A Note on Variational Principle of Subsets for Nonautonomous Dynamical Systems

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Abstract

This paper,we introduce measure-theoretic for Borel probability measures to characterize upper and lower Katok measure-theoretic entropies in discrete type and the measure-theoretic entropy for arbitrary Borel probability measure in nonautonomous case. Then we establish new variational principles for Bowen topological entropy for nonautonomous dynamical systems.

Keywords: Nonautonomous ; Measure-theoretical entropies ; Variational principles

1 Introduction

As an important invariant of topological conjugacy, the notion of topological entropy was introduced by Adler, Konheim and McAndrew [1] in 1965 [3]. Topological entropy is a key tool to measure the complexity of a classical dynamical system, i.e. the exponential growth rate of the number of distinguishable orbits of the iterates of an endomorphism of a compact metric space.In 1973,Bowen [2] introduced the topological entropy $h^B_{top}(T, Z)$ for any set Z in a topological dynamical system X,in a way resembling Hausdorff dimension, where X is a compact metric space and $T : X \to X$ is a continuous selfmap. Bowen topological entropy plays a key role in topological dynamics and dimension theory [2]. In 2012, Feng and Huang [6] showed that there is certain variational relation between Bowen topological entropy and measure-theoretic entropy for arbitrary non-invariant compact set of a topological dynamical system (X, T). Following the idea of Brin and

Katok [8], they defined the measure-theoretic entropy for Borel probability measure on X for their results.

In contrast with the autonomous discrete, in contrast with the autonomous discrete case [13], the properties of the entropies for the nonautonomous dynamical systems have not been fully investigated. In order to have a good understanding of the topological entropy of a skew product of dynamical systems (as we know that the calculation of its topological entropy can be transformed into that of its fibers), Kolyada and Snoha [4] proposed the concept of topological entropy in 1996 for a nonautonomous dynamical system determined by a sequence of maps.A nonautonomous discrete dynamical systems (in short: NADDS) is a natural generalization of a classical dynamical systems, its dynamics is determined by a sequence of continuous self-maps $f_n : X \to X$ where $n \in \mathbb{N}$, defined on a compact metric space X.

By a nonautonomous dynamical system(NADDS for short) we understand a pair $(X, \{f_n\}_{n=1}^{\infty})$, where X is a compact metric space endowed with a metric d and $\{f_n\}_{n=1}^{\infty}$, is a sequence of continuous maps from X to X. In 2013, Kawan [11] generalized the classical notion of measure-theoretical entropy established by Kolomogorov and Sinai to NADSs, and proved that the measure-theoretical entropy can be estimated from above by its topological entropy. Following the idea of Brin and Katok [8] and Zhou [7] introduced the measure-theoretical entropy in nouautonomous case and established a variational principle for the first time. More results related to entropy for NADSs were developed in [12]. In this paper, We introduce ideas of Wang [9] to nonautonomous systems to establish new variational principles for Bowen topological entropy for nonautonomous dynamical systems.

Give a NADDS (X,$\{f_n\}_{n=1}^{\infty}$).For each $n \in \mathbb{N}_+$,the Bowen metric d_n on x is defined by $d_n(x, y) = \max_{0 \le i \le n-1} d(f_1^i(x), f_1^i(y))$. For every $\epsilon > 0$, we denote by $B_n(x, \epsilon)$ the open ball of radius ϵ in the metric d_n around x, i.e., $B_n(x, \epsilon) = \{y \in X : d_n(x, y) < \epsilon\}$.

We also consider a nonautonomous dynamical system (for short NADS) (X, ϕ) where (X, d) is a compact metric space and $\phi : [0, +\infty) \times X \to X$ is a continuous map with $\phi(0, x) = x$ for $x \in X$. We want to know whether there is certain variational relation of entropy for nonautonomous dynamical systems. For our study, we need to define the measure-theoretic entropy for arbitrary Borel probability measure in nonautonomous case.

Given a NADS (X, ϕ). For any $t \in [0, +\infty)$, the tth Bowen metric d_t^{ϕ} on X is defined by

$$d_t^{\phi}(x, y) = \max_{0 \le s \le t} \{d(\phi(s, x), \phi(s, x))\}$$

For every $\varepsilon > 0$, we denote by $B_t^{\phi}(x,\varepsilon)$ the open ball of radius ε in the metric d_t^{ϕ} around x, i.e.,

$$B_t^{\phi}(x,\varepsilon) = \{y \in X : d_t^{\phi}(x,y) < \varepsilon$$

.

Write $\phi^i(x) := \phi(i,x)$ for $i = 1, 2, \cdots$ and $x \in X$. In this case ,we take $f_n(x) = \phi^n(x)$, then $\{\phi^n\}_{n=1}^{\infty}$ is a NADDS.

Let $M(X)$ denote the set of all Borel probability measures on X, $Z \subset X$ and $\mu \in M(X)$,$(X, \{f_n\}_{n=1}^{\infty})$ is a NADDS.

(1) A set $E \subset Z$ is said to be an (n, ε, Z)-separated set if $x, y \in E$ with $x \neq y$ implies $d_n^{\phi}(x,y) > \varepsilon$. Let $r_n(\varepsilon, Z)$ denote the maximum cardinality of (n, ε, Z)-separated set.

(2)A set $F \subset Z$ is said to be an (n, ε, Z)-spanning set if for any $x \in X$, there exists $y \in F$ with $d_n^{\phi}(x,y) \leq \varepsilon$. Let $s_n(\varepsilon, Z)$ denote the minimum cardinality of (n, ε, Z)-spanning sets.

(3) A set $F \subset X$ is said to be a $(\mu, n, \varepsilon, \delta)$-spanning set if the union $\bigcup\limits_{x \in F} B_n(x,\varepsilon)$ has μ-measure more than or equal to $1-\delta$. Let $r_n(\mu, \varepsilon, \delta)$ denote the minimum cardinality of $(\mu, n, \varepsilon, \delta)$-spanning sets.

(4) We introduce a useful set:$X_{\mu,\delta} = \{Z \subset X : \mu(Z) \geq 1 - \delta\}$. Then it is clear that

$$r_n(\mu, \varepsilon, \delta) = \inf_{Z \in X_{\mu,\delta}} r_n(\varepsilon, Z)$$

An open cover of X is a family of open subsets of X, whose union is X. For two covers $\mathcal{U}$ and $\mathcal{V}$ we say that $\mathcal{U}$ is a refinement of $\mathcal{V}$ if for each $U \in \mathcal{U}$ there is $V \in \mathcal{V}$ with $U \in V$. For$n \in \mathbb{N}$ and open covers $\mathcal{U}_1, \mathcal{U}_2, \cdots, \mathcal{U}_n$ of X we denote

$$\bigvee_{i=1}^{n} \mathcal{U}_i = \{A_1 \cap A_2 \cap \cdots \cap A_n : A_1 \in \mathcal{U}_1, A_2 \in \mathcal{U}_2, \cdots, A_n \in \mathcal{U}_n\}$$

.

Note that $\bigvee\limits_{i=1}^{n} \mathcal{U}_i$ is also an open cover of X. We denote by $\mathcal{N}(\mathcal{U})$ the minimal cardinality of all subcovers chosen from U. Set

$$f_i^0 = id_X, f_i^n = f_{i+(n-1)} \circ f_{i+(n-2)} \circ \cdots \circ f_{i+1} \circ f_i, f_i^{-n} = (f_i^n)^{-1}$$

for all $i, n \in \mathbb{N}$, where id_X is the identity map on X. Let

$$h_{top}(\{f_n\}_{n=1}^{\infty}, \mathcal{U}) = \limsup_{n \to \infty} \frac{\log \mathcal{N}(\bigvee\limits_{i=0}^{n} f_1^{-i}\mathcal{U})}{n}.$$

The *topological entropy* is defined by

$$h_{top}(X,\{f_n\}_{n=1}^{\infty}) = \{h_{top}(\{f_n\}_{n=1}^{\infty},\mathcal{U}) : \mathcal{U} \text{ is an open cover of } X\}.$$

It was proved in [1] that for every NADS, we have

$$h_{top}(X,\{f_n\}_{n=1}^{\infty}) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{\log s_n(\varepsilon,X)}{n} = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{\log r_n(\varepsilon,X)}{n}.$$

Following the idea of Katok [1],we give the following.

Definition 1.1. Let $\mu \in M(X)$. The NADDS *Katok measure-theoretical lower and upper entropies* of μ are defined respectively by

$$\underline{h}_{\mu}^{K}(\{f_n\}_{n=1}^{\infty}) = \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\lim_{n\to\infty}\inf\frac{1}{n}\log r_n(\mu,\varepsilon,\delta),$$
$$\overline{h}_{\mu}^{K}(\{f_n\}_{n=1}^{\infty}) = \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\lim_{n\to\infty}\sup\frac{1}{n}\log r_n(\mu,\varepsilon,\delta)$$

In this paper, we introduce many quantities for Borel probability measure $\mu \in M(X)$, respectively denoted by $e_{\mu}(\{f_n\}_{n=1}^{\infty})$,$\underline{e}_{\mu}(\{f_n\}_{n=1}^{\infty})$,$\overline{e}_{\mu}(\{f_n\}_{n=1}^{\infty})$, $e_{\mu}^{*}(\{f_n\}_{n=1}^{\infty})$, and so on. According to the relations of the several types of NADS topological entropies, it is natural to consider relationship of some new quantities and Katok measure-theoretical lower and upper entropies. Therefore, we have the first main result.

Our main result is as follows.

Theorem 1.2. *Let* $(X,\{f_n\}_{n=1}^{\infty})$ *be a NADDS,*$\mu \in M(X)$*. Then following statements hold.*

(i) *For any* $Z \subseteq X$, $h_{top}^{B}(\{f_n\}_{n=1}^{\infty},Z) \le h_{top}^{P}(\{f_n\}_{n=1}^{\infty},Z)$.

(ii) $\overline{h}_{\mu}^{K}(\{f_n\}_{n=1}^{\infty}) = \bar{e}_{\mu}(\{f_n\}_{n=1}^{\infty})$.

(iii) $\underline{h}_{\mu}^{K}(\{f_n\}_{n=1}^{\infty}) = \underline{e}_{\mu}(\{f_n\}_{n=1}^{\infty})$.

(iv) $e_{\mu}(\{f_n\}_{n=1}^{\infty}) \le \underline{e}_{\mu}(\{f_n\}_{n=1}^{\infty}) \le \bar{e}_{\mu}(\{f_n\}_{n=1}^{\infty})$.

(v) $e_{\mu}(\{f_n\}_{n=1}^{\infty}) \le e_{\mu}^{*}(\{f_n\}_{n=1}^{\infty}) = \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\inf_{Z\in X_{\varepsilon,\delta}} h_{top}^{P}(\{f_n\}_{n=1}^{\infty},Z,\varepsilon)$.

where the definitions of these notions will be given in Section 2.

Theorem 1.3. *Let* $(X,\{f_n\}_{n=1}^{\infty})$ *be a NADDS. If* $K \subset X$ *is a non-empty and compact, then*

$$h_{top}^{B}(\{f_n\}_{n=1}^{\infty},K) = \sup\{e_{\mu}(\{f_n\}_{n=1}^{\infty})\colon \mu \in M(X), \mu(K) = 1\}.$$

Theorem 1.4. *Let (X,ϕ) be a NADS,$\mu \in M(X)$. Then following statements hold.*

(i) *For any $Z \subseteq X$, $h^B_{top}(\phi, Z) \leq h^P_{top}(\phi, Z)$.*

(ii) $e_\mu(\phi) \leq \underline{e}_\mu(\phi) \leq \bar{e}_\mu(\phi)$.

(iii) $e_\mu(\phi) \leq e^*_\mu(\phi) = \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\inf_{Z\in X_{\varepsilon,\delta}} h^P_{top}(\phi, Z, \varepsilon)$.

Theorem 1.5. *Let (X,ϕ) be a NADS. If $K \subset X$ is a non-empty and compact, then*

$$h^B_{top}(\phi, K) = \sup\{e_\mu(\phi)\colon \mu \in M(X), \mu(K) = 1\}.$$

2 Preliminaries

2.1 NADDS

In this subsection, let $(X, \{f_n\}_{n=1}^\infty)$ be a NADDS, next we introduced NADDS's entropies. Following, we give some definitions of several NADDS topological entropies of subsets.

Definition 2.1. Let $Z \subset X$, $s \geq 0$, $N \in \mathbb{N}$ and $\varepsilon > 0$, define

$$M^s_{N,\varepsilon}(\{f_n\}_{n=1}^\infty, Z) = \inf \sum_i \exp(-sn_i),$$

where the infimum is taken over all finite or countable families $\{B_{n_i}(x_i, \varepsilon)\}$ such that $x_i \in X$, $n_i \geq N$ and $\bigcup_i B_{n_i}(x_i, \varepsilon) \supseteq Z$. The quantity $M^s_{N,\varepsilon}(\{f_n\}_{n=1}^\infty, Z)$ does not decrease as N increase and ε decreases, hence the following limits exist:

$$M^s_\varepsilon(\{f_n\}_{n=1}^\infty, Z) = \lim_{N\to\infty} M^s_{N,\varepsilon}(\{f_n\}_{n=1}^\infty, Z),$$
$$M^s(\{f_n\}_{n=1}^\infty, Z) = \lim_{\varepsilon\to 0} M^s_\varepsilon(\{f_n\}_{n=1}^\infty, Z).$$

Bowen's topological entropy $h^B_{top}(\{f_n\}_{n=1}^\infty, Z)$ is defined as a critical value of the parameters s, where $M^s(\{f_n\}_{n=1}^\infty, Z)$ jumps from ∞ to 0, i.e.

$$M^s(\{f_n\}_{n=1}^\infty, Z) = \begin{cases} 0, & s > h^B_{top}(\{f_n\}_{n=1}^\infty, Z), \\ \infty, & s < h^B_{top}(\{f_n\}_{n=1}^\infty, Z). \end{cases}$$

Definition 2.2. Let $Z \subseteq X$. For $s \geq 0$,$N \in \mathbb{N}$ and $\varepsilon > 0$, define

$$P^s_{N,\varepsilon}(\{f_n\}_{n=1}^\infty, Z) = \sup \sum_i \exp(-sn_i),$$

where the supremum is taken over all finite or countable pairwise disjoint families$\{\overline{B}_{n_i}(x_i,\varepsilon)\}$ such that $x_i \in Z$, $n_i \geq N$ for all i, where $\{\overline{B}_{n_i}(x_i,\varepsilon)\} := \{y \in X : d_n(x,y) \leq \varepsilon\}$.
The quantity $P^s_{N,\varepsilon}(\{f_n\}_{n=1}^{\infty}, Z)$ does not decrease as N,ε decrease, hence the following limit exists:

$$P^s_{\varepsilon}(\{f_n\}_{n=1}^{\infty}, Z) = \lim_{N\to\infty} P^s_{N,\varepsilon}(\{f_n\}_{n=1}^{\infty}, Z).$$

Define

$$\mathcal{P}^s_{\varepsilon}(\{f_n\}_{n=1}^{\infty}, Z) = \inf\{\sum_{i=1}^{\infty} P^s_{\varepsilon}(\{f_n\}_{n=1}^{\infty}, Z_i) : \bigcup_{i=1}^{\infty} Z_i \supseteq Z\}.$$

There exists a critical value of the parameters s, which we will denote by $h^P_{top}(\{f_n\}_{n=1}^{\infty}, Z, \varepsilon)$, where $P^s_{\varepsilon}(\{f_n\}_{n=1}^{\infty}, Z)$ jumps from ∞ to 0, i.e.

$$\mathcal{P}^s(\{f_n\}_{n=1}^{\infty}, Z) = \begin{cases} 0, & s > h^P_{top}(\{f_n\}_{n=1}^{\infty}, Z, \varepsilon), \\ \infty, & s < h^P_{top}(\{f_n\}_{n=1}^{\infty}, Z, \varepsilon). \end{cases}$$

Note that $h^P_{top}(\{f_n\}_{n=1}^{\infty}, Z, \varepsilon)$ increases when ε decreases. We call

$$h^P_{top}(\{f_n\}_{n=1}^{\infty}, Z) := \lim_{\varepsilon\to 0} h^P_{top}(\{f_n\}_{n=1}^{\infty}, Z, \varepsilon)$$

the packing topological entropy of Z.

Definition 2.3. Let $Z \subseteq X$. For $s \geq 0$,$N \in \mathbb{N}$ and $\varepsilon > 0$, define

$$R^s_{N,\varepsilon}(\{f_n\}_{n=1}^{\infty}, Z) = \inf \sum_i \exp(-sN),$$

where the infimum is taken over all finite or countable families $\{B_N(x_i,\varepsilon)\}$ such that $x_i \in X$, and $\bigcup_i B_N(x_i,\varepsilon) \supseteq Z$. Let

$$\underline{R}^s_{\varepsilon}(\{f_n\}_{n=1}^{\infty}, Z) = \liminf_{N\to\infty} R^s_{N,\varepsilon}(\{f_n\}_{n=1}^{\infty}, Z),$$

$$\overline{R}^s_{\varepsilon}(\{f_n\}_{n=1}^{\infty}, Z) = \limsup_{N\to\infty} R^s_{N,\varepsilon}(\{f_n\}_{n=1}^{\infty}, Z)$$

and

$$\underline{Ch}_Z(\{f_n\}_{n=1}^{\infty}, \varepsilon) = \inf\{s : \underline{R}^s_{\varepsilon}(\{f_n\}_{n=1}^{\infty}, Z) = 0\} = \sup\{s : \underline{R}^s_{\varepsilon}(\{f_n\}_{n=1}^{\infty}, Z) = +\infty\},$$

$$\overline{Ch}_Z(\{f_n\}_{n=1}^{\infty}, \varepsilon) = \inf\{s : \overline{R}^s_{\varepsilon}(\{f_n\}_{n=1}^{\infty}, Z) = 0\} = \sup\{s : \underline{R}^s_{\varepsilon}(\{f_n\}_{n=1}^{\infty}, Z) = +\infty\}.$$

The lower and upper capacity topological entropies of $\{f_n\}_{n=1}^{\infty}$ restricted to Z are defined respectively by

$$\underline{Ch}_Z(\{f_n\}_{n=1}^{\infty}) = \lim_{\varepsilon\to 0} \underline{Ch}_Z(\{f_n\}_{n=1}^{\infty}, \varepsilon),$$

$$\overline{Ch}_Z(\{f_n\}_{n=1}^{\infty}) = \lim_{\varepsilon\to 0} \overline{Ch}_Z(\{f_n\}_{n=1}^{\infty}, \varepsilon).$$

Following we introduce several measure-theoretic definition.

Definition 2.4. Let $\mu \in M(X)$, $s \geq 0$, $N \in \mathbb{N}$ $\varepsilon > 0$ and $0 < \delta < 1$, define

$$M^s_{N,\varepsilon}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) = \inf \sum_i \exp(-sn_i),$$

where the infimum is taken over all finite or countable families $\{B_{n_i}(x_i, \varepsilon)\}$ such that $x_i \in X$, $n_i \geq N$ and $\mu(\bigcup_i B_{n_i}(x_i, \varepsilon)) \geq 1 - \delta$. The quantity $M^s_{N,\varepsilon}(\{f_n\}_{n=1}^{\infty}, \mu, \delta)$ does not decrease as N increase, hence the following limit exist:

$$M^s_{\varepsilon}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) = \lim_{N\to\infty} M^s_{N,\varepsilon}(\{f_n\}_{n=1}^{\infty}, \mu, \delta).$$

Using standard method, we have following is well- defined:

$$\begin{aligned} e_{\mu}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta) &= \inf\{s : M^s_{\varepsilon}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) = 0\} \\ &= \sup\{s : M^s_{\varepsilon}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) = +\infty\}, \end{aligned}$$

defined

$$e_{\mu}(\{f_n\}_{n=1}^{\infty}) = \lim_{\delta\to 0} \lim_{\varepsilon\to 0} e_{\mu}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta).$$

Definition 2.5. Let $\mu \in M(X)$, $s \geq 0$, $N \in \mathbb{N}$, $\varepsilon > 0$ and $0 < \delta < 1$, put

$$R^s_{N,\varepsilon}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) = \inf \sum_i \exp(-sN),$$

where the infimum is taken over all finite or countable families $\{B_N(x_i, \varepsilon)\}$ such that $x_i \in X$, and $\mu(\bigcup_i B_N(x_i, \varepsilon)) \geq 1 - \delta$. Let

$$\underline{R}^s_{\varepsilon}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) = \liminf_{N\to\infty} R^s_{N,\varepsilon}(\{f_n\}_{n=1}^{\infty}, \mu, \delta),$$

$$\overline{R}^s_{\varepsilon}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) = \limsup_{N\to\infty} R^s_{N,\varepsilon}(\{f_n\}_{n=1}^{\infty}, \mu, \delta).$$

Using standard method, we have following is well- defined:

$$\begin{aligned} \underline{e}_{\mu}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta) &= \inf\{s : \underline{R}^s_{\varepsilon}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) = 0\} \\ &= \sup\{s : \underline{R}^s_{\varepsilon}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) = +\infty\} \\ \overline{e}_{\mu}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta) &= \inf\{s : \overline{R}^s_{\varepsilon}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) = 0\} \\ &= \sup\{s : \overline{R}^s_{\varepsilon}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) = +\infty\}, \end{aligned}$$

define

$$\underline{e}_{\mu}(\{f_n\}_{n=1}^{\infty}) = \lim_{\delta\to 0} \lim_{\varepsilon\to 0} \underline{e}_{\mu}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta),$$

$$\overline{e}_{\mu}(\{f_n\}_{n=1}^{\infty}) = \lim_{\delta\to 0} \lim_{\varepsilon\to 0} \overline{e}_{\mu}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta).$$

Definition 2.6. Let $\mu \in M(X)$, $s \geq 0$, $N \in \mathbb{N}$ $\varepsilon > 0$ and $0 < \delta < 1$, put

$$\mathcal{P}_{\varepsilon}^{s}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) = \inf\{\sum_{i=1}^{\infty} P_{\varepsilon}^{s}(\{f_n\}_{n=1}^{\infty}, Z_i) : \mu(\bigcup_{i=1}^{\infty} Z_i) \geq 1 - \delta\},$$

where $P_{\varepsilon}^{s}(\{f_n\}_{n=1}^{\infty}, Z_i)$ is defined in Definition 2.2. There exists a critical value of s such that

$$\begin{aligned} e_{\mu}^{*}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta) &= \inf\{s : \mathcal{P}_{\varepsilon}^{s}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) = 0\} \\ &= \sup\{s : \mathcal{P}_{\varepsilon}^{s}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) = +\infty\}. \end{aligned}$$

Define

$$e_{\mu}^{*}(\{f_n\}_{n=1}^{\infty}) = \lim_{\delta \to 0} \lim_{\varepsilon \to 0} e_{\mu}^{*}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta).$$

2.2 NADS

In this subsection, let (X, ϕ) be a NADS, next we introduced NADS's entropies.

Definition 2.7. Let $Z \subset X$, $s \geq 0$, $N \in \mathbb{N}$ and $\varepsilon > 0$, define

$$M_{N,\varepsilon}^{s}(\phi, Z) = \inf \sum_{i} \exp(-st_i),$$

where the infimum is taken over all finite or countable families $\{B_{t_i}^{\phi}(x_i, \varepsilon)\}$ such that $x_i \in X$, $t_i \geq N$ and $\bigcup_{i} B_{t_i}^{\phi}(x_i, \varepsilon) \supseteq Z$. The quantity $M_{N,\varepsilon}^{s}(\phi, Z)$ does not decrease as N increase and ε decreases, hence the following limits exist:

$$\begin{aligned} M_{\varepsilon}^{s}(\phi, Z) &= \lim_{N \to \infty} M_{N,\varepsilon}^{s}(\phi, Z), \\ M^{s}(\phi, Z) &= \lim_{\varepsilon \to 0} M_{\varepsilon}^{s}(\phi, Z). \end{aligned}$$

Bowen's topological entropy $h_{top}^{B}(\phi, Z)$ is defined as a critical value of the parameters s, where $M^{s}(\phi, Z)$ jumps from ∞ to 0, i.e.

$$M^{s}(\phi, Z) = \begin{cases} 0, & s > h_{top}^{B}(\phi, Z), \\ \infty, & s < h_{top}^{B}(\phi, Z). \end{cases}$$

Other topological entropy definitions are similar to the discrete case definition.

Definition 2.8. Let $\mu \in M(X)$, $s \geq 0$, $N \in \mathbb{N}$ $\varepsilon > 0$ and $0 < \delta < 1$, define

$$M_{N,\varepsilon}^{s}(\phi, \mu, \delta) = \inf \sum_{i} \exp(-st_i),$$

where the infimum is taken over all finite or countable families $\{B^{\phi}_{t_i}(x_i,\varepsilon)\}$ such that $x_i \in X$, $t_i \geq N$ and $\mu(\bigcup_i B^{\phi}_{t_i}(x_i,\varepsilon)) \geq 1-\delta$. The quantity $M^s_{N,\varepsilon}(\phi,\mu,\delta)$ does not decrease as N increase, hence the following limit exist:

$$M^s_{\varepsilon}(\phi,\mu,\delta) = \lim_{N\to\infty} M^s_{N,\varepsilon}(\phi,\mu,\delta).$$

Using standard method, we have following is well- defined:

$$e_\mu(\phi,\varepsilon,\delta) = \inf\{s : M^s_{\varepsilon}(\phi,\mu,\delta) = 0\} = \sup\{s : M^s_{\varepsilon}(\phi,\mu,\delta) = +\infty\},$$

defined

$$e_\mu(\phi) = \lim_{\delta\to 0}\lim_{\varepsilon\to 0} e_\mu(\phi,\varepsilon,\delta).$$

Other measure-theoretic entropy definitions are similar to the discrete case definition.

3 Proof of Theorem

3.1 Proof of Theorem 1.2

Proposition 3.1. *Let $0<\delta<1$, $\mu \in M(X)$, $\{Z_i\}_{i=1}^{\infty}$ be a family of Borel subsets of X with $\mu(\bigcup_{i=1}^{\infty} Z_i) \geq 1-\delta$. For any $\epsilon > 0$, $M^s_{\epsilon}(\{f_n\}_{n=1}^{\infty},\mu,\delta) \leq \sum_{i=1}^{\infty} M^s_{\epsilon}(\{f_n\}_{n=1}^{\infty}, Z_i)$.*

Proof. For any $\varepsilon>0$,N,$i\in\mathbb{N}$, there exists $N_i > N$ such that

$$M^s_{N_i,\epsilon}(\{f_n\}_{n=1}^{\infty}, Z_i) < M^s_{\epsilon}(\{f_n\}_{n=1}^{\infty}, Z_i) + \frac{\varepsilon}{2^i}.$$

Hence, there exists a countable family $\{B_{n^i_j}(x^i_j,\epsilon)\}_{j=1}^{\infty}$ such that $n^i_j \geq N_i$, $x^i_j \in X$, $\{B_{n^i_j}(x^i_j,\epsilon)\} \supseteq Z_i$,

$$\sum_{j=1}^{\infty} e^{-sn^i_j} < M^s_{\epsilon}(\{f_n\}_{n=1}^{\infty}, Z_i) + \frac{\varepsilon}{2^i}.$$

Since $\mu(\bigcup_{i=1}^{\infty}) \geq 1-\delta$, we have $\mu(\bigcup_{i\geq 1}\bigcup_{j\geq 1}(B_{n^i_j}(x^i_j,\epsilon))) \geq 1-\delta$. Hence

$$M^s_{\epsilon}(\{f_n\}_{n=1}^{\infty},\mu,\delta) \leq \sum_{i\geq 1}\sum_{j\geq 1} e^{-sn^i_j} < \sum_{i=1}^{\infty} M^s_{\epsilon}(\{f_n\}_{n=1}^{\infty}, Z_i).$$

□

Now we are ready to prove the main result.

Proof. (i) Let $Z \subseteq x$ and assume be $0 < s < h_{top}^{B}(\{f_n\}_{n=1}^{\infty}, Z)$. For any $n \in \mathbb{N}$ and $\varepsilon > 0$, let $R = R_n(\{f_n\}_{n=1}^{\infty}, Z, \varepsilon)$ be the largest number so that there is a disjoint family $\{\overline{B}_n(x_i, \varepsilon)\}_{i=1}^{R}$ with $x_i \in Z$. Then it is easy to see that for any $\delta > 0$,

$$\bigcup_{i=1}^{R} \overline{B}_n(x_i, 2\varepsilon + \delta) \supseteq Z,$$

which implies that

$$M_{n,2\varepsilon+\delta}^{s}(\{f_n\}_{n=1}^{\infty}, Z) \leq Re^{-ns} \leq P_{n,\varepsilon}^{s}(\{f_n\}_{n=1}^{\infty}, Z)$$

for any $s \geq 0$, and hence $M_{2\varepsilon+\delta}^{s}(\{f_n\}_{n=1}^{\infty}, Z) \leq P_{\varepsilon}^{s}(\{f_n\}_{n=1}^{\infty}, Z)$, we have $M_{2\varepsilon+\delta}^{s}(\{f_n\}_{n=1}^{\infty}, Z) \leq \mathcal{P}_{\varepsilon}^{s}(\{f_n\}_{n=1}^{\infty}, Z)$. Since $0 < s < h_{top}^{B}(\{f_n\}_{n=1}^{\infty}, Z)$, we have $M^{s}(\{f_n\}_{n=1}^{\infty}, Z) = \infty$ and thus $M_{2\varepsilon+\delta}^{s}(\{f_n\}_{n=1}^{\infty}, Z) \geq 1$ when ε and δ are small enough. Hence $\mathcal{P}_{\varepsilon}^{s}(\{f_n\}_{n=1}^{\infty}, Z) \geq 1$ and $h_{top}^{P}(\{f_n\}_{n=1}^{\infty}, Z, \varepsilon) \geq s$ when ε is small. Therefore $h_{top}^{P}(\{f_n\}_{n=1}^{\infty}, Z) = \lim\limits_{\varepsilon\to 0} h_{top}^{P}(\{f_n\}_{n=1}^{\infty}, Z, \varepsilon) \geq s$. This implies that $h_{top}^{B}(\{f_n\}_{n=1}^{\infty}, Z) \leq h_{top}^{P}(\{f_n\}_{n=1}^{\infty}, Z)$.

(ii) Denote

$$\overline{h}_{\mu}^{K}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta) = \limsup_{n\to\infty} \frac{1}{n} \log r_n(\mu, \varepsilon, \delta)$$

then $\overline{h}_{\mu}^{K}(\{f_n\}_{n=1}^{\infty}) = \lim\limits_{\delta\to 0}\lim\limits_{\varepsilon\to 0} \overline{h}_{\mu}^{K}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta)$. We first prove that

$$\overline{e}_{\mu}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta) \leq \overline{h}_{\mu}^{K}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta)$$

for any $0 < \delta < 1$ and $\varepsilon > 0$, using like-Huasdorff dimension method. For any $s > \overline{h}_{\mu}^{K}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta)$ and $Z \in X_{\mu,\delta}$, let F is a (n, ε, Z)-spanning set, then

$$R_{n,\varepsilon}^{s}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) \leq \sum_{x\in F} \exp(-sn) = \sharp F \cdot e^{-sn}$$

which follows that

$$R_{n,\varepsilon}^{s}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) \leq e^{-sn} \cdot \inf_{Z\in X_{\mu,\delta}} r_n(\varepsilon, Z).$$

Hence

$$R_{n,\varepsilon}^{s}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) \leq e^{-sn} \cdot r_n(\mu, \varepsilon, \delta) = e^{-n(s-\frac{1}{n}\log r_n(\mu,\varepsilon,\delta))}.$$

Since $\overline{h}_{\mu}^{K}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta) = \limsup\limits_{n\to\infty} \frac{1}{n} \log r_n(\mu, \varepsilon, \delta) < s$, we have

$$\limsup_{n\to\infty} R_{n,\varepsilon}^{s}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) = 0.$$

For $s > \overline{h}_{\mu}^{K}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta)$ we get $\overline{R}_{n,\varepsilon}^{s}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) = 0$ and $\overline{e}_{\mu}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta) \leq s$. Hence $\overline{e}_{\mu}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta) \leq \bar{h}_{\mu}^{K}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta)$.
Next we prove $\overline{e}_{\mu}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta) \geq \bar{h}_{\mu}^{K}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta)$ for any $0 < \delta < 1$ and $\varepsilon > 0$ by showing $\overline{h}_{\mu}^{K}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta) \leq s$ whenever $s > \bar{e}_{\mu}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta)$. For such a s, we have $\overline{R}_{n,\varepsilon}^{s}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) = 0$. Then there exists $N \in \mathbb{N}$ such that $R_{n,\varepsilon}^{s}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) < 1$ for any $n \geq N$. Fix$n \geq N$, we can find a finite family $\{B_n(x_i, \varepsilon)\}_{i\in I}$ such that $x_i \in X$,

$$\mu(\bigcup_{i\in I} B_n(x_i, \varepsilon)) \geq 1 - \delta \quad \text{and} \quad \sharp I \cdot e^{-sn} < 1$$

So $r_n(\mu, \varepsilon, \delta) \leq e^{sn}$ for any $n \geq N$. Hence $\bar{h}_{\mu}^{K}(\{f_n\}_{n=1}^{\infty}, \varepsilon, \delta) \leq s$.
(iii) The proof of (iii) is similar to (ii).
(iv) The proof of (iv) is a consequence of definition.
(v) We first show that $e_{\mu}(\{f_n\}_{n=1}^{\infty}) \leq e_{\mu}^{*}(\{f_n\}_{n=1}^{\infty})$. Let $s < e_{\mu}(\{f_n\}_{n=1}^{\infty})$, $0 < \delta < 1$ and $\{Z_i\}_{i=1}^{\infty}$ be a family of Borel subsets of X with $\mu(\bigcup_{i=1}^{\infty} Z_i) \geq 1 - \delta$. For any $i, n \in \mathbb{N}$ and $\epsilon > 0$, let $R_n^i = R_n(Z_i, \varepsilon)$ be the largest number such that there is a disjoint family $\{\overline{B}_n(x_j^i, \epsilon)\}_{j=1}^{R_n}$ with $x_j^i \in Z_i$. Then we can verify that for any $\theta > 0$,

$$\{B_{n_j^i}(x_j^i, 2\epsilon + \theta)\} \supseteq Z_i.$$

It following that $M_{n,2\epsilon+\theta}^{s}(\{f_n\}_{n=1}^{\infty}, Z_i) \leq e^{-sn} \cdot R_n^i \leq P_{n,\epsilon}^{s}(\{f_n\}_{n=1}^{\infty}, Z_i)$ and $M_{2\epsilon+\theta}^{s}(\{f_n\}_{n=1}^{\infty}, Z_i) \leq P_{\epsilon}^{s}(\{f_n\}_{n=1}^{\infty}, Z_i)$. Therefore, by the Proposition3.1, we have $M_{2\epsilon+\theta}^{s}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) \leq \mathcal{P}_{\epsilon}^{s}(\{f_n\}_{n=1}^{\infty}, \mu, \delta)$. As $s < e_{\mu}(\{f_n\}_{n=1}^{\infty})$, we can get $< e_{\mu}(\{f_n\}_{n=1}^{\infty}, 2\epsilon + \theta, \delta)$ when ϵ, θ, δ are small enough. This implies that $M_{2\epsilon+\theta}^{s}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) = \infty$ and thus $\mathcal{P}_{\epsilon}^{s}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) = \infty$. Therefore, it can be deduced that $e_{\mu}^{*}(\{f_n\}_{n=1}^{\infty}) \geq s$. So the desired inequality holds.
Now we proved that $e_{\mu}^{*}(\{f_n\}_{n=1}^{\infty}) = \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \inf_{Z\in X_{\varepsilon,\delta}} h_{top}^{P}(\{f_n\}_{n=1}^{\infty}, Z, \varepsilon)$. Let $e_{\mu}^{*}(\{f_n\}_{n=1}^{\infty}) > s$, then there exists $\epsilon', \delta' > 0$ such that $e_{\mu}^{*}(\{f_n\}_{n=1}^{\infty}, \epsilon, \delta) \geq s$ for any $\epsilon \in (0, \epsilon')$ and $\delta \in (0, \delta')$. Thus, $\mathcal{P}_{\epsilon}^{s}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) = \infty$. For any $Z \in X_{\mu,\delta}$ and any $\{Z_i\}_{i\geq 1}$ with $\bigcup_{i=1}^{\infty} Z_i \supseteq Z$, we have $\mu(\bigcup_{i=1}^{\infty} Z_i) \geq 1 - \delta$. It follows from $\mathcal{P}_{\epsilon}^{s}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) = \infty$ that $\sum_{i=1}^{\infty} P_{\epsilon}^{s}(\{f_n\}_{n=1}^{\infty}, Z_i) = \infty$. So $\mathcal{P}_{\epsilon}^{s}(\{f_n\}_{n=1}^{\infty}, Z) = \infty$, which gives that $h_{top}^{P}(\{f_n\}_{n=1}^{\infty}, Z, \varepsilon) \geq s$.
On the other hand, let $s < \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \inf_{Z\in X_{\varepsilon,\delta}} h_{top}^{P}(\{f_n\}_{n=1}^{\infty}, Z, \varepsilon)$. Then there exist $\epsilon', \delta' > 0$ such that$h_{top}^{P}(\{f_n\}_{n=1}^{\infty}, Z, \varepsilon) > s$ for any $\epsilon \in (0, \epsilon')$, $\delta \in (0, \delta')$ and $Z \in X_{\mu,\delta}$. Thus, we have $\mathcal{P}_{\epsilon}^{s}(\{f_n\}_{n=1}^{\infty}, Z) = \infty$. Fix $\{Z_i\}_{i\geq 1}$ with $\mu(\bigcup_{i=1}^{\infty} Z_i) \geq 1-\delta$ and write $Z = \bigcup_{i=1}^{\infty} Z_i$, then $Z \in X_{\mu,\delta}$. So, $\sum_{i=1}^{\infty} P_{\epsilon}^{s}(\{f_n\}_{n=1}^{\infty}, Z_i) =$

∞, which yields that $\mathcal{P}_{\epsilon}^{s}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) = \infty$. Furthermore, we can get $e_{\mu}^{*}(\{f_n\}_{n=1}^{\infty}, \epsilon, \delta) \geq s$ and $e_{\mu}^{*}(\{f_n\}_{n=1}^{\infty}) \geq s$. □

3.2 Proof of Theorem 1.3

Proposition 3.2. *For $\mu \in M(X)$, it holds that*

$$\underline{h}_{\mu}(\{f_n\}_{n=1}^{\infty}) \leq e_{\mu}(\{f_n\}_{n=1}^{\infty}) \leq \inf\{h_{top}^{B}(\{f_n\}_{n=1}^{\infty}, K) : \mu(K) = 1\}.$$

Proof. The second inequality is a direct consequence of the definition and we only deduce the first one. For $s > 0$ with $\underline{h}_{\mu}(\{f_n\}_{n=1}^{\infty}) > s$. By a standard procedure, there exist $A \subset X$ with $\mu(A) > 0$ and $N \in \mathbb{N}$ such that

$$\mu(B_n(x, \varepsilon)) < e^{-ns}, \quad \forall x \in A, n \geq N$$

Pick $\delta \in (0, \mu(A))$. Let $\{B_{n_i}(x_i, \frac{\varepsilon}{2})\}_{i \in I}$ be a countable family such that $n_i \geq N, x_i \in X$ and $\mu(\bigcup_{i \in I} B_{n_i}(x_i, \frac{\varepsilon}{2})) \geq 1 - \delta$ that intersects A, if taking $y_i \in B_{n_i}(x_i, \frac{\varepsilon}{2}) \cap A$, then one has $B_{n_i}(x_i, \frac{\varepsilon}{2}) \subseteq B_{n_i}(y_i, \varepsilon)$ and thus

$$\mu(B_{n_i}(x_i, \frac{\varepsilon}{2})) \leq B_{n_i}(y_i, \varepsilon) \leq e^{-n_i s}$$

Then we have

$$\begin{aligned}\sum_{i \in I} e^{-n_i s} \geq \sum_{i \in I} \mu(B_{n_i}(y_i, \varepsilon) \cap A) &\geq \sum_{i \in I} \mu(B_{n_i}(x_i, \frac{\varepsilon}{2}) \cap A) \\ &\geq \mu(\bigcup_{i} B_{n_i}(x_i, \frac{\varepsilon}{2}) \cap A) = \mu(A) > 0\end{aligned}$$

Hence $M_{\frac{\varepsilon}{2}}^{s}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) \geq M_{N,\frac{\varepsilon}{2}}^{s}(\{f_n\}_{n=1}^{\infty}, \mu, \delta) \geq \mu(A)$. By Bowen's definition, we can derive that $e_{\mu}(\{f_n\}_{n=1}^{\infty}, \frac{\varepsilon}{2}, \delta) \geq s$ and moreover $\underline{h}_{\mu}(\{f_n\}_{n=1}^{\infty}) \leq e_{\mu}(\{f_n\}_{n=1}^{\infty})$. □

Definition 3.3. Let $\mu \in M(X)$. The *NADS (x, ϕ) measure-theoretical lower entropies of μ* is defined by

$$\underline{h}_{\mu}(\phi,) = \int \underline{h}_{\mu}(\phi, x) d\mu(x)$$

where

$$\underline{h}_{\mu}(\phi, x) = \lim_{\varepsilon \to 0} \liminf_{t \to \infty} -\frac{1}{t} \log \mu(B_t^{\phi}(x, \varepsilon)).$$

Lemma 3.4 ([5, Theorem 1.4])**.** *Let $(X, \{f_n\}_{n=1}^{\infty})$ be a $NADDS$. If $K \subseteq X$ is non-empty and compact,then*

$$h_{top}^{B}(\{f_n\}_{n=1}^{\infty}, K) = \sup\{\underline{h}_{\mu}(\{f_n\}_{n=1}^{\infty}) \colon \mu \in M(X), \mu(K) = 1\}.$$

Next we prove Theorem 1.3.

Proof. By the Proposition, we have

$$\begin{aligned}\sup\{\underline{h}_\mu(\{f_n\}_{n=1}^\infty)\colon \mu\in M(X),\mu(K)=1\} &\le \sup\{e_\mu(\{f_n\}_{n=1}^\infty)\colon \mu\in M(X),\mu(K)=1\}\\ &\le \inf\{h_{top}^B(\{f_n\}_{n=1}^\infty,K) : \mu\in M(X)\mu(K)=1\}.\end{aligned}$$

Combining with lemma,

$$h_{top}^B(\{f_n\}_{n=1}^\infty,K)=\sup\{\underline{h}_\mu(\{f_n\}_{n=1}^\infty)\colon \mu\in M(X),\mu(K)=1\}.$$

the conclusion can be proved. □

Using the same proof method of Theorem 1.2 and, Theorem 1.3, we have result of Theorem 1.4 and Theorem 1.5.

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