Special Smarandache curves according to Bishop frame in Euclidean space-time

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Abstract. In this paper, we introduce some special Smarandache curves according to Bishop frame in Euclidean 3-space E^3. Also, we study Frenet-Serret invariants of a special case in E^3. And we give an example to illustrate these curves.

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1 Introduction

In the theory of curves in the Euclidean and Minkowski spaces, one of the interesting problems is the characterization of a regular curve. In the solution of the problem, the curvature functions κ and τ of a regular curve have an effective role. It is known that the shape and size of a regular curve can be determined by using its curvatures κ and τ [7, 8, 9]. For instance, Bertrand curves and Mannheim curves arise from this relationship. Another example is the Smarandache curves. They are the objects of Smarandache geometry, that is, a geometry which has at least one Smarandachely denied axiom [1]. The axiom is said to be Smarandachely denied if it behaves in at least two different ways within the same space. Smarandache geometries are connected with the theory of relativity and the parallel universes.

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By definition, if the position vector of a curve β is composed by the Frenet frame's vectors of another curve α, then the curve β is called a Smarandache curve [10]. Special Smarandache curves in the Euclidean and Minkowski spaces are studied by some authors [6, 11]. For instance, the special Smarandache curves according to Darboux frame in E^3 are characterized in [5].

In this work, we study special Smarandache curves according to Bishop frame in the Euclidean space. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

2 Preliminaries

The Euclidean 3-space E^3 provided with the standard flat metric given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E^3. Recall that, the norm of an arbitrary vector $v \in \mathrm{E}^3$ is given by $\|v\| = \sqrt{|\langle v, v\rangle|}$. α is called an unit speed curve if velocity vector α' of satisfies $\|\alpha'\| = 1$. For vectors $u, v \in \mathrm{E}^3$ it is said to be orthogonal if and only if $\langle u, v\rangle = 0$. Let $\alpha = \alpha(s)$ be a regular curve in E^3. If the tangent vector field of this curve forms a constant angle with a constant vector field U, then this curve is called a general helix or an inclined curve. The sphere of radius $r > 0$ and with center in the origin in the space E^3 is defined by

$$S^2 = \{p = (p_1, p_2, p_3) \in \mathrm{E}^3 : \langle p, p\rangle = r^2\}.$$

Denote by $\{T, N, B\}$ the moving Frenet-Serret frame along the curve α in the space E^3. For an arbitrary curve $\alpha \in \mathrm{E}^3$, with first and second curvature, κ and τ respectively, the Frenet-Serret formulas is given by [7, 9].

$$\begin{pmatrix} T'(s) \\ N'(s) \\ B'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}. \tag{1}$$

where $\langle T, T\rangle = \langle N, N\rangle = \langle B, B\rangle = 1$, $\langle T, N\rangle = \langle T, B\rangle = \langle N, B\rangle = 0$. Then, we write Frenet invariants in this way: $T(s) = \alpha'(s)$, $\kappa(s) = \|T'(s)\|$, $N(s) = T'(s)/\kappa(s)$, $B(s) = T(s) \times N(s)$ and $\tau(s) = \langle N(s), B'(s)$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. One can express Bishop of an orthonormal frame along a curve simply by parallel transporting each component of the frame [2]. The tangent vector and any convenient arbitrary basis for the remainder of the frame are used (for details, see [3]). The Bishop frame is expressed as [2, 4].

$$\begin{pmatrix} T'(s) \\ N_1'(s) \\ N_2'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & 0 \\ -k_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{pmatrix}. \tag{2}$$

Here, we shall call the set $\{T, N_1, N_2\}$ as Bishop trihedra and $k_1(s)$ and $k_2(s)$ as Bishop curvatures. The relation matrix may be expressed as

$$\begin{pmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(s) & -\sin\theta(s) \\ 0 & \sin\theta(s) & \cos\theta(s) \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}, \tag{3}$$

where

$$\begin{cases} \theta(s) = arc\tan\left(\dfrac{k_2}{k_1}\right), \;\; k_1 \neq 0 \\ \tau(s) = -\dfrac{d\theta(s)}{ds} \\ \kappa(s) = \sqrt{k_1^2(s) + k_2^2(s)} \end{cases} \tag{4}$$

Here, Bishop curvatures are defined by

$$\begin{cases} k_1(s) = \kappa(s)\cos\theta(s) \\ k_2(s) = \kappa(s)\sin\theta(s) \end{cases} \tag{5}$$

Let $\alpha = \alpha(s)$ be a regular non-null curve parametrized by arc-length in Euclidean 3-space E^3 with its Bishop frame $\{T, N_1, N_2\}$. Then TN_1, TN_2, N_1N_2 and TN_1N_2-Smarandache curve of α are defined, respectively as follows [10]:

$$\xi = \xi(s^*) = \frac{1}{\sqrt{2}}\Big(T(s) + N_1(s)\Big),$$
$$\xi = \xi(s^*) = \frac{1}{\sqrt{2}}\Big(T(s) + N_2(s)\Big),$$
$$\xi = \xi(s^*) = \frac{1}{\sqrt{2}}\Big(N_1(s) + N_2(s)\Big),$$
$$\xi = \xi(s^*) = \frac{1}{\sqrt{3}}\Big(T(s) + N_1(s) + N_2(s)\Big).$$

3 Special Smarandache curves according to Bishop frame in E^3

Definition 3.1. A regular curve in Euclidean space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve.

In the light of the above definition, we adapt it to regular curves according to Bishop frame in the Euclidean space as follows:

Definition 3.2. Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{T, N_1, N_2\}$ be its moving Bishop frame. TN_1-Smarandache curves are defined by

$$\xi = \xi(s^*) = \frac{1}{\sqrt{2}}\Big(T(s) + N_1(s)\Big). \tag{6}$$

Let us investigate Frenet-Serret invariants of TN_1-Smarandache curves according to $\alpha = \alpha(s)$. By differentiating Eqn. (6) with respected to s and using Eqn. (2), we get

$$\xi' = \frac{d\xi}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\Big(- k_1 T + k_1 N_1 + k_2 N_2\Big), \tag{7}$$

hence

$$T_\xi = \frac{-k_1 T + k_1 N_1 + k_2 N_2}{\sqrt{2k_1^2 + k_2^2}}, \tag{8}$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{2k_1^2 + k_2^2}{2}}. \tag{9}$$

In order to determine the first curvature and the principal normal of the curve ξ, we formalize

$$T'_\xi = \frac{dT_\xi}{ds^*}\frac{ds^*}{ds} = \dot{T}_\xi \frac{ds^*}{ds} = \frac{\zeta_1 T + \zeta_2 N_1 + \zeta_3 N_2}{\big(2k_1^2 + k_2^2\big)^{\frac{3}{2}}}, \tag{10}$$

where

$$\begin{cases} \zeta_1 = \big[k_1(2k_1k_1' + k_2k_2') - (2k_1^2 + k_2^2)(k_1' + k_1^2 + k_2^2)\big], \\ \zeta_2 = \big[(2k_1^2 + k_2^2)(k_1' - k_1^2) - k_1(2k_1k_1' + k_2k_2')\big], \\ \zeta_3 = \big[(2k_1^2 + k_2^2)(k_2' - k_1k_2) - k_2(2k_1k_1' + k_2k_2')\big]. \end{cases} \tag{11}$$

Then, we have

$$\dot{T}_\xi = \frac{\sqrt{2}}{\big(2k_1^2 + k_2^2\big)^2}\Big(\zeta_1 T + \zeta_2 N_1 + \zeta_3 N_2\Big). \tag{12}$$

So, the first curvature and the principal normal vector field are respectively given by

$$\kappa_\xi = \|\dot{T}_\xi\| = \frac{\sqrt{2}\sqrt{\zeta_1^2+\zeta_2^2+\zeta_3^2}}{\left(2k_1^2+k_2^2\right)^2}. \tag{13}$$

and

$$N_\xi = \frac{\zeta_1 T + \zeta_2 N_1 + \zeta_3 N_2}{\sqrt{\zeta_1^2+\zeta_2^2+\zeta_3^2}}. \tag{14}$$

On other hand, we express

$$T_\xi \times N_\xi = \frac{1}{p\,q}\begin{vmatrix} T & N_1 & N_2 \\ -k_1 & k_1 & k_2 \\ \zeta_1 & \zeta_2 & \zeta_3 \end{vmatrix} \tag{15}$$

where $p = \sqrt{2k_1^2+k_2^2}$ and $q = \sqrt{\zeta_1^2+\zeta_2^2+\zeta_3^2}$. So, the binormal vector is

$$B_\xi = \frac{1}{p\,q}\Big\{\big[k_1\zeta_3 - k_2\zeta_2\big]T + \big[k_1\zeta_3 + k_2\zeta_1\big]N_1 + k_1\big[\zeta_1+\zeta_2\big]N_2\Big\}. \tag{16}$$

In order to calculate the torsion of the curve ξ, we differentiate Eqn. (7) with respected to s, we have

$$\xi'' = \frac{1}{\sqrt{2}}\Big\{-\big[k_1' + k_1^2 + k_1k_2 + \big]T + \big[k_1' - k_1^2\big]N_1 + \big[k_2' - k_1k_2\big]N_2\Big\}. \tag{17}$$

and thus

$$\xi''' = \frac{\nu_1 T + \nu_2 N_1 + \nu_2 N_2}{\sqrt{2}} \tag{18}$$

where

$$\begin{cases} \nu_1 = -\big[k_1'' + k_1'(3k_1+k_2) + k_2'(k_1+k_2) - k_1(k_1^2+k_2^2)\big], \\ \nu_2 = k_1'' - k_1(k_1^2 + 3k_1' + k_1k_2), \\ \nu_3 = k_2'' - k_1k_2' - k_2(k_1^2 + 2k_1' + k_1k_2). \end{cases} \tag{19}$$

The torsion is then given by:

$$\tau_\xi = \frac{\sqrt{2}\big[(k_1^2-k_1')(k_1\nu_3+k_2\nu_1) + k_1(k_2'-k_1k_2)(\nu_1+\nu_2) + (k_1^2+k_1'+k_1k_2)(k_1\nu_3-k_2\nu_2)\big]}{(k_1k_2'-k_1'k_2)^2 + \big[k_1k_2' + k_2(k_1'+k_1k_2)\big]^2 + k_1^2(2k_1^2+k_1k_2)^2} \tag{20}$$

Corollary 3.1. Let $\alpha = \alpha(s)$ be a curve lying fully in E^3 with the moving frame $\{T, N, B\}$. If the torsion $\tau = 0$, then the Bishop curvatures becomes constant and the TN_1-Smarandache curve is a circular helix.

Definition 3.3. Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{T, N_1, N_2\}$ be its moving Bishop frame. TN_2-Smarandache curves are defined by

$$\xi = \xi(s^*) = \frac{1}{\sqrt{2}}\Big(T(s) + N_2(s)\Big). \tag{21}$$

Remark 3.1. The Frenet-Serret invariants of TN_2-Smarandache curves can be easily obtained by the apparatus of the regular curve $\alpha = \alpha(s)$.

Definition 3.4. Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{T, N_1, N_2\}$ be its moving Bishop frame. N_1N_2-Smarandache curves are defined by

$$\xi = \xi(s^*) = \frac{1}{\sqrt{2}}\Big(N_1(s) + N_2(s)\Big). \tag{22}$$

Remark 3.2. The Frenet-Serret invariants of N_1N_2-Smarandache curves can be easily obtained by the apparatus of the regular curve $\alpha = \alpha(s)$.

Definition 3.5. Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{T, N_1, N_2\}$ be its moving Bishop frame. TN_1N_2-Smarandache curves are defined by

$$\xi = \xi(s^*) = \frac{1}{\sqrt{3}}\Big(T(s) + N_1(s) + N_2(s)\Big). \tag{23}$$

Remark 3.3. The Frenet-Serret invariants of TN_1N_2-Smarandache curves can be easily obtained by the apparatus of the regular curve $\alpha = \alpha(s)$.

Example 3.1. Let $\alpha(s) = \frac{1}{\sqrt{2}}\big(-\cos s, -\sin s, s\big)$ be a curve parametrized by arc length. Then it is easy to show that $T(s) = \frac{1}{\sqrt{2}}\big(\sin s, -\cos s, 1\big)$, $\kappa = \frac{1}{\sqrt{2}} \neq 0$, $\tau = -\frac{1}{\sqrt{2}} \neq 0$ and $\theta(s) = \frac{1}{\sqrt{2}}s + c$, $c = constant$. Here, we can take $c = 0$. From Eqn. (4), we get $k_1(s) = \frac{1}{\sqrt{2}}\cos\big(\frac{s}{\sqrt{2}}\big)$, $k_2(s) = \frac{1}{\sqrt{2}}\sin\big(\frac{s}{\sqrt{2}}\big)$. From Eqn. (1), we get $N_1(s) = \displaystyle\int k_1(s)T(s)ds$, $N_2(s) = \displaystyle\int k_2(s)T(s)ds$, the we have

$$\begin{aligned}
N_1(s) = \Bigg(&\frac{\sqrt{2}}{4(1+\sqrt{2})}\cos\Big((1+\sqrt{2})s\Big) - \frac{\sqrt{2}}{4(1-\sqrt{2})}\cos\Big((1-\sqrt{2})s\Big),\\
&-\frac{\sqrt{2}}{4(1+\sqrt{2})}\sin\Big((1+\sqrt{2})s\Big) - \frac{\sqrt{2}}{4(1-\sqrt{2})}\sin\Big((1-\sqrt{2})s\Big), \frac{\sqrt{2}}{2}\sin\Big(\frac{s}{\sqrt{2}}\Big)\Bigg)\\
N_2(s) = \Bigg(&\frac{\sqrt{2}}{4(1+\sqrt{2})}\sin\Big((1+\sqrt{2})s\Big) - \frac{\sqrt{2}}{4(1-\sqrt{2})}\sin\Big((1-\sqrt{2})s\Big),\\
&\frac{\sqrt{2}}{4(1+\sqrt{2})}\cos\Big((1+\sqrt{2})s\Big) + \frac{\sqrt{2}}{4(1-\sqrt{2})}\cos\Big((1-\sqrt{2})s\Big), \frac{\sqrt{2}}{2}\cos\Big(\frac{s}{\sqrt{2}}\Big)\Bigg).
\end{aligned}$$

In terms of definitions, we obtain special Smarandache curves, see Figures 2 - 5.

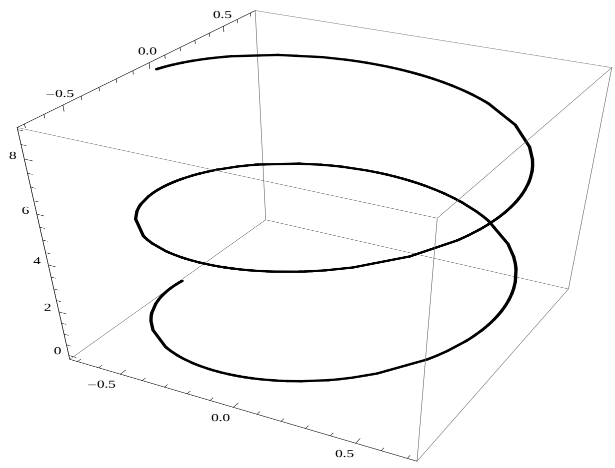

Figure 1: The curve $\alpha = \alpha(s)$.

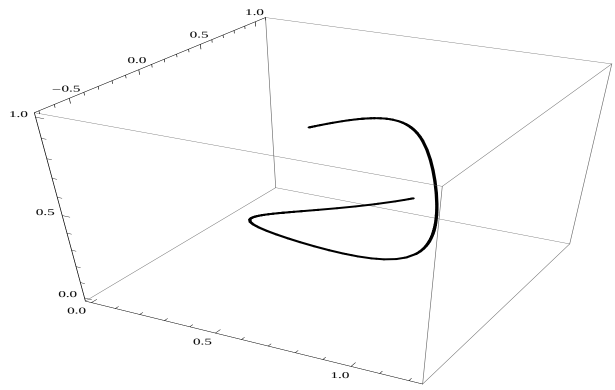

Figure 2: TN_1-Smarandache curve.

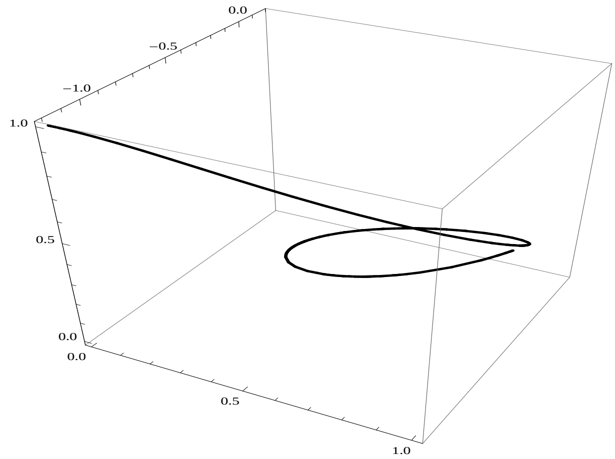

Figure 3: TN_2-Smarandache curve.

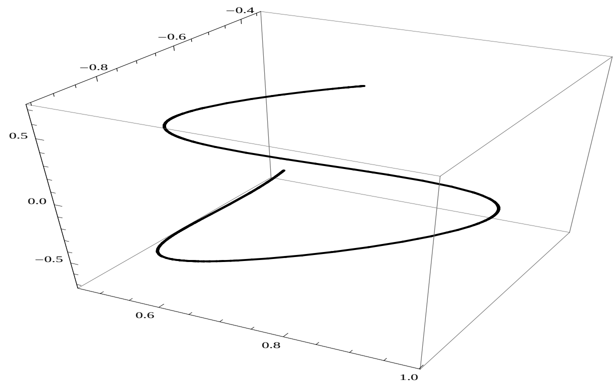

Figure 4: N_1N_2-Smarandache curve.

4 Conclusion

Consider a curve $\alpha = \alpha(s)$ parametrized by arc length in Euclidean 3-space E^3 that the curve$\alpha(s)$ is sufficiently smooth so that the Bishop frame adapted to it is defined. In this paper, we study the problem of constructing Frenet-Serret invariants $\{T_\xi, N_\xi, B_\xi, \kappa_\xi, \tau_\xi\}$ from a given some special curve ξ according to Bishop frame in Euclidean 3-space E^3 that posses this curve as Smarandache curve. We list an example to illustrate the discussed curves. Finally, we hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

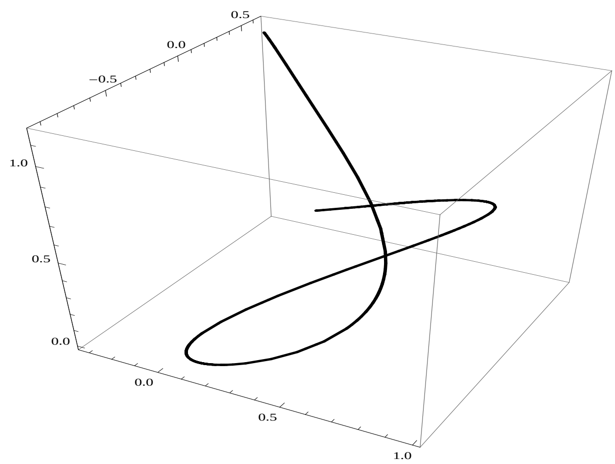

Figure 5: TN_1N_2-Smarandache curve.

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