda+3)^{n_8+1}$. □

Proof of Theorem 1.7. We proceed by induction on n:
For $n = 1$, we get

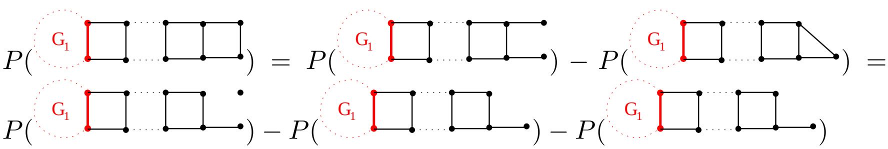


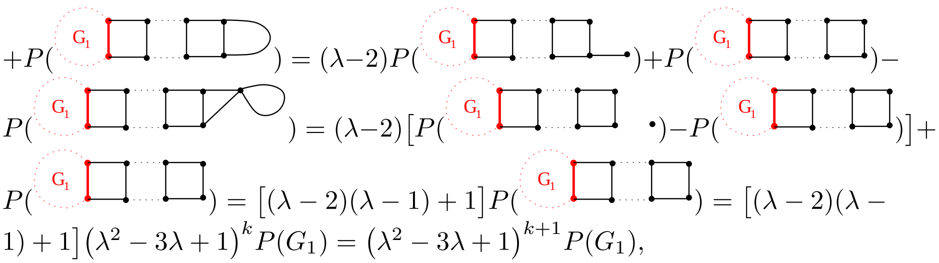
$\big(\lambda^2-3\lambda+1\big)P(G_1)$.

Suppose the result holds for $n = k$, that is



Now for $n = k + 1$, we have

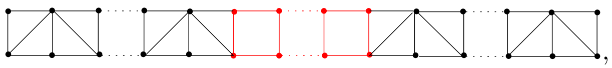




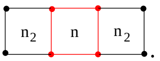
as was required. □

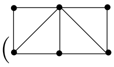
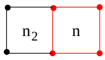
Proof of Lemma 1.9. We give only proof of part (b), which is the most difficult; other parts have similar proofs.

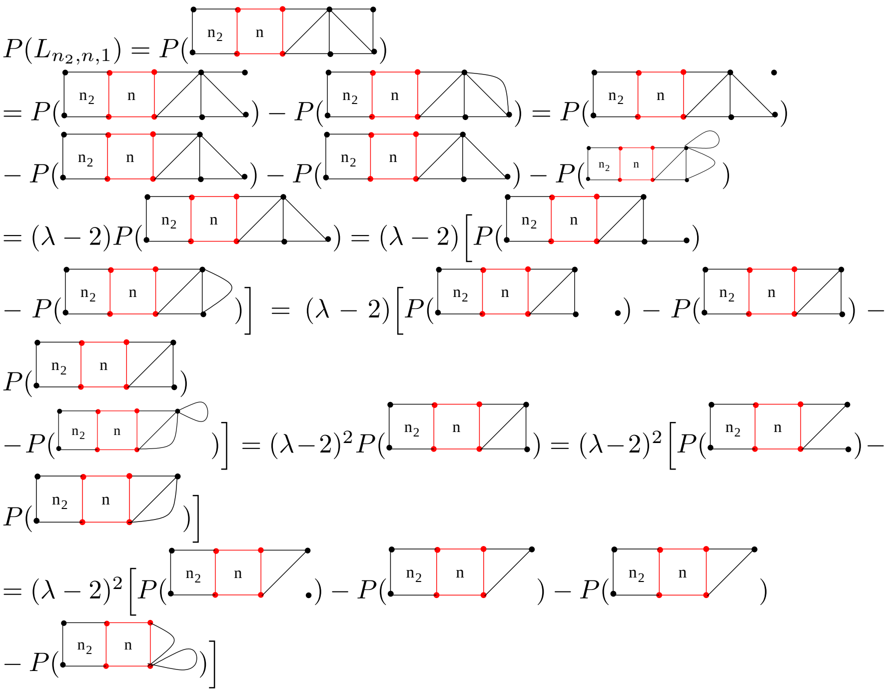
Instead of the long ladder

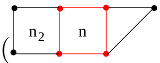
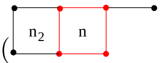
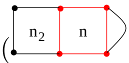


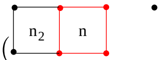
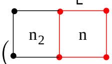
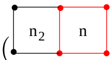
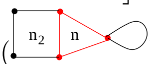
we shall use a short form as

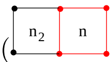


When a single unit () of L_{n_2} is appended to , we get

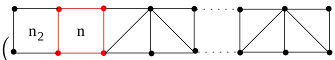
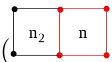


$= (\lambda-2)^3 P($  $) = (\lambda-2)^3 \Big[P($  $) - P($  $) \Big]$

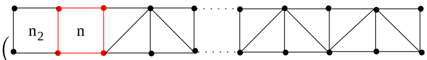
$= (\lambda-2)^3 \Big[P($  $) - P($  $) - P($  $) + P($  $) \Big]$

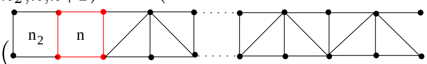
$= (\lambda-2)^4 P($  $).$

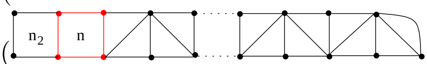
Now suppose the result holds when the appended ladder (L_{n_2}) has k units, that is

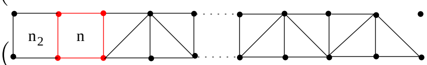
$P(L_{n_2,n,k}) = P($  $) = (\lambda-2)^{4k} P($  $).$

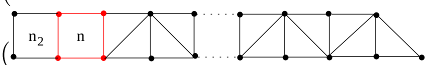
If the appended ladder has $k+1$ units, then we receive

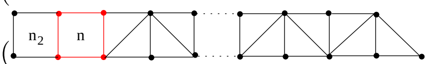
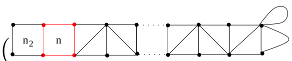
$P(L_{n_2,n,k+1}) = P($  $)$

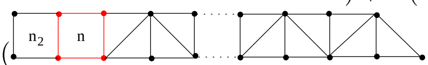
$= P($  $)$

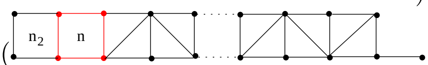
$- P($  $)$

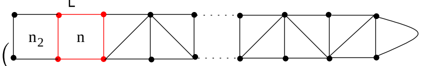
$= P($  $)$

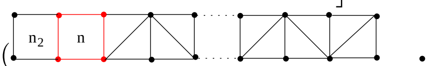
$- P($  $)$

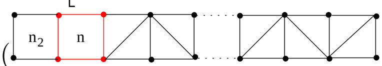
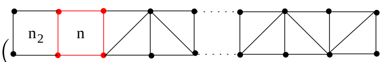
$- P($  $) + P($  $)$

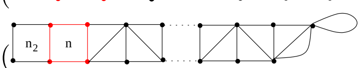
$= (\lambda-2) P($  $)$

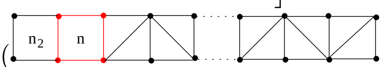
$= (\lambda-2) \Big[P($  $)$

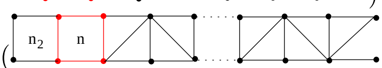
$- P($  $) \Big]$

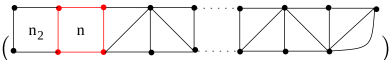
$= (\lambda-2) \Big[P($  $)$

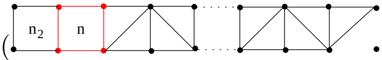
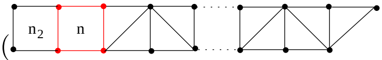
$- P($  $) - P($  $)$

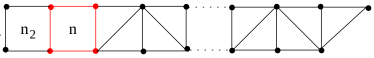
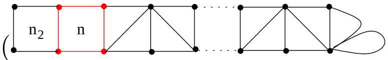
$+ P($  $) \Big]$

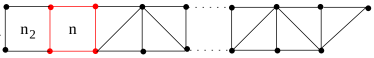
$= (\lambda-2)^2 P($  $)$

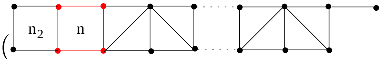
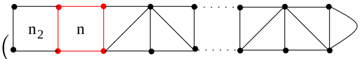
$= (\lambda-2)^2 \Big[P($  $)$

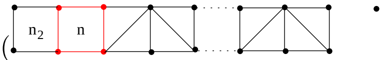
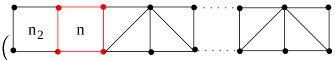
$-P($  $)\big]$

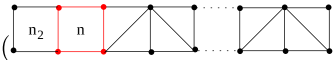
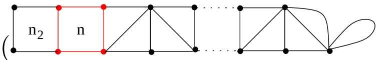
$= (\lambda-2)^2 \big[P($  $) - P($  $)$

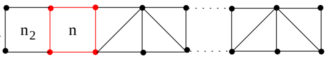
$-P($  $) + P($  $)\big]$

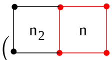
$= (\lambda - 2)^3 P($  $)$

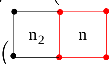
$= (\lambda-2)^3 \big[P($  $) - P($  $)\big]$

$= (\lambda-2)^3 \big[P($  $) - P($  $)$

$-P($  $) + P($  $)\big]$

$= (\lambda - 2)^4 P($  $)$

$= (\lambda - 2)^4 (\lambda - 2)^{4k} P($  $)$ (Inductive step!)

$= (\lambda - 2)^{4(k+1)} P($  $)$,

which is the desired result. □

Proof of Theorem 1.10. In each case, apply recursively Lemma 1.9 and Theorem 1.5 k times and then use $P(L_{n_i})$. □

Proof of Theorem 1.11. 1. Obvious; just see Theorem 1.5.

2. Simply observe that if $n_1 = 2n_2$, then $P(L_{n_1}) = P(L_{n_2})$. These are not chromatically unique because $L_{n_1} \ncong L_{n_2}$; just observe that there are vertices of degree 5 in L_{n_2} but are not in L_{n_1}.
3. It is obvious; simply observe that $P(L_{n_1,n}) = P(L_{n_5})$, $P(L_{n_2,n}) = P(L_{n_6})$, $P(L_{n_3,n}) = P(L_{n_7})$, and $P(L_{n_4,n}) = P(L_{n_8})$ while $L_{n_1,n} \ncong L_{n_5}$, $L_{n_2,n} \ncong L_{n_6}$, $L_{n_3,n} \ncong L_{n_7}$, and $L_{n_4,n} \ncong L_{n_8}$.

□

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