The Cantor Ternary Set Formula I-Basic Approach

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Abstract. Much of modern set theory was started by Georg Cantor (1845-1918). Among his major contributions was the introduction of the notion of the cantor set,which consists of points along a single line segment with a number of remarkable and deep properties. The following paper traces a brief history of topology in relation to the cantor set,defines the cantor set, states some of its basic properties and uses the basic approach in constructing the cantor ternary set.

Keywords: topology, open middle-third, interval, cantor set, geometry

Introduction

Overview

This introductory chapter seeks to present a very brief history of topology,and then consider a specific topic under topology-the cantor ternary set. The basic approach used by Cantor would be used in constructing a number of terms.

Background study

The field of topology had always been a hidden and embedded part of geometry. Up until the Konigsberg bridges problem, no one had had the courage of discussing geometry without measurement. And although, topology clearly dotted pure mathematics areas such as algebra and analysis, it remained unnoticed and unapproached. However, the ground breaking effort from Euler to the Konigsberg bridges problem in 1736 not only solved the problem but birthed an appreciation of geometry of different type where distance seemed irrelevant. Today its known as rubber-sheet geometry and the first person who used the word topology was Listing [1]. Topology literally means the study of position/location clearly seen from Euler's description. Topology may be defined as the study of shapes, including their properties, deformations applied to them, mappings between them and configurations composed of them.[2]

Cantor Set

The Cantor set is a set of points lying on a single line segment with a number of remarkable and deep properties [3]. The set obtained by repeatedly deleting the open-middle third from the closed interval [0, 1] is termed as the **Cantor Ternary Set**.

Some Properties of the Cantor Set

1. The cantor set has no interval
2. The cantor set is non-empty
3. The cantor set is closed and nowhere dense
4. The cantor set is compact
5. The cantor set is perfect and totally disconnected
6. The cantor set is uncountable [4]

Main Work

We begin with the closed real interval C_o, where the nth set formula is given as:

$$C_n = \frac{C_{n-1}}{3} U(\frac{2}{3} + \frac{C_{n-1}}{3})$$

Where $n = 1, 2, 3, \ldots$ and $C_o = [0, 1]$
$\Rrightarrow$ When n=1

$$C_1 = \frac{C_0}{3} U(\frac{2}{3} + \frac{C_0}{3}) \tag{1}$$

$$C_1 = (\frac{0}{3}, \frac{1}{3}) U(\frac{2}{3} + [\frac{0}{3}, \frac{1}{3}]) \tag{2}$$

Open middle third $I_1 = (\frac{1}{3}, \frac{2}{3})$

$$C_1 = [0, \frac{1}{3}]U[\frac{2}{3}, 1]$$

$\Rrightarrow$ When n=2

$$C_2 = \frac{C_1}{3}U(\frac{2}{3} + \frac{C_1}{3}) \quad (3)$$

$$C_2 = \frac{[0, \frac{1}{3}]U[\frac{2}{3}, 1]}{3}U(\frac{2}{3} + \frac{[0, \frac{1}{3}]U[\frac{2}{3}, 1]}{3}) \quad (4)$$

Open middle third $I_2 = (\frac{1}{9}, \frac{2}{9})U(\frac{7}{9}, \frac{8}{9})$

$$C_2 = [0, \frac{1}{9}]U[\frac{2}{9}, \frac{1}{3}]U[\frac{2}{3}, \frac{7}{9}]U[\frac{8}{9}, 1]$$

$\Rrightarrow$ When n=3

$$C_3 = \frac{C_2}{3}U(\frac{2}{3} + \frac{C_2}{3}) \quad (5)$$

$$C_3 = \frac{[0, \frac{1}{9}]U[\frac{2}{9}, \frac{1}{3}]U[\frac{2}{3}, \frac{7}{9}]U[\frac{8}{9}, 1]}{3}U(\frac{2}{3} + \frac{[0, \frac{1}{9}]U[\frac{2}{9}, \frac{1}{3}]U[\frac{2}{3}, \frac{7}{9}]U[\frac{8}{9}, 1]}{3}) \quad (6)$$

Open middle third $I_3 = (\frac{1}{27}, \frac{2}{27})U(\frac{7}{27}, \frac{8}{27})U(\frac{19}{27}, \frac{20}{27}U(\frac{25}{27}, \frac{26}{27})$

$$C_3 = [0, \frac{1}{27}]U[\frac{2}{27}, \frac{1}{9}]U[\frac{2}{9}, \frac{7}{27}]U[\frac{8}{27}, \frac{1}{3}]U[\frac{2}{3}, \frac{19}{27}]U[\frac{20}{27}, \frac{7}{9}]U[\frac{8}{9}, \frac{25}{27}]U[\frac{26}{27}, 1]$$

$\Rrightarrow$ When n=4

$$C_4 = \frac{C_3}{3}\mathbf{U}(\frac{2}{3} + \frac{C_3}{3})$$

$$C_4 = \frac{[0, \frac{1}{27}]U[\frac{2}{27}, \frac{1}{9}]U[\frac{2}{9}, \frac{7}{27}]U[\frac{8}{27}, \frac{1}{3}]U[\frac{2}{3}, \frac{19}{27}]U[\frac{20}{27}, \frac{7}{9}]U[\frac{8}{9}, \frac{25}{27}]U[\frac{26}{27}, 1]}{3}\mathbf{U}$$

$$(\frac{2}{3} + \frac{[0, \frac{1}{27}]U[\frac{2}{27}, \frac{1}{9}]U[\frac{2}{9}, \frac{7}{27}]U[\frac{8}{27}, \frac{1}{3}]U[\frac{2}{3}, \frac{19}{27}]U[\frac{20}{27}, \frac{7}{9}]U[\frac{8}{9}, \frac{25}{27}]U[\frac{26}{27}, 1]}{3})$$

Much concentration is needed as it is becoming more and more cumbersome already.
Open middle third $I_4 = (\frac{1}{81}, \frac{2}{81})U(\frac{7}{81}, \frac{8}{81})U(\frac{19}{81}, \frac{20}{81})U(\frac{25}{81}, \frac{26}{81})U(\frac{61}{81}, \frac{62}{81})U(\frac{73}{81}, \frac{74}{81})U(\frac{79}{81}, \frac{80}{81})$

$$C_4 = [0, \frac{1}{81}]U[\frac{2}{81}, \frac{1}{27}]U[\frac{2}{27}, \frac{7}{81}]U[\frac{8}{81}, \frac{1}{9}]U[\frac{2}{9}, \frac{19}{81}]U[\frac{20}{81}, \frac{7}{27}]U[\frac{8}{27}, \frac{25}{81}]U[\frac{26}{81}, \frac{1}{3}]U$$

$$[\frac{2}{3}, \frac{55}{81}]U[\frac{56}{81}, \frac{19}{27}]U[\frac{20}{27}, \frac{61}{81}]U[\frac{62}{81}, \frac{7}{9}]U[\frac{8}{9}, \frac{73}{81}]U[\frac{74}{81}, \frac{25}{27}]U[\frac{26}{27}, \frac{79}{81}]U[\frac{80}{81}, 1]$$

Remark 1:As n increases,per the formula it gets more and more cumbersome and care must be taken to note the technicalities that arise as seen throughout.The danger may be that one may leave out a particular middle third which would in effect affect the answer,

Let's see when n=5

$$C_5 = \frac{C_4}{3}U(\frac{2}{3} + \frac{C_4}{3})$$

We write out C_4 from the above and substitute into C_5,

Open middle third I_5 is:
$(\frac{1}{243}, \frac{2}{243})U(\frac{7}{243}, \frac{8}{243})U(\frac{19}{243}, \frac{20}{243})U(\frac{25}{243}, \frac{26}{243})U(\frac{61}{243}, \frac{62}{243})U(\frac{73}{243}, \frac{74}{243})U(\frac{79}{243}, \frac{80}{243})U(\frac{163}{243}, \frac{164}{243})U$

$(\frac{169}{243}, \frac{170}{243})U(\frac{181}{243}, \frac{182}{243})U(\frac{187}{243}, \frac{188}{243})U(\frac{217}{243}, \frac{218}{243})U(\frac{223}{243}, \frac{224}{243})U(\frac{235}{243}, \frac{236}{243})U(\frac{241}{243}, \frac{242}{243})$

$C_5 = [0, \frac{1}{243}]U[\frac{2}{243}, \frac{1}{81}]U[\frac{2}{81}, \frac{7}{243}]U[\frac{8}{243}, \frac{1}{27}]U[\frac{2}{27}, \frac{19}{243}]U[\frac{20}{243}, \frac{7}{81}]U[\frac{8}{81}, \frac{25}{243}]U[\frac{26}{243}, \frac{1}{9}]U[\frac{2}{9}, \frac{55}{243}]U$

$[\frac{56}{243}, \frac{19}{81}]U[\frac{20}{81}, \frac{61}{243}]U[\frac{62}{243}, \frac{7}{27}]U[\frac{8}{27}, \frac{73}{243}]U[\frac{74}{243}, \frac{25}{81}]U[\frac{26}{81}, \frac{79}{243}]U[\frac{80}{243}, \frac{1}{3}]U[\frac{2}{3}, \frac{163}{243}]U[\frac{164}{243}, \frac{55}{81}]U$

$[\frac{56}{81}, \frac{169}{243}]U[\frac{170}{243}, \frac{19}{27}]U[\frac{20}{27}, \frac{181}{243}]U[\frac{182}{243}, \frac{61}{81}]U[\frac{62}{81}, \frac{187}{243}]U[\frac{188}{243}, \frac{7}{9}]U[\frac{8}{9}, \frac{217}{243}]U[\frac{218}{243}, \frac{73}{81}]U$

$[\frac{74}{81}, \frac{223}{243}]U[\frac{224}{243}, \frac{25}{27}]U[\frac{26}{27}, \frac{235}{243}]U[\frac{236}{243}, \frac{79}{81}]U[\frac{80}{81}, \frac{241}{243}]U[\frac{242}{243}, 1]$

Conclusion

From the above text,we can say that the basic approach can be used to construct the ternary set. This approach can however be modified to reduce:

- computational effort and
- computational time

References

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