New characterization of vector field on Weil bundles

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Abstract

Let M be a paracompact smooth manifold, A a Weil algebra and M^A the associated Weil bundle. In this paper, we give another definition and characterization of vector field on M^A.

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1 Introduction

In what follows we denote A, a Weil algebra i.e a local algebra in the sense of André Weil, M a smooth manifold, $C^\infty(M)$ the algebra of smooth functions on M, M^A the manifold of infinitely near points of kind A and $\pi_M : M^A \longrightarrow M$ be the projection which assigns every infinitely near point to $x \in M$ to its origin x. The triplet (M^A, π_M, M) defines a bundle called bundle of infinitely near points on M of kind A or simply weil bundle[13],[7],[9],[5],[12].

If $f : M \longrightarrow \mathbb{R}$ is a smooth function, then the application

$$f^A : M^A \longrightarrow A, \xi \longmapsto \xi(f)$$

is also smooth. The set, $C^\infty(M^A, A)$ of smooth functions on M^A with values on A, is a commutative algebra over A with unit and the application

$$C^\infty(M) \longrightarrow C^\infty(M^A, A), f \longmapsto f^A$$

is an injective homomorphism of algebras. Then, we have:

$$\begin{aligned}(f+g)^A &= f^A + g^A;\\ (\lambda \cdot f)^A &= \lambda \cdot f^A;\\ (f \cdot g)^A &= f^A \cdot g^A.\end{aligned}$$

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The map

$$A \times C^{\infty}(M^A) \longrightarrow C^{\infty}(M^A, A), (a, F) \longmapsto a \cdot F : \xi \longmapsto a \cdot F(\xi)$$

is bilinear and induces one and only one linear map

$$\sigma : A \otimes C^{\infty}(M^A) \longrightarrow C^{\infty}(M^A, A).$$

When $(a_\alpha)_{\alpha=1,2,...,\dim A}$ is a basis of A and when $(a^*_\alpha)_{\alpha=1,2,...,\dim A}$ is a dual basis of the basis $(a_\alpha)_{\alpha=1,2,...,\dim A}$, the application

$$\sigma^{-1} : C^{\infty}(M^A, A) \longrightarrow A \otimes C^{\infty}(M^A), \varphi \longmapsto \sum_{\alpha=1}^{\dim A} a_\alpha \otimes (a^*_\alpha \circ \varphi)$$

is an isomorphism of A-algebras. That isomorphism does not depend of a choisen basis and the application

$$\gamma : C^{\infty}(M) \longrightarrow A \otimes C^{\infty}(M^A), f \longmapsto \sigma^{-1}(f^A),$$

is a homomorphism of algebras.
If (U, φ) is a local chart of M with coordinate system $(x_1, ..., x_n)$, the map

$$\varphi^A : U^A \longrightarrow A^n, \xi \longmapsto (\xi(x_1), ..., \xi(x_n))$$

is a bijection from U^A onto an open set of A^n. In addition, if $(U_i, \varphi_i)_{i\in I}$ is an atlas of M^A, then $(U^A_i, \varphi^A_i)_{i\in I}$ is also an atlas of M^A [2].

2 Other defintion of a vector field on Weil bundles

Let M be a smooth manifold of dimension n, A a Weil algebra and M^A a Weil bundle associated. In this paper, we give another chatacterization of vector fields on M^A. We show that, the tangent bundle TM^A is locally trivial with typical fiber A^n; we also give a writing of a vector field on M^A, in coordinate neighborhood system. Moreover, we verify easily that the $C^{\infty}(M^A, A)$-module $\mathfrak{X}(M^A)$ of vector field on M^A is a Lie algebra over A.

2.1 Tangent vectors at M^A

Let $(a_\alpha)_{\alpha=1,...,\dim A}$ be a basis of the Weil algebra A. For any $\varphi \in C^{\infty}(M^A, A)$, we have

$$\varphi = \sum_{\alpha=1}^{\dim A} a^*_\alpha \circ \varphi \cdot a_\alpha.$$

When $\xi \in M^A$, the map

$$\widetilde{\xi} : C^{\infty}(M^A, A) \longrightarrow A, \varphi \longmapsto \varphi(\xi),$$

is a homomorphism of A-algebras.
We denote $Der_{A,\widetilde{\xi}}[C^\infty(M^A, A), A]$ the set of $\widetilde{\xi}$-derivations which are A-linear i.e. the set of maps

$$v : C^\infty(M^A, A) \longrightarrow A$$

such that

1. v is A-linear;
2. $v(\varphi\cdot\psi) = v(\varphi)\cdot\widetilde{\xi}(\psi)+\widetilde{\xi}(\varphi)\cdot v(\psi) = v(\varphi)\cdot\psi(\xi)+\varphi(\xi)\cdot v(\psi)$, for any $\varphi, \psi \in C^\infty(M^A, A)$.

Proposition 1. For any $\xi \in M^A$, $Der_{A,\xi}\left[C^\infty(M^A, A), A\right]$ is a module over A.

Theorem 2. For any $\xi \in M^A$, the following assertions are equivalent:

1. A tangent vector at $\xi \in M^A$ is a $\mathbb{R}$-linear map

$$u : C^\infty(M^A) \longrightarrow \mathbb{R}$$

such that for any $F, G \in C^\infty(M^A)$,

$$u(F \cdot G) = u(F) \cdot G(\xi) + F(\xi) \cdot u(G);$$

2. A tangent vector at $\xi \in M^A$ is an A-linear map

$$v : C^\infty(M^A, A) \longrightarrow A$$

such that for any $\varphi, \psi \in C^\infty(M^A, A)$,

$$v(\varphi \cdot \psi) = v(\varphi) \cdot \psi(\xi) + \varphi(\xi) \cdot v(\psi);$$

3. A tangent vector at $\xi \in M^A$ is a $\mathbb{R}$-linear map

$$w : C^\infty(M) \longrightarrow A$$

such that for any $f, g \in C^\infty(M)$,

$$w(f \cdot g) = w(f) \cdot \xi(g) + \xi(f) \cdot w(g).$$

Proof. 1. (1) $\Longrightarrow$ (2)

Let $u : C^\infty(M) \longrightarrow A$ be a tangent vector at M^A and

$$v : C^\infty(M^A, A) \xrightarrow{\sigma^{-1}} A \otimes C^\infty(M^A) \xrightarrow{id_A \otimes u} A \otimes \mathbb{R} = A.$$

For any $\varphi, \psi \in C^\infty(M^A, A)$, and for $a \in A$, we have:

$$\begin{aligned} v(\varphi + \psi) &= \left[(id_A \otimes u) \circ \sigma^{-1}\right](\varphi + \psi) = \left[(id_A \otimes u)\right](\sigma^{-1}(\varphi + \psi)) \\ &= \left[(id_A \otimes u)\right](\sigma^{-1}(\varphi) + \sigma^{-1}(\psi)) \\ &= \left[(id_A \otimes u)\right](\sigma^{-1}(\varphi)) + \left[(id_A \otimes u)\right](\sigma^{-1}(\psi)) \\ &= \left[(id_A \otimes u) \circ \sigma^{-1}\right](\varphi) + \left[(id_A \otimes u) \circ \sigma^{-1}\right](\psi) \\ &= v(\varphi) + v(\psi), \end{aligned}$$

$$\begin{aligned} v(a \cdot \varphi) &= \left[(id_A \otimes u) \circ \sigma^{-1}\right](a \cdot \varphi) \\ &= \left[(id_A \otimes u)\right](\sigma^{-1}(a \cdot \varphi)) \\ &= \left[(id_A \otimes u)\right](a \cdot \sigma^{-1}(\varphi)) \\ &= a \cdot \left[(id_A \otimes u)\right](\sigma^{-1}(\varphi)) \\ &= a \cdot \left[(id_A \otimes u) \circ \sigma^{-1}\right](\varphi) \\ &= a \cdot v(\varphi) \end{aligned}$$

and

$$\begin{aligned} v(\varphi \cdot \psi) &= \left[(id_A \otimes u) \circ \sigma^{-1}\right](\varphi \cdot \psi) = \left[(id_A \otimes u)\right](\sigma^{-1}(\varphi \cdot \psi)) \\ &= \left[(id_A \otimes u)\right](\sigma^{-1}(\varphi) \cdot \sigma^{-1}(\psi)) \\ &= \left[(id_A \otimes u)\right](\sigma^{-1}(\varphi)) \cdot \psi(\xi) + \varphi(\xi) \cdot \left[(id_A \otimes u)\right](\sigma^{-1}(\psi)) \\ &= \left[(id_A \otimes u) \circ \sigma^{-1}\right](\varphi) \cdot \psi(\xi) + \varphi(\xi) \cdot \left[(id_A \otimes u) \circ \sigma^{-1}\right](\psi) \\ &= v(\varphi) \cdot \psi(\xi) + \varphi(\xi) \cdot v(\psi). \end{aligned}$$

2. (2) $\Longrightarrow$ (3) Let $v : C^\infty(M^A, A) \longrightarrow A$ be a tangent vector at $\xi \in M^A$. Let

$$w : C^\infty(M) \longrightarrow C^\infty(M^A, A), f \longmapsto v(f^A).$$

For any $f, g \in C^\infty(M)$, and for $\lambda \in \mathbb{R}$, we have:

$$\begin{aligned} w(f+g) &= v\left[(f+g)\right]^A = v(f^A + g^A) = v(f^A) + w(g^A) \\ &= w(f) + w(g), \end{aligned}$$

$$\begin{aligned} w(\lambda f) &= v\left[(\lambda \cdot f)\right]^A = v(\lambda \cdot f^A) = \lambda \cdot v(f^A) \\ &= \lambda \cdot w(f). \end{aligned}$$

$$\begin{aligned} w(f \cdot g) &= v\left[(f \cdot g)\right]^A = v(f^A \cdot g^A) = v(f^A) \cdot g^A(\xi) + f^A(\xi) \cdot w(g^A) \\ &= w(f) \cdot \xi(g) + \xi(f) \cdot w(g). \end{aligned}$$

3. (3) $\Longrightarrow$ (1) The implication holds from the following result: the map

$$Der_\xi\left[C^\infty(M^A),\mathbb{R}\right]\longrightarrow Der_\xi\left[C^\infty(M),A\right], v\longmapsto (id_A\otimes v)\circ\gamma,$$

is an isomorphism of vector spaces see [9].

□

In what follows, we denote $T_\xi M^A$ the set of A-linear maps $v : C^\infty(M^A, A) \longrightarrow A$ such that for any $\varphi,\psi\in C^\infty(M^A, A)$,

$$v(\varphi\cdot\psi) = v(\varphi)\cdot\psi(\xi)+\varphi(\xi)\cdot v(\psi)$$

that is to say

$$T_\xi M^A = Der_{A,\widetilde{\xi}}[C^\infty(M^A, A), A].$$

Remark 1. For $v\in T_\xi M^A$, we have $v\left[C^\infty(M^A)\right]\subset\mathbb{R}$.

2.2 Vector fields on M^A

The set, $Der_A[C^\infty(M^A, A)]$, of derivations which are A-linear is a $C^\infty(M^A, A)$-module.

Theorem 3. The following assertions are equivalent:

1. A vector field on M^A is a differentiable section of the tangent bundle (TM^A, π_{M^A}, M^A).
2. A vector field on M^A is a derivation of $C^\infty(M^A)$.
3. A vector field on M^A is a derivation of $C^\infty(M^A, A)$ which is A-linear.
4. A vector field on M^A is a linear map $X : C^\infty(M) \longrightarrow C^\infty(M^A, A)$ such that

$$X(f\cdot g) = X(f)\cdot g^A + f^A\cdot X(g),\quad \text{for any } f, g\in C^\infty(M).$$

Proof. (1) $\Longrightarrow$ (2) Let $U : M^A \longrightarrow TM^A$ be a differential section of the tangent bundle (TM^A, π_{M^A}, M^A).

1. Let

$$W : C^\infty(M^A)\longrightarrow C^\infty(M^A)$$

such that $[W(F)](\xi) = [U(\xi)](F)$ for any $F\in C^\infty(M^A)$ and $\xi\in M^A$.

- For any $F, G\in C^\infty(M^A)$, and for $\lambda\in\mathbb{R}$, we have:

$$\begin{aligned}[W(F+G)](\xi) &= [U(\xi)](F+G) = [U(\xi)](F)+[U(\xi)](G)\\ &= [W(F)](\xi)+[W(G)](\xi)\\ &= [W(F)+W(G)](\xi))\end{aligned}$$

for any $\xi \in M^A$, then $W(F+G) = W(F) + W(G)$;

$$\begin{aligned} [W(\lambda \cdot F)](\xi) &= [U(\xi)](\lambda \cdot F) = \lambda \cdot [W(F)](\xi) \\ &= [\lambda \cdot W(F)](\xi) \end{aligned}$$

for any $\xi \in M^A$, then $W(\lambda \cdot F) = \lambda \cdot W(F)$;

$$\begin{aligned} [W(F \cdot G)](\xi) &= [U(\xi)](F \cdot G) = [U(\xi)](F) \cdot G(\xi) + F(\xi) \cdot [U(\xi)](G) \\ &= [W(F)(\xi)] \cdot G(\xi) + F(\xi) \cdot [W(G)](\xi) \\ &= [W(F).G + F \cdot W(G)](\xi)) \end{aligned}$$

for any $\xi \in M^A$, then $W(F \cdot G) = W(F).G + F \cdot W(G)$.

2. (2) $\Longrightarrow$ (3) Let W be a vector field on M^A considered as a derivation of $C^\infty(M^A)$. Let

$$X : C^\infty(M^A, A) \xrightarrow{\sigma^{-1}} A \otimes C^\infty(M^A) \overset{id_A \otimes W}{\longrightarrow} A \otimes C^\infty(M^A) \xrightarrow{\sigma} C^\infty(M^A, A).$$

For any $\varphi, \psi \in C^\infty(M^A)$, we have:

$$\begin{aligned} X(\varphi + \psi) &= \left[\sigma \circ (id_A \otimes W) \circ \sigma^{-1}\right](\varphi + \psi) = [\sigma \circ (id_A \otimes W)](\sigma^{-1}(\varphi + \psi)) \\ &= [\sigma \circ (id_A \otimes W)](\sigma^{-1}(\varphi) + \sigma^{-1}(\psi)) \\ &= [\sigma \circ (id_A \otimes W)](\sigma^{-1}(\varphi)) + \sigma \circ [(id_A \otimes W)](\sigma^{-1}(\psi)) \\ &= \left[\sigma \circ (id_A \otimes W) \circ \sigma^{-1}\right](\varphi) + \left[\sigma \circ (id_A \otimes W) \circ \sigma^{-1}\right](\psi) \\ &= X(\varphi) + X(\psi); \end{aligned}$$

$$\begin{aligned} X(\lambda \cdot \varphi) &= \left[\sigma \circ (id_A \otimes W) \circ \sigma^{-1}\right](\lambda \cdot \varphi) = [\sigma \circ (id_A \otimes W)](\sigma^{-1}(\lambda \cdot \varphi)) \\ &= [\sigma \circ (id_A \otimes W)](\lambda \cdot \sigma^{-1}(\varphi)) = \lambda \cdot [\sigma \circ (id_A \otimes W)](\sigma^{-1}(\varphi)) \\ &= \lambda \cdot \left[\sigma \circ (id_A \otimes W) \circ \sigma^{-1}\right](\varphi) \\ &= \lambda \cdot X(\varphi); \end{aligned}$$

and

$$\begin{aligned} X(\varphi \cdot \psi) &= \left[\sigma \circ (id_A \otimes W) \circ \sigma^{-1}\right](\varphi \cdot \psi) \\ &= [\sigma \circ (id_A \otimes W)](\sigma^{-1}(\varphi \cdot \psi)) \\ &= [\sigma \circ (id_A \otimes W)](\sigma^{-1}(\varphi) \cdot \sigma^{-1}(\psi)) \\ &= [\sigma \circ (id_A \otimes W)](\sigma^{-1}(\varphi)) \cdot \psi + \varphi \cdot [(id_A \otimes W)](\sigma^{-1}(\psi)) \\ &= \left[\sigma \circ (id_A \otimes W) \circ \sigma^{-1}\right](\varphi) \cdot \psi + \varphi \cdot \left[\sigma \circ (id_A \otimes W) \circ \sigma^{-1}\right](\psi) \\ &= X(\varphi) \cdot \psi + \varphi \cdot X(\psi). \end{aligned}$$

3. (3) $\Longrightarrow$ (4) Let X be a vector field on M^A considered as a derivation of $C^\infty(M^A, A)$ which is A-linear. Let

$$Y : C^\infty(M) \longrightarrow C^\infty(M^A, A), f \longmapsto f^A.$$

- For any $f, g \in C^\infty(M)$, we have:

$$\begin{aligned} Y(f+g) &= X\left[(f+g)^A\right] \\ &= X(f^A + g^A) \\ &= X(f^A + X(g^A) \\ &= Y(f) + Y(g). \end{aligned}$$

- For any $f, g \in C^\infty(M)$, we have:

$$\begin{aligned} Y(f \cdot g) &= X\left[(f \cdot g)^A\right] \\ &= X\left(f^A \cdot g^A\right) \\ &= X(f^A) \cdot g^A + X(g^A) \cdot f^A \\ &= Y(f) \cdot g^A + Y(g) \cdot f^A \\ &= Y(f) + Y(g). \end{aligned}$$

4. (4) $\Longrightarrow$ (1) For that implication see, corollary 6 in [2].

$$Der_\xi\left[C^\infty(M^A), \mathbb{R}\right] \longrightarrow Der_\xi\left[C^\infty(M), A\right], v \longmapsto (id_A \otimes v) \circ \gamma,$$

is an isomorphism of vector spaces see [9].

□

Remark 2. For any $X \in \mathfrak{X}(M^A)$, we have $X\left[C^\infty(M^A)\right] \subset C^\infty(M^A)$.

Theorem 4. The map

$$\mathfrak{X}(M^A) \times \mathfrak{X}(M^A) \longrightarrow \mathfrak{X}(M^A), (X, Y) \longmapsto [X, Y] = X \circ Y - Y \circ X$$

is skew-symmetric A-bilinear and defines a structure of A-Lie algebra over $\mathfrak{X}(M^A)$.

In all what follows, we denotes $\mathfrak{X}(M^A)$, the set of A-linear maps

$$X : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)$$

such that

$$X(\varphi \cdot \psi) = X(\varphi) \cdot \psi + \varphi \cdot X(\psi), \quad \text{for any}\, \varphi, \psi \in C^\infty(M^A, A)$$

that is to say

$$\mathfrak{X}(M^A) = Der_A[C^\infty(M^A, A)].$$

2.3 Prolongations to M^A of vector fields on M

Proposition 5. If $\theta : C^\infty(M) \longrightarrow C^\infty(M)$, is a vector field on M, then there exists one and only one A-linear derivation

$$\theta^A : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A),$$

such that

$$\theta^A(f^A) = [\theta(f)]^A,$$

for any $f \in C^\infty(M)$.

Proof. If $\theta : C^\infty(M) \longrightarrow C^\infty(M)$,is a vector field on M, then the map

$$C^\infty(M) \longrightarrow C^\infty(M^A, A), f \longmapsto [\theta(f)]^A$$

is a vector field on M^A. Thus, according to the equivalent $(2) \Longleftrightarrow (3)$ of the theorem 2.1.2, there exists one and only one vector field on M^A

$$\theta^A : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A),$$

such that

$$\theta^A(f^A) = [\theta(f)]^A,$$

for any $f \in C^\infty(M)$. □

Proposition 6. If $\theta, \theta_1, \theta_2$ are vector fields on M and if $f \in C^\infty(M)$, then we have:

1. $(\theta_1 + \theta_2)^A = \theta_1^A + \theta_2^A$;
2. $(f \cdot \theta)^A = f^A \cdot \theta^A$;
3. $[\theta_1, \theta_2]^A = \left[\theta_1^A, \theta_2^A\right]$.

Corollary 7. The map

$$\mathfrak{X}(M) \longrightarrow Der_A[C^\infty(M^A, A)], \theta \longmapsto \theta^A$$

is an injective homomorphism of $\mathbb{R}$-Lie algebras.

Proposition 8. If $\mu : A \longrightarrow A$, is a $\mathbb{R}$-endomorphism, and $\theta : C^\infty(M) \longrightarrow C^\infty(M)$ a vector field on M, then

$$\theta^A(\mu \circ f^A) = \mu \circ [\theta(f)]^A,$$

for any $f \in C^\infty(M)$.

2.3.1 Vector fields on M^A deduced from derivations of A

Proposition 9. If d is a derivation of A, then there exists one and only, one A-linear derivation

$$d^* : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)$$

such that

$$d^*(f^A) = (-d) \circ f^A,$$

for any $f \in C^\infty(M)$.

Proof. If d is a derivation of A, then the map

$$C^\infty(M) \longrightarrow C^\infty(M^A, A), f \longmapsto (-d) \circ f^A,$$

is a vector field on M^A. Thus, according to the equivalent (2) $\Longleftrightarrow$ (3) of the theorem 2.1.2, there exists one and only, one A-linear derivation

$$d^* : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)$$

such that

$$d^*(f^A) = (-d) \circ f^A,$$

for any $f \in C^\infty(M)$. □

Proposition 10. For any $f \in C^\infty(M)$,

$$d^*(\mu \circ f^A) = -\mu \circ -d \circ f^A.$$

Proposition 11. If d, d_1, d_2 are vector fields on M and if $f \in C^\infty(M)$, then we have:

1. $(d_1 + d_2)^* = d_1^* + d_2^*$;
2. $(a \cdot d)^* = a \cdot d^*$;
3. $[d_1, d_2]^* = [d_1^*, d_2^*]$.
4. $\left[d^*, \theta^A\right] = 0$.

2.4 Writing of a vector field on M^A in local coordinate system.

Let U is a coordinate neighborhood of M at x with coordinate system $(x_1, ..., x_n)$. Then according to [7], $\left(\frac{\partial}{\partial x_1}\right)^A(\xi), \left(\frac{\partial}{\partial x_2}\right)^A(\xi), ..., \left(\frac{\partial}{\partial x_n}\right)^A(\xi)$ is an A-basis of an A-free module $T_\xi M^A$ of dimension n. For $v \in T_\xi M^A$, we have:

$$v = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial x_i}\right)^A(\xi).$$

Proposition 12. There exists a canonical diffeomorphism

$$\Theta : T(M^A) \longrightarrow TM^A, v \longmapsto \Theta(v),$$

such that, for any $f \in C^\infty(M)$, we have:

1. $[\Theta(v)]\,(f \circ \pi_M) = [\pi_{M^A}(v)]\,(f)$.
2. $[\Theta(v)]\,(df) = v\left(f^A\right)$.

Proof. Let

$$\Theta : T(M^A) \xrightarrow{\Theta_1} (M^A)^{\mathbb{D}} \xrightarrow{\Theta_2} M^{A\otimes\mathbb{D}} \xrightarrow{\Theta_3} (M^{\mathbb{D}})^A \xrightarrow{\Theta_4} (M^A)^{\mathbb{D}} \xrightarrow{\Theta_5} TM^A,$$

where

$$\Theta_1(v) : C^\infty(M^A) \longrightarrow \mathbb{D}, F \longmapsto F(\xi) + v(F)\cdot\varepsilon;$$

$$\Theta_2 : \Theta_1(v) \longmapsto (id_A \otimes \Theta_1(v)) \circ \gamma_A;$$

$$\Theta_3 : (id_A \otimes \Theta_1(v)) \circ \gamma_A \longmapsto \Theta_3\left[(id_A \otimes \Theta_1(v)) \circ \gamma_A\right] : f \longmapsto \sum_{\alpha=1}^{\dim A} [\Theta_1(v)]\left(a_\alpha^* \circ f^A\right) \otimes a_\alpha;$$

$$\Theta_4 : \Theta_3\left[(id_A \otimes \Theta_1(v)) \circ \gamma_A\right] \longmapsto \eta$$

such that

$$(id_{\mathbb{D}} \otimes \eta) \circ \gamma_{\mathbb{D}} = \Theta_3\left[(id_A \otimes \Theta_1(v)) \circ \gamma_A\right].$$

Thus for any $f \in C^\infty(M)$, we have:

$$\left[(id_{\mathbb{D}} \otimes \eta) \circ \gamma_{\mathbb{D}}\right](f) = \Theta_3\left[(id_A \otimes \Theta_1(v)) \circ \gamma_A\right](f)$$

and

$$\begin{aligned}
1 \otimes \eta(1^* \circ f^{\mathbb{D}}) + \varepsilon \otimes \eta(\varepsilon^* \circ f^{\mathbb{D}}) &= \sum_{\alpha=1}^{\dim A} [\Theta_1(v)]\left(a_\alpha^* \circ f^A\right) \otimes a_\alpha \\
&= \sum_{\alpha=1}^{\dim A} a_\alpha \otimes \left(a_\alpha^* \circ f^A\right)(\xi) + \sum_{\alpha=1}^{\dim A} a_\alpha \otimes v\left[\left(a_\alpha^* \circ f^A\right)\cdot\varepsilon\right] \\
&= 1 \otimes \left(\sum_{\alpha=1}^{\dim A} a_\alpha^*\left[f^A(\xi)\right]\cdot a_\alpha\right) + \varepsilon \otimes \sum_{\alpha=1}^{\dim A} v\left[\left(a_\alpha^* \circ f^A\right)\cdot a_\alpha\right] \\
& \quad 1 \otimes \left(\sum_{\alpha=1}^{\dim A} a_\alpha^*\left[\xi(f)\right]\cdot a_\alpha\right) + \varepsilon \otimes v(f^A),
\end{aligned}$$

then the identification

$$\eta(1^* \circ f^{\mathbb{D}}) = \xi(f)$$

and

$$\eta(\varepsilon^* \circ f^{\mathbb{D}}) = v(f^A).$$

It follows that

$$\begin{aligned} \Theta(v) &= [\Theta_5 \circ \Theta_4 \circ \Theta_3 \circ \Theta_2 \circ \Theta_1](v) \\ &= \Theta_5(\eta) \\ &= \eta \circ \varphi^* \end{aligned}$$

with $\varphi : M^{\mathbb{D}} \longrightarrow TM, \xi \longmapsto v$, such that for $f \in C^\infty(M)$, $\xi(f) = f(p) + v(f)$ where $p \in M, v \in T_pM$ and $\xi \in M_p^A$. The map φ is a diffeomorphism. Then, Θ is a diffeomorphism as composition of diffeomorphisms. Moreover,

$$[\Theta(v)](f \circ \pi_M) = \eta[\varphi^*(f \circ \pi_M)] = \eta(1^* \circ f^{\mathbb{D}}) = \xi(f) = \pi_{M^A}(v)$$

and

$$[\Theta(v)](df) = \eta[\varphi^*(df)] = \eta(\varepsilon^* \circ f^{\mathbb{D}}) = v(f^A).$$

□

Proposition 13. The map

$$\theta : TU^A \longrightarrow U^A \times A^n, v \longmapsto \left(\pi_{M^A}(v), v\left(x_1^A\right), ..., v(x_n^A)\right)$$

is a diffeomorphism. For $\xi \in U^A$,

$$\theta_{|T_\xi U^A} : T_\xi U^A \longrightarrow \{\xi\} \times A^n$$

is an isomorphism of A-modules.

Proof. Let $\pi_M : TM \longrightarrow M$ and $\pi_{M^A} : TM^A \longrightarrow M^A$ be the projections of TM and TM^A on M, M^A respectively.
The bundle TM being locally trivial then for any $x \in M$ there exists an open coordinate neighborhood U of x in M and a local diffeomorphism $h_U : \pi_M^{-1}(U) \longrightarrow U \times \mathbb{R}^n$ such that the following diagram

$$\begin{array}{ccc} \pi_M^{-1}(U) & \xrightarrow{h_U} & U \times \mathbb{R}^n \\ \pi_{M_{|\pi_M^{-1}(U)}} \downarrow & \swarrow pr_1 & \\ U & & \end{array}$$

commute i.e $pr_1 \circ h_U = \pi_{M|\pi_M^{-1}(U)}$. Thus, let

$$T(U^A) \xrightarrow{\Theta} (TU)^A \xrightarrow{(h_U)^A} [U \times \mathbb{R}^n]^A \xrightarrow{\phi_1} U^A \times (\mathbb{R}^n)^A \xrightarrow{\phi_2} U^A \times (\mathbb{R}^n \otimes A) \xrightarrow{\phi_3} U^A \otimes A^n$$

where

$$\Theta : v \mapsto \Theta(v);$$
$$(h_U)^A : \Theta(v) \mapsto \Theta(v) \circ h_U^*;$$
$$\phi_1 : \Theta(v) \circ h_U^* \mapsto \left((pr_1)^A[\Theta(v) \circ h_U^*], (pr_2)^A[\Theta(v) \circ h_U^*]\right);$$
$$\phi_2 : \left((pr_1)^A[\Theta(v) \circ h_U^*], (pr_2)^A[\Theta(v) \circ h_U^*]\right) \mapsto \left((pr_1)^A[\Theta(v) \circ h_U^*], \sum_{i=1}^n e_i \otimes \left[(pr_2)^A[\Theta(v) \circ h_U^*]\right](e_i^*)\right);$$
$$\phi_3 : \left((pr_1)^A[\Theta(v) \circ h_U^*], \sum_{i=1}^n e_i \otimes \left[(pr_2)^A[\Theta(v) \circ h_U^*]\right](e_i^*)\right) \mapsto \left(\pi_{M^A}(v), \left(v(x_1^A), ..., v(x_n^A)\right)\right)$$

with $e_i^* \circ pr_2 \circ h_U = dx_i$.

It follows that,

$$\theta(v) = [\phi_3 \circ \phi_2 \circ \phi_1 \circ (h_U)^A \circ \Theta\left(v(x_1^A), ..., v(x_n^A)\right)](v) = (\pi_{M^A}(v),)$$

hence θ is a diffeomorphism as composition of diffeomorphisms.
Besides, for $\xi \in U^A$, $\theta_{|T_\xi U^A}$ is an isomorphism of A-modules. Indeed: it follows from θ that $\theta_{|T_\xi U^A}$ is bijective and in addition, if $v = \sum_{i=1}^n \lambda_i \left(\frac{\partial}{\partial x_i}\right)^A (\xi)$ and $w = \sum_{i=1}^n \mu_i \left(\frac{\partial}{\partial x_i}\right)^A (\xi)$ are element of $T_\xi U^A$ and $a \in A$, we have first

$$\begin{aligned}
\theta_{|T_\xi U^A}(v+w) &= \theta_{|T_\xi U^A}\left(\sum_{i=1}^n (\lambda_i + \mu_i)\left(\frac{\partial}{\partial x_i}\right)^A (\xi)\right) \\
&= (\lambda_1 + \mu_1, ..., \lambda_n + \mu_n) \\
&= (\lambda_1, ..., \lambda_n) + (\mu_1, ..., \mu_n) \\
&= \theta_{|T_\xi U^A}(v) + \theta_{|T_\xi U^A}(w).
\end{aligned}$$

and secondly, we have

$$\theta_{|T_\xi U^A}(a \cdot v) = (a \cdot \lambda_1, ..., a \cdot \lambda_n) = a \cdot (\lambda_1, ..., \lambda_n) = a \cdot \theta_{|T_\xi U^A}(v).$$

□

That result leads to state:

Corollary 14. The tangent bundle TM^A is locally trivial with typical fiber A^n.

Proposition 15. If $X : M^A \longrightarrow TM^A$ is a vector field on M^A and if U is a coordinate neighborhood of M with coordinate neighborhood $(x_1, ..., x_n)$, then there exists some functions $f_i \in C^\infty(U^A, A)$ for $i = 1, ..., n$ such that

$$X_{|U^A} = \sum_{i=1}^n f_i \left(\frac{\partial}{\partial x_i^A}\right)^A.$$

Suggestion for notations.

When (U, φ) is local chart and $(x_1, ..., x_n)$ his local coordinate system. The map

$$U^A \longrightarrow A^n, \xi \longmapsto (\xi(x_1), ..., \xi(x_n)),$$

is a diffeomorphism from U^A onto an open set on A^n.
As

$$\left(\frac{\partial}{\partial x_i}\right)^A : C^\infty(U^A, A) \longrightarrow C^\infty(U^A, A)$$

is such that

$$\left(\frac{\partial}{\partial x_i}\right)^A (x_j^A) = \delta_{ij},$$

we can denote

$$\frac{\partial}{\partial x_i^A} = \left(\frac{\partial}{\partial x_i}\right)^A .$$

If $v \in T_\xi M^A$, we can write

$$v = \sum_{i=1}^n v(x_i^A) \frac{\partial}{\partial x_i^A}|_\xi \ .$$

If $X \in \mathfrak{X}(M^A) = Der_A[C^\infty(M^A, A)]$, we have

$$X_{|U^A} = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i^A}.$$

with $f_i \in C^\infty(U^A, A)$ for $i = 1, 2, ..., n$.

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