Noether Theorem of a Kind of Complete Singular Integral Equation with Hilbert Kernel on Open Arcs

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Abstract. We considered a kind of singular integral equation with Hilbert kernel on open arcs lying in a period strip. By regularization method, we transformed the equation into Fredholm equation and then obtained the solvable Noether theorem for this kind of complete singular integral equations with Hilbert kernel on open arcs.

1. Introduction

In [1,2,3,4,5], the authors discussed the singular integral equation with Cauchy kernel on the real half-line on the real line, or on a complicated contour. In [6], the authors discussed Numerical solution of a singular integral equation with Cauchy kernel in the plane contact problem. While the solvable Noether theorem for complete singular integral equations with Hilbert kernel on open arcs, has few been discussed. Here we study a kind of complete singular integral equation with Hilbert kernel on open arcs, by using the regularization method for this kind of singular integral equation, we transform the equation into Fredholm equation and give the solvable Noether theorem for the singular integral equation.

2. The complete singular integral equation with Hilbert kernel on open arcs

Suppose that $L=\sum_j L_j$ is a finite set of nonintersecting open arcs, with period $a\pi$ lying entirely in the in the same periodic region $S:|\operatorname{Re} z|<a\pi/2$, and being positively oriented. We consider the following the singular integral equation with Hilbert kernel for a Hölder continuous function $\varphi(t)$ on L

$$K\varphi \equiv A(t_0)\varphi(t_0)+\frac{1}{a\pi i}\int_L K(t_0,t)\cot\frac{t-t_0}{a}\varphi(t)dt = f(t_0),\quad t_0\in L. \tag{1}$$

where $A(t), B(t), f(t)\in H(L)$, $K(t_0,t)\in H_0(L\times L)$ are given functions with $A^2(t)-B^2(t)\neq 0$. We assume the different points $c_1,c_2,\cdots,c_n$ are the all nodes on L (including the end-points of L_j and the discontinuity points of $A(t), B(t)$ and $f(t)$). Without loss of generality, no special condition is needed except for the requirement that $\varphi\in h=h(c_1,c_2,\cdots,c_q)$ $(q\le n)$, that is φ is bounded near $c_1,c_2,\cdots$ and c_q, while φ has a singularity of order less than 1 at any other nodes. For L, we permit it to arrive at the boundary of the region S, and in this case we regarded c and $c\pm a\pi$ as the same one.

If we let $B(t)=K(t,t)$, then (1) can be rewritten as followed

$$K\varphi \equiv A(t_0)\varphi(t_0)+\frac{B(t_0)}{a\pi i}\int_L \cot\frac{t-t_0}{a}\varphi(t)dt+\int_L k(t_0,t)\varphi(t)dt = f(t_0), t_0\in L, \tag{2}$$

where

$$k(t_0,t)=[K(t_0,t)-K(t,t)]\cot\left[(t-t_0)/a\right], \tag{3}$$

Since $K(t_0,t)\in H^{\alpha}$ ($0<\alpha\le 1$), we have

$$\left|k(t_0,t)\right|\le A\big/\left|t-t_0\right|^{\lambda}\ ,\ \ 0\le\lambda=1-\alpha<1\,. \tag{4}$$

When $k(t_0,t)$ satisfies (4), $k\varphi\equiv\int_L k(t_0,t)\varphi(t)dt$ becomes an F—I operator, and

$$K^0\varphi\equiv A(t_0)\varphi(t_0)+\frac{B(t_0)}{a\pi i}\int_L\cot\frac{t-t_0}{a}\varphi(t)dt\,, \tag{5}$$

is actually the characteristic operator of operator K. Therefore, Eq.(1) can be rewritten as

$$(K^0+k)\varphi=f\,, \tag{6}$$

$$K'\psi\equiv A(t_0)\psi(t_0)-\frac{1}{a\pi i}\int_L K(t,t_0)\psi(t)\cot\frac{t-t_0}{a}dt=g(t_0),\ t_0\in L \tag{7}$$

or

$$K'\psi\equiv A(t_0)\psi(t_0)-\frac{1}{a\pi i}\int_L B(t)\psi(t)\cot\frac{t-t_0}{a}dt\pm\int_L k(t,t_0)\psi(t)dt=g(t_0), \tag{7$'$}$$

is the associative equation of Eq. (1). Setting

$$S(t)=A(t)+B(t)\,,\quad D(t)=A(t)-B(t)\,, \tag{8}$$

$$\kappa_j=\frac{1}{2\pi i}\,[\log\frac{A(t)-B(t)}{A(t)+B(t)}]_{L_j}=\frac{1}{2\pi}[\arg\frac{D}{S}]_{L_j}\,, \tag{9}$$

thus the index of Eq. (1) is $\kappa=\sum_{j=1}^{p}\kappa_j$.

3. Regularization for the singular integral equation with Hilbert kernel on open arcs

We may rewrite (2) as follows

$$K^0\varphi=f(t_0)-k\varphi\,, \tag{10}$$

when t_0 and t_1 are on the same smooth arc (including endpoints), we have

$$\left|k(t_0,t)\right|=\left|[K(t_0,t)-K(t_0,t_0)]\cot\frac{t-t_0}{a}\right|\le A\left|t-t_0\right|^{\alpha}\left|\frac{a}{t-t_0}+\sum_{n=1}^{\infty}b_n(t-t_0)^n\right|\le\frac{M}{\left|t-t_0\right|^{\lambda}}\,(0\le\lambda=1-\alpha<1),$$

so $\int_L k(t_0,t)\varphi(t)dt$ is F—I operator, and $k(t_0,t)\in H_0(L)$, therefore $f-k\varphi\in H_0$. Let

$$X(z)=\prod_{j=1}^{p}(\tan\frac{z}{a}-\tan\frac{c_j}{a})^{\lambda_j}\,e^{\Gamma(z)}\,, \tag{11}$$

$$K^*f\equiv A^*(t_0)f(t_0)-\frac{B^*(t_0)Z(t_0)}{a\pi i}\int_L\frac{f(t)}{Z(t)}\cot\frac{t-t_0}{a}dt\,. \tag{12}$$

If $\kappa\ge 0$, then φ must satisfies

$$\varphi(t_0)=K^*(f-k\varphi)+B^*(t_0)Z(t_0)P_\kappa(\tan\frac{t_0}{a})\,, \tag{13}$$

that is

$$\varphi(t_0)+K^*k\varphi=f^*(t_0)\,, \tag{13$'$}$$

where

$$f^*(t_0)=K^*f+B^*(t_0)Z(t_0)P_\kappa(\tan\frac{t_0}{a})\,. \tag{14}$$

Obviously, $f^*(t_0)\in h$. On the contrary, if $\varphi\in h$ (for some P_κ) satisfies (13)′, then it must satisfy (13), that is φ is the solution for (2).

If $\kappa<0$, if and only if

$$\begin{cases}\int_L \frac{f(t)-k\varphi}{Z(t)}(\tan^{j-1}\frac{t}{a}+\tan^{j+1}\frac{t}{a})dt=0, \ j=1,\cdots,-\kappa-1, \\ \int_L \frac{f(t)-k\varphi}{Z(t)}(\sin\frac{v}{2}-\cos\frac{v}{2}\tan\frac{t}{a})dt=0,\end{cases} \tag{15}$$

Eq.(13) has solution in class h, and its unique solution is $\varphi(t_0)=K^*(f-k\varphi)$, that is (13)′ with $P_\kappa \equiv 0$.

Since k is a Fredholm operator, we have

$$\int_L k\varphi\left[\tan^{j-1}\frac{t}{a}+\tan^{j+1}\frac{t}{a}\right]\Big/Z(t)dt=\int_L \varphi(t)k'\left[\tan^{j-1}\frac{t}{a}+\tan^{j+1}\frac{t}{a}\right]\Big/Z(t)\,dt\,, \quad j=1,\cdots,-\kappa-1$$

thus

$$\begin{cases}\int_L \frac{\tan^{j-1}\frac{t}{a}+\tan^{j+1}\frac{t}{a}}{Z(t)}f(t)dt=\int_L \varphi(t)P_j(t)dt, \ j=1,\cdots,-\kappa-1, \\ \int_L \frac{\sin\frac{v}{2}-\cos\frac{v}{2}\tan\frac{t}{a}}{Z(t)}f(t)dt=\int_L \varphi(t)P_0(t)dt,\end{cases} \tag{15$'$}$$

where

$$P_j(t)=k'\left\{\left[\tan^{j-1}\tfrac{t}{a}+\tan^{j+1}\tfrac{t}{a}\right]\Big/Z(t)\right\}=\int_L [\tan^{j-1}\tfrac{t_1}{a}+\tan^{j+1}\tfrac{t_1}{a}]/Z(t)dt_1\,, j=1,\cdots,-\kappa-1\,,$$

$$P_0(t)=k'\left\{\left[\sin\frac{v}{2}-\cos\frac{v}{2}\tan\frac{t}{a}\right]\Big/Z(t)\right\}=\int_L \frac{k(t_1,t)}{Z(t)}[\sin\frac{v}{2}-\cos\frac{v}{2}\tan\frac{t_1}{a}]dt_1\,,$$

where $P_j(t)(j=0,1,\cdots,\kappa-1)$ are the known functions in class H_0, and when $\kappa \le 0$ the Eq. (2) in class h is equivalent to Eq. (13)′ in class h (with $P_\kappa \equiv 0$ in (14)) and the additive condition (15) or (15)′ by the same reason to the case that when $\kappa \ge 0$.

Now, Let us demonstrate K^*k is a weak Fredholm operator. Because

$$K^*f \equiv A^*(t_0)f(t_0)-\frac{B^*(t_0)Z(t_0)}{a\pi i}\int_L \frac{f(t)}{Z(t)}\cot\frac{t-t_0}{a}dt\,,$$

we have

$$K^*k\varphi=A^*(t_0)(k\varphi)(t_0)-\frac{B^*(t_0)Z(t_0)}{a\pi i}\int_L \cot\frac{t-t_0}{a}\frac{\mathrm{d}t}{Z(t)}\cdot\int_L k(t,t_1)\varphi(t_1)dt_1\cdot$$

Since the order of Hilbert integral and common integral can be exchanged, we obtain

$$K^*k\varphi=A^*(t_0)(k\varphi)(t_0)-B^*(t_0)Z(t_0)\int_L k^*(t_0,t_1)\varphi(t_1)dt_1\,, \tag{16}$$

where

$$k^*(t_0,t)=\frac{1}{a\pi i}\int_L \frac{k(t,t_1)}{Z(t)}\cot\frac{t-t_0}{a}dt\,, \tag{17}$$

has at the most less than one order singularity at the node t_0 and so that it is a weak Fredholm kernel.

Applying Fredholm Theorem, we obtain the following results.

Theorem 3.1 Equation

$$\varphi(t_0)+K^*k\varphi=f^*(t_0)\ \ ,\ \ t_0\in L \tag{18}$$

is solvable in class h if and only if

$$\int_L f^*(t)w_j(t)dt=0\,, \quad j=1,2,\cdots,v\,, \tag{19}$$

where $\{w_j(t)\}_1^v$ is the set of linearly independent solutions of the homogeneous associative equation

$$w(t_0)+k'K^{*'}w=0 \tag{20}$$

in class h'.

When the solvable condition (19) is satisfied, the solution of (18) in class h can be written as

$$\varphi(t_0)=\Gamma f^*+\sum_{j=1}^{v}C_j x_j(t_0), \tag{21}$$

where $C_1,\cdots,C_v$ are arbitrary complex constants, $\{\chi_j(t)\}_1^v$ is the set of linearly independent solutions of the homogeneous equation of Eq.(18), while

$$\Gamma f^*\equiv f^*(t_0)+\int_L \Gamma(t_0,t)f^*(t)dt,$$

and $\Gamma(t_0,t)$ is the generalized resolvent kernel.

4. Noether theorem of a kind of singular integral equation with Hilbert kernel on open arcs

Applying the results above and following [1], we may obtain the following result.

Theorem 4.1 (**Noether Theorem**)

(a) The number of the linearly independent solutions of the homogeneous equation for (2) ($f\equiv 0$) in class $h=h(c_1,\cdots,c_q)$ is finite, denoted by l.

(b) Eq.(2) is solvable in class $h=h(c_1,\cdots,c_q)$ if and only if

$$\int_L f\psi_j dt=0 \quad, j=1,\cdots,l'$$

is satisfied, where $\{\psi_j\}^{l'}$ is the set of linearly independent solutions of the homogeneous associative equation

$$K^{0\prime}\psi\equiv A(t_0)\psi(t_0)-\frac{1}{a\pi i}\int_L B(t)\psi(t)\cot\frac{t-t_0}{a}=0$$

in class h'.

(c) If κ is the index of Eq.(2) in class $h=h(c_1,c_2,\cdots,c_q)$, then $l-l'=\kappa$.

Proof

(a) is true because that the Fredholm Eq. regularized from $K\varphi=0$ has only a finite number of linearly independent solutions and the solutions of the latter must be the solutions of Eq. $K\varphi=0$ .And **(c)** is clearly true.Now we aim to prove **(b)**.

Case 1. $\kappa\geq 0$. Eq. (2) in class h is equivalent to Eq. (13)′, where

$$f^*(t_0)=K^*f+B^*(t_0)Z(t_0)P_\kappa(\tan\frac{t_0}{a})$$

with the coefficients of the polynomial

$$P_\kappa(\tan\frac{t_0}{a})=A_0\tan^\kappa\frac{t}{a}+A_1\tan^{\kappa-1}\frac{t}{a}+\ldots+A_{\kappa-1}\tan\frac{t}{a}+A_\kappa$$

satisfying

$$-\cos\frac{v}{2}(A_\kappa-A_{\kappa-2}+\ldots)+\sin\frac{v}{2}(A_{\kappa-1}-A_{\kappa-3}+\ldots)=\frac{\sin\frac{v}{2}}{a\pi i}\int_L\frac{f(t)}{Z(t)}dt. \tag{22}$$

Without loss of generality, we let $A_1,A_2,...,A_\kappa$ are arbitrary constants, then A_0 is determined by (22). On the other hand, the solvable conditions for Eq.(13)′ are given by (19), we put (14) for $f^*(t_0)$ into (19), and introduce the note

$$\delta_j=\int_L w_j K^* f dt,\ \text{j=1,2,…,v}, \tag{23}$$

then (19) can be rewritten as

$$\sum_{k=1}^{\kappa}\gamma_{jk}A_k=\delta_j\ ,\quad j=1,\ldots,v, \tag{24}$$

where (γ_{jk}) is a given constant matrix independent to $f(t)$.

It is always true for the operator K that $\int_L fKgdt = \int_L gK'fdt$ as long as both f and g have no integrable singularity at the same knot. Thus, we may rewrite δ_j as

$$\delta_j = \int_L fK^{*'} w_j dt = \int_L fw_j^* dt, \tag{25}$$

where

$$w_j^*(t) = K^{*'} w_j, j = 1,2,\ldots,v \tag{26}$$

are linearly independent and satisfies $w_j^* \in h'$.

Set rank $(\delta_{jk}) = \rho(\rho \le v, \rho \le \kappa)$. Without loss of generality, we suppose that the ρ-order determinant of (δ_{jk}) in the top left is not 0. Then the compatibile conditions are given by

$$\begin{vmatrix} \gamma_{11} & \cdots & \gamma_{1\rho} & \delta_1 \\ \vdots & & \vdots & \vdots \\ \gamma_{\rho 1} & \cdots & \gamma_{\rho\rho} & \delta_\rho \\ \gamma_{\rho+j,1} & \cdots & \gamma_{\rho+j,\rho} & \delta_{\rho+j} \end{vmatrix} = 0, \quad j = 1,2,\ldots,v-\rho, \tag{27}$$

or

$$\delta_{\rho+j} + \sum_{k=1}^{\rho} a_{jk}\delta_k = 0, \quad j = 1,2,\ldots,v-\rho, \tag{27$'$}$$

where (a_{jk}) is constant matrix independent to $f(t)$.

Put (25) for δ_j into $(27)'$, we obtain the solvable conditions for Eq. $(13)'$ as follows

$$\int_L \lambda_j(t) f(t) dt = 0, \quad j = 1,2,\ldots,v-\rho, \tag{28}$$

where

$$\lambda_j(t) = w_{\rho+j}^*(t) + \sum_{k=1}^{\rho} a_{jk} w_k^*(t), \quad j = 1,2,\ldots,v-\rho, \tag{29}$$

are definite, linearly independent functions, belonging to h'.

Suppose that the solvable conditions are satisfied, then Eq.(13) is solvable and（27）is also satisfied, thus we can express $\{A_j\}_1^\rho$ using arbitrary constants $\{A_k\}_{\rho+1}^v$. Put the expression for $\{A_j\}_1^\rho$ into (24), we get

$$A_j = \sum_{k=1}^{\kappa-\rho} B_{jk} A_{\rho+k} + \sum_{k=1}^{\kappa} \Gamma_{jk}\delta_k, \quad j = 1,2,\ \ldots,\rho \tag{30}$$

where (B_{jk}) and (Γ_{jk}) are all given constant matrix, independent to f (t). Put the expression (30) for $\{A_j\}_1^\rho$ into the right side of Eq.(13), we see Eq.(13) is always solvable for arbitrary constants $\{A_k\}_{\rho+1}^v$. From (25) , we obtain by (14) and (23) the general solutions for Eq.(13)

$$\varphi(t_0) = \Gamma^* K^* f + C_1\chi_1(t_0) + C_2\chi_2(t_0) + \cdots C_{\kappa+v-\rho}\chi_{\kappa+v-\rho}(t_0), \tag{31}$$

where χ_2, χ_2, …, χ_v and $C_1, C_2, \ldots, C_v$ are same to (21), χ_{V+1}, χ_{v+2}, $\cdots, \chi_{\kappa+v-\rho}$ are some definite functions, independent to f and belonging to the class h, and for the sake of consistency, we have rewritten $A_{\rho+1}$, $A_{\rho+2}$ … A_κ as $C_{v+1}, C_{v+2}, \ldots,\ C_{\kappa+v-\rho}$ respectively, and have put the terms relating to f into the expression $\Gamma^* K^* f$ with $\Gamma^* F \equiv F(t_0) + \int_L \Gamma^*(t_0,t) F(t) dt$, where $\Gamma^*(t_0,t)$ and $\Gamma(t_0,t)$ are same except some definite terms.

In particular, the homogeneous Eq.(2) $(f \equiv 0)$ is equivalent to

$$\varphi(t_0) + K^* k\varphi = b^*(t_0) Z(t_0) P_\kappa(t_0) \tag{32}$$

and the general solution for it is

$$\varphi(t_0) = C_1\chi_1(t_0) + C_2\chi_2(t_0) + \ldots C_{\kappa+\nu-\rho}\chi_{\kappa+\nu-\rho}(t_0), \tag{33}$$

where $\{C_j\}_1^{\kappa+\nu-\rho}$ are arbitrary constants, and $\{\chi_j(t)\}_1^{\kappa+\nu-\rho}$ is a full set of linear independent solutions for $K\varphi = 0$.

Case 2. $\kappa < 0$. At this time, $P_\kappa \equiv 0$ and the solvable conditions for Eq.(13)$'$ is $\delta_j = 0$ $(j = 1,2,\cdots,\nu)$, i.e.

$$\int_L \lambda_j(t) f(t) dt = 0, j = 1,2,\cdots,\nu, \tag{34}$$

where $\lambda_j \in h'(j = 1,2,\cdots,n)$ are definite and linearly independent. Assume (34) is valid, then Eq.(13)$'$ is solvable in class h if and only if the solvable condition

$$\int_L \lambda_j(t) f(t) dt = 0, \quad j = \nu+1, \nu+2, \ldots, \nu+\sigma. \ (\sigma \le -\kappa, \lambda_j \in h') \tag{35}$$

Combining (34) and (35), we obtain the following solvable condition for Eq.(2) in class h.

Now we come to the result Ⅱ. $\forall \varphi \in h, \psi \in h'$, we see $\int_L \psi K\varphi dt = \int_L \varphi K'\psi dt$ and the solvable necessary condition for $K\varphi = f$ in class h is

$$\int_L f\psi_j dt = 0, \quad j = 1,2\ldots l', \tag{36}$$

where $\{\psi_j\}_1^{l'}$ is the is the set of linearly independent solutions of the homogeneous associative equation $K'\psi = 0$ in class h'.

By the preceding discussion, we see $K\varphi = f$ is solvable in class h if and only if both (34) and (35), i.e.

$$\int_L \lambda_j(t) f(t) dt = 0, \quad j = 1,\ldots,m \quad \left(m \in \mathbb{N}^+, \lambda_j \in h'\right) \tag{37}$$

are satisfied. Assume (36) is valid, then for any $g(t) \in H$ with $g(c_j) = 0$ $(j = 1,2,\ldots,n)$, we have $Kg \in H_0$. Equation $K\varphi = Kg$ in class H and then in class h must be solvable. So we have

$$0 = \int_L \lambda_j Kg dt = \int_L gK'\lambda_j dt, \ j = 1,\ldots,m.$$

By arbitrariness of $g(t)$, we see $K'\lambda_j = 0$, that is, $\lambda_j \in h'$ is a solution for $K'\psi = 0$. So every λ_j must be a linear combination of $\{\psi_k\}_1^{l'}$. Now by (36) and (37), we see the result Ⅱ is true.

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