**Mathematics of Harmony:**

 **Constructing Musical Chords Using Means**

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**ABSTRACT**

The purpose of this paper is twofold. Our first goal is to prove the well-known chain of inequalities involving the harmonic, geometric, logarithmic, identric, and arithmetic means using nothing more than the Mean Value Theorem of basic calculus.

Of course, these results are all quite familiar and several proofs of them and their generalizations have been given – a short list includes Hardy, Littlewood, and Pólya (1964); Carlson (1965); Carlson and Tobey (1968); Beckenbach and Bellman (1971); Alzer (1985a, 1985b); Bullen, Mitrinović, and Vasić (1988), and Mercer (2003). Here, we present a unified approach and give the proofs as corollaries of one elementary theorem.

Our second goal is to apply these means to music, namely to chord construction. We will show that every note in any scale starting by a given tone (and consequently any chord based on that tone) can be expressed as an appropriate mean of the tone, the tone an octave higher, or the tone two octaves higher.

**Keywords:** Pythagorean means, logarithmic mean, identric mean, major chords, minor chords

**AMS Subject Classification Code:** 97M80

1. **Introduction**

The exploration of the connections between mathematics and music constitutes a field of study that is filled with theoretically captivating and intellectually challenging concepts.

Although usually when this topic is mentioned the first thing that comes to mind is that both disciplines depend heavily on symbols to express their creations, this, in my opinion, is a rather trivial connection and does not represent either the intrinsic aspects or the profundity of the links between the two. Some other commonly mentioned connections such as problems associated with tuning and expression of sound waves via Fourier series belong to physics more than mathematics.

In my estimation, the first and foremost organic connection between the two disciplines lies in the fact that they are two of the most unadulterated, most abstract, and most aesthetic forms of human creativity. This special place occupied by mathematics and music amongst all other human activities was elegantly noted by the English mathematician and philosopher Alfred North Whitehead (1861-1947)

The science of pure mathematics, in its modern developments, may claim to be the most original creation of the human spirit. Another claimant for this position is music. (Whitehead 1927, 29)

Equally significant is the fact that mathematics and music both contend with a difficult yet rewarding pursuit of complex patterns and harmony – in fact these are at the crux of both endeavors. As G. H. Hardy (1877-1947) articulated in his 1941 pamphlet *A Mathematician’s Apology*

A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made of ideas. His patterns, like the painter's or the poet's must be beautiful; the ideas, like the colors or the words, must fit together in a harmonious way.

Or as Albert Einstein (1879-1955) remarked

In art, and in the higher ranges of science, there is a feeling of harmony which underlies all endeavors. There is no true greatness in art or science without that sense of harmony.

That is probably why the set of mathematicians and the set of musicians have an astonishingly substantial intersection. For instance, the German/English astronomer William Herschel (1738-1822) was a violinist, organist, conductor, and composer; the German linguist and mathematician Hermann Günther Grassmann (1809-1877) was a piano player, singer, and composer; the Hungarian mathematician János Bolyai (1802-1860) was an exceptionally gifted violinist; the English logician and mathematician Augustus De Morgan (1806-1871) was an excellent flute player; the Russian composer Alexander Borodin (1833-1887) was a professor of chemistry; the noted English astronomer, physicist, and mathematician Sir James Jeans (1877-1946) was a versatile organ player; Albert Einstein was a violinist who played with several symphony orchestras[[1]](#footnote-1).

As such, there are, of course, numerous books and papers that explore the profound connections between mathematics and music, for instance, Helmholtz (1870); Schoenberg (1950); Coxeter (1968); Xenakis (1972); Witlich (1975); Brindle (1987); Morris (1987); McCartin (1998); Dunne, Edward and McConnell (1999); and Izmirli (2011a, 2011b), to name a few.

In this paper we will explore the possibility of expressing the notes in any scale starting with a tone as an appropriate mean of , (that is, tone an octave higher) or (that is, tone two octaves higher.)

1. **The Pythagorean Means**

For a sequence of numbers we will let





and



to denote the well-known *arithmetic*, *geometric*, and *harmonic* *means*, also called the *Pythagorean means* ().

The Pythagorean means have the obvious properties:

1. is independent of order
2. 
3. 
4. is always a solution of a simple equation. In particular, the arithmetic mean of two numbers  and can be defined via the equation



The harmonic mean satisfies the same relation with reciprocals, that is, it is a solution of the equation



The geometric mean of two numbers  and can be visualized as the solution of the equation



This follows because

1. **The Logarithmic and Identric Means**

The *logarithmic mean* of two non-negative numbers  and  is defined as follows:

 and for positive distinct numbers and

The following are some basic properties of the logarithmic means:

1. Logarithmic mean can be thought of as the mean-value of the function over the interval .
2. The logarithmic mean can also be interpreted as the area under an exponential curve.

Since

we have

Using this representation it is easy to show that

1. We have the identity

which follows easily:

To define the logarithmic mean of positive numbers , we first recall the definition of divided differences for a function at points , denoted as

For

and for and ,

We now define

So for example for *n* = 2, we get

The *identric mean* of two distinct positive real numbers is defined as:

with 

The slope of the secant line joining the points and on the graph of the function is the natural logarithm of .

1. **The Main Theorem**

**Theorem 1.** *Suppose is a function with a strictly increasing derivative. Then*

 *for all in .*

 *Let be defined by the equation*

*Then,*

*is the sharpest form of the above inequality.*

***Proof.*** By the Mean Value Theorem, for all in , we have

for some between and . Assuming without loss of generality by the assumption of the theorem we have

Integrating both sides with respect to , we have

and the inequality of the theorem follows.

Let us now put

Note that

Moreover, since

there exists an in such that .

Since is strictly increasing, we have

for

and

for

Thus, is a minimum of and for all in

1. **Proof of the Mean Inequalities**

We will now prove the well-known chain of inequalities



as special cases of Theorem 1. All of these are strict inequalities unless, of course, the numbers are the same, in which case all means are equal to the common value of the two numbers.

Let us now assume that

Let us let The condition of the Theorem 1 is satisfied. Solving the equation

we find

where

Hence the left-hand side of the inequality becomes

Thus we have

implying

or

Let us let . The condition of Theorem 1 is satisfied. We can easily compute the of the theorem from the equation

as

Our inequality becomes

Implying,

that is

Now let . Again the condition of Theorem 1 is satisfied. The of the theorem can be computed from the equation

as

where

Since

Thus,

where

Consequently our inequality becomes

implying

that is,

Finally, let us put . Again the condition of Theorem 1 is satisfied. Since in this case

the of the theorem can be computed as

The right-hand side of the inequality becomes

The integral on the left-hand side of our inequality yields

implying

or

Thus, we now have for

1. **Chord Construction Using Means**

That the Pythagorean means were musically significant was first pointed out by the Roman philosopher Anicius Manlius Severinus Boethius (c. 480 – 524). Since his results involve some ratios of frequencies of notes let us list these frequencies here. We will use the sharp notation instead of the flat notation in the enharmonic spelling.

 (Unison)

(Minor second )

 (Major second )

 (Minor third)

 (Major third )

(Perfect fourth)

(Augmented fourth )

 (Perfect fifth

 (Minor sixth )

 (Major sixth )

 (Minor seventh )

 (Major seventh)

 (Octave)

Boethius elucidated the simple yet noteworthy fact that the arithmetic mean of tones separated by an octave is the perfect fifth (in terms of frequencies):



Later, Italian music theorist Gioseffo Zarlino (1517-1590) discovered yet a more significant relation. The arithmetic mean (*medietas*) of the tone and its fifth yields the major third, whereas the harmonic mean of the tone and its fifth yields the minor third, i.e.,



and



The triad is represented by the ***arithmetic progression***  (= 4 : 5: 6) is called the ***major triad***, and the triad represented by the ***harmonic progression***  (= 10: 12: 15) is called the ***minor triad***. Zarlino discovered these relations using the lengths of the stings (which of course are the reciprocals of the frequencies) and got the opposites of the above results. Thus, he ended up calling the major triad ***harmonica***, and the minor triad ***arithmetica***.

So we can use Pythagorean means to form major and minor triads (chords). Let us call the tone , the tone an octave higher .

Then the above equations are:

 , the major fifth

, the major third

, the minor third.

Thus the major chord is

and the minor chord is

So using (c representing middle c) our formulas yield = () as the major triad and as the minor triad.

In fact, any note in the scale can be expressed in terms of means and hence any possible chord can be expressed in terms of means. To this end, straightforward computations yield that

where is the major second an octave higher and is the tone two octaves higher. In the formula for , one should, of course, replace by and by .

Hence we have the following:

*Let be a note with frequency . Any chord (which is any combination of notes) based on the tone can be formed by a combination of the means of , that is by computing the appropriate means of the frequencies*

Let us conclude this paper by a question first posed by the 19th century English mathematicianJames Joseph Sylvester(1814-1897)in his paper *On Newton's Rule for the Discovery of Imaginary Roots* (*Collected Mathematical Papers, Vol. 2, p. 419* ):

May not music be described as mathematics of the sense, mathematics as music of the reason?”

**References**

Alzer H. 1985a. Über einen Wert, der zwischen dem geometrischen und dem artihmetischen Mittel zweier Zahlen liegt. In *Elemente Math*., 40 (1985), p. 22-24.

Alzer H. 1985b. Ungleichungen für . In *Elemente Math.*, 40 (1985), p. 120-123.

Barbour, Murray J. 1951. *Tuning and Temperament: A Historical Survey*. New York: Simon and Schuster, Inc.

Beckenbach E. F. and R. Bellman. 1971. *Inequalities.* Third Edition. Ergebnisse der Mathematik, Band 30, Berlin and New York: Springer-Verlag.

Brindle, Reginald S. 1987. *The New Music: The Avant-garde since 1945*. (Second Edition.) Oxford and New York: Oxford University Press.

Bullen, P. S., D. S. Mitrinović, and P. M. Vasić. 1988. *Means and Their Inequalities.* Dordrecht: D. Reidel Publishers.

Carlson, B. C. 1965. A hypergeometric mean value. In *Proc. Am. Math. Soc*. 16 (1965), p. 759-766.

Carlson B. C. and M. D. Tobey. 1968. A property of the hypergeometric mean value. In *Proc. Am. Math. Soc.* 19 (1968), p. 255-262.

Coxeter, H. M. S. 1968. Music and Mathematics. In *The Mathematics Teacher****.***

Dunne, Edward and Mark McConnell. 1999. Pianos and Continued Fractions*.*In*Mathematics Magazine****,*** Vol 72., No 2, April 1999.

Hardy G. H., J. E. Littlewood, and G. Pólya. *Inequalities.* 1964. Second Edition. London: Cambridge University Press (Originally published in 1934).

Helmholtz, Hermann. 1870. *Die Lehre von den Tonempfindungen als physiologische Grundlage für die Theorie der Musik.*

Izmirli, I. 2011a. Categorizing Musical Pieces by Their Correlation Coefficients. In *Journal of Interdisciplinary Mathematics*, Vol.14, Issue 4, August 2011 pp. 407 – 424.

Izmirli, I. 2011b. Directional Pitch Spaces. In B. Sriraman & V. Freiman (Eds), *Interdisciplinarity for the 21st Century: Proceedings of the 3rd International Symposium on Mathematics and its Connections to Arts and Sciences*, Moncton 2009. Monograph 11 in The Montana Mathematics Enthusiast Monographs in Mathematics Education, Information Age Publishing, Charlotte, NC, pp. 327-346.

McCartin, Brian J. 1998. Prelude to Mathematical Geometry. In *The College Mathematics Journal,* Vol 29, No 5, November 1998.

Mercer, Peter R. 2003. Refined Arithmetic, Geometric, and Harmonic Mean Inequalities. In *Rocky Mountain Journal of Mathematics*, vol. 33, number 4, pp. 1459-1464.

Morris, Robert D. 1987. *Composition with Pitch Classes: A Theory of Compositional Design*. New Haven, Conn.: Yale University Press.

Schoenberg*,* Arnold*.* 1950.Composition with Twelve Tones*.* In *Style and Idea*tr. Dika Newlin. New York: Philosophical Library.

Whitehead, Alfred North. 1927. *Science and the Modern World.* New York: The Macmillan Company.

Witlich, Gary E. (Editor) 1975. *Aspects of Twentieth-Century Music****.*** Englewood Cliffs, NJ: Prentice-Hall Inc.

Xenakis, Iannis. 1972. *Formalized Music*. Bloomington: Indiana University Press.

1. There is an anecdote about a critic who wrote after Einstein played his violin at a charity concert, “He did tolerably well, but I cannot understand how his name comes to be so well known all over the world.” [↑](#footnote-ref-1)