Deterministic discounting of risky cash-flows

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May 3, 2017

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Abstract

A model to value risky cash-flows through discounting at deterministic rates is presented. The analysis mainly concerns with the valuation of project's levered cash-flows under default risky debt and general tax shield assumptions. Deterministic unlevered and levered rates as well as a deterministic Weighted Average Cost of Capital (dWACC) are defined and the relevant relationships among them are derived. The model allows to account for the risk of cash-flows in a proper way and produce exact results as in the stochastic discounting method. To illustrate the model, a numerical example about the evaluation of a two-period investment project with default risky debt is provided. The proposed approach is general and represents a first step toward a bridge between stochastic models for capital budgeting and more traditional capital budgeting techniques based on discounted cash-flow analysis.

Keywords: Risky cash-flow, risky debt, tax shield, present value, WACC

JEL Code: G31, G32, G33.

1 Introduction

Valuing risky cash-flows is a very important task in capital budgeting decisions to determine correctly the market value of a levered firm or a levered project. Due to its mathematical tractability, the risk-adjusted discount rate method (Modigliani and Miller, 1963) is a widely applied pricing technique, especially in the context of the Adjusted Present Value (APV) approach (Myers, 1977) or in the Weighted Average Cost of Capital (WACC) method of valuation (Miles and Ezzel, 1980). In the standard applications of the risk-adjusted discount rate method, deterministic discount rates are used under the strong assumption that the risk of the cash-flow does not vary across the different states of the world (Trigeorgis, 1996). Although in many cases such an assumption can be accepted, it cannot be assumed in general.

The evaluation of risky cash-flows is a difficult task: it can be accomplished determining first the level of risk of the cash-flow and then the cost of capital, i.e. the appropriate discount rate that accounts for the systematic risk of the cash-flow (Fama, 1977; Miles and Ezzel, 1985). In general, the discount rate is a random process that must properly track the variability of the cash-flow risk at any time and in each state of the world. To quote some examples, in default risky debt the default event modifies the risk profile of the debt and of the tax-shield in a significant way. Real options provide further examples in which the cash-flows generated by investment projects exhibit variable risky profiles (Trigeorgis, 1996). In many practical applications such well recognized stochasticity is treated within some approximating scheme in which deterministic discount rates are used (Koziol, 2014; Molnár and Nyborg, 2013).

This paper provides a general model to evaluate risky cash-flows generated by levered investment projects (or by levered firms) through deterministic discounting. The proposed approach is not an approximating scheme, it properly accounts for the risk of cash-flows and produces exact results as in the stochastic discounting method. From this point of view, this paper can be viewed as a first step toward a bridge between stochastic models for capital budgeting (Dixit and Pindyck, 1994) and more traditional capital budgeting techniques based on discounted cash-flow analysis.

The contribution of the paper is twofold. The first one regards stochastic discounting. Within the context of the risk-adjusted method, a general approach to value project's levered cash-flows under general tax shield assumptions is provided. Relationship between levered rates and unlevered rates, as well as, relationships between the WACC and unlevered rates and between the WACC and levered rates are derived under general tax shield assumptions. Such WACC formulas extend the classical Modigliani-Miller formula (1963) and the Harrison-Pringle formula (1985). Our result also generalizes the formula derived by Farber, Gillet and Szafarz (2006).

As a second contribution, a general model to value risky cash-flows generated by levered investment projects (or by levered firms) through deterministic discounting is proposed. Deterministic unlevered rates and levered rates (as well as deterministic cost of debt and of the tax shield) will be defined. Using deterministic discount rates the valuation process is then provided. It will be proved that such an approach properly accounts for the level of risk of the cash-flows and that it produces the same cash-flows present values as those obtained in the stochastic discounting method.

Three are the main results. The first one provides a formal relationship between deterministic discount rates and stochastic discount rates. It will be shown that deterministic rates can be expressed as weighted averages of stochastic discount rates. A further representation of deterministic rates is given in terms of cash-flows without involving stochastic rates. Such relations allow to account for the level of risk of the cash-flow in a proper way. As a second result, we derive a generalization of Proposition II of the Modigliani-Miller theorem (Modigliani and Miller, 1958): it provides a relationship between deterministic unlevered rates and deterministic levered rates, debt rates and tax shield rates. The third result regards the definition of the deterministic Weighted Average Cost of Capital (dWACC). The derivation of general relationships between dWACC and deterministic unlevered rates and between dWACC and deterministic levered rates is then provided. Such relations allow to prove the equivalence between the Adjusted Present Value and the Weighted Average Cost of Capital approach also in the deterministic discounting model. The equivalence is general and is valid, of course, also in the case of a growing leveraged firm (Dempsey, 2013; Massari *et al.*, 2007).

To illustrate the deterministic discounting model, a numerical example of the evaluation of a two-period investment project is provided. The relevant cash-flows are defined on a stochastic lattice to simulate levered project in presence of default risky debt. The analysis is performed by using both, the stochastic discounting approach and the deterministic discounting model.

The paper is organized as follows. Section 2 provides a brief review of the stochastic discounting approach for evaluating risky cash-flows. WACC formulas are presented under general tax shield assumptions (the derivation of the main formulas presented in this section is provided in Appendix A). Section 3 contains the deterministic discounting model (the derivation of the main formulas presented this section is given in Appendix A). A numerical example is proposed in Section 4. Some remarks conclude the paper.

2 Stochastic discounting of risky cash-flows

In this Section, after a brief review of some basics results which play a crucial role in determining present values of risky cash-flows, we provide a general formula for the WACC under any tax-shield assumption.

Let us denote by $\{F_t^U\}_{t=1}^m$ a collection of measurable random variables with respect a given filtration of a given probability space (Duffie, 1998), describing the *unlevered* cash-flow, or simply the *free* cash-flow, generated by a firm (or by a single investment project) [1]. The present value of the cash-flow at time t, V_t^U, i.e. the unlevered value at time t of the firm (or of the investment project) generating the free cash-flow $\{F_t^U\}_{t=1}^m$, can be obtained according to the following recursive relation:

$$V_t^U = \frac{\mathbb{E}_t[F_{t+1}^U + V_{t+1}^U]}{1 + r_t^U}, \tag{1}$$

where the discount rate, r_t^U, is a $t-$measurable random variable which accounts for the risk of the unlevered cash-flow in the interval $[t, t+1]$, and $\mathbb{E}_t$ denotes the expectation operator with respect to the information available at time t. The discount rate, r_t^U, is also named unlevered cost of capital or unlevered rate. Let us suppose that the firm (or a given investment project) is partially financed by (risky) debt. If we denote by $\{F_t^D\}_{t=1}^m$ the debt repayments, the outstanding debt at time t, V_t^D, is related to the value of the debt at time $t+1$, V_{t+1}^D, by the recursive relation:

$$V_t^D = \frac{\mathbb{E}_t[F_{t+1}^D + V_{t+1}^D]}{1 + r_t^D}, \tag{2}$$

where r_t^D is a $t-$measurable random variable accounting for the cost of debt. In presence of debt, the amount of equity invested in the project is remunerated by the *levered* cash-flow (or equity cash-flow), $\{F_t^S\}_{t=1}^m$, obtained by subtracting the debt service from the

[1] In general, the free cash-flow can be defined according to

$$F_t^U = Rev_t - C_t - T_t - I_t - \Delta WC_t,$$

obtained by subtracting from revenues Rev_t, costs C_t, taxes T_t, capital expenditures I_t, and time variation of working capital ΔWC_t, all referred to the operating year t.

free cash-flow and taking into account the tax shield for interest payments, $\{F_t^{TS}\}_{t=1}^m$,

$$F_t^S = F_t^U - F_t^D + F_t^{TS}. \tag{3}$$

The value of the equity at time t, V_t^S, can be defined by the following recursive relation:

$$V_t^S = \frac{\mathbb{E}_t[F_{t+1}^S + V_{t+1}^S]}{1 + r_t^S}, \tag{4}$$

where the $t-$measurable random variable r_t^S is the levered cost of capital (cost of equity). On the same basis, the present value of the tax shield, V_t^{TS}, is related to $\{F_t^{TS}\}_{t=1}^m$ by the following relation:

$$V_t^{TS} = \frac{\mathbb{E}_t[F_{t+1}^{TS} + V_{t+1}^{TS}]}{1 + r_t^{TS}}, \tag{5}$$

where the $t-$measurable random variable r_t^{TS} accounts for the risk of the tax shield cash-flow. Equations (1), (2), (4) and (5) show the same algebraic structure:

$$V_t^X = \frac{\mathbb{E}_t[F_{t+1}^X + V_{t+1}^X]}{1 + r_t^X}, \tag{6}$$

where X stands for U (unlevered), D (debt), S (equity) and TS (tax shield). The cost of capital, r_t^X, can be determined using, for example, multi-factor models as the Arbitrage Pricing Theory (APT) (Ross, 1976) or the Capital Asset Pricing Model (CAPM) (Sharpe, 1984). Alternatively, the cost of capital can be obtained within the context of risk-neutral valuation (Harrison and Kreps, 1979). We recall that, under a risk-neutral probability measure (martingale measure), the value at time t, V_t^X, of the risky cash-flow $\{F_t^X\}$ can be expressed as follows:

$$V_t^X = \frac{\mathbb{E}_t^*[F_{t+1}^X + V_{t+1}^X]}{1 + r^f}, \tag{7}$$

where E_t^* denotes the conditional expectation operator under the risk-neutral measure and r_f is the risk-free rate. Comparing Equation (7) with Equation (6), it is straightforward to obtain the following representation of the cost of capital:

$$1 + r_t^X = (1 + r^f)\frac{\mathbb{E}_t[F_{t+1}^X + V_{t+1}^X]}{\mathbb{E}_t^*[F_{t+1}^X + V_{t+1}^X]}. \tag{8}$$

Equity rates are not independent quantities, but they are related to unlevered rates, debt rates and tax shield rates by the linear combination:

$$r_t^S = r_t^U + (r_t^U - r_t^D)\frac{V_t^D}{V_t^S} - (r_t^U - r_t^{TS})\frac{V_t^{TS}}{V_t^S}. \tag{9}$$

Equation (9) provides a generalization of Proposition II of the Modigliani-Miller theorem which properly accounts for the increasing risk of equity in a leveraged firm. It can be obtained, after some algebraic manipulations, by substituting Equation (3) into Equation (4) and using Proposition I of the Modigliani-Miller theorem:

$$V_t \equiv V_t^S + V_t^D = V_t^U + V_t^{TS}. \tag{10}$$

Appendix A contains a detailed proof of the above result.

The Weighted Average Cost of Capital (WACC)

The present value of a leveraged firm (or a leveraged investment project) can be also obtained by discounting the free cash-flow at the so called Weighted Average Cost of Capital (WACC). The WACC rates are defined by the following recursive relation:

$$V_t = \frac{\mathbb{E}_t[F_{t+1}^U + V_{t+1}]}{1 + r_t^W}. \tag{11}$$

WACC rates are not independent quantities, and they are related to unlevered rates, debt rates and tax shield rates by the linear combination:

$$r_t^W = r_t^U - (r_t^U - r_t^{TS})\frac{V_t^{TS}}{V_t} - \frac{\mathbb{E}_t[F_{t+1}^{TS}]}{V_t}. \tag{12}$$

Finally, WACC rates can be also expressed in terms of equity rates and debt rates,

$$r_t^W = \frac{V_t^S}{V_t} r_t^S + \frac{V_t^D}{V_t} r_t^D - \frac{\mathbb{E}_t[F_{t+1}^{TS}]}{V_t}. \tag{13}$$

Appendix A contains a detailed proof of such a result.

Equation (12) and Equation (13) are general and are valid, for example, also in presence of default risky debt and under general tax shield assumptions. In this sense, they extend the classical Modigliani-Miller formula (1963) and the Harrison-Pringle formula (1985).

Since in general $\mathbb{E}_t[F_{t+1}^{TS}] \neq T_c r_t^D D_t$ (T_c is the corporate tax rate), such relations also extends the formula derived by Farber, Gillet and Szafarz (2006). A similar reation has been derived by Cooper and Nyborg (2008) under a constant debt ratio policy.

Valuing present values of risky cash-flows in a multi-period stochastic environment is a difficult task. Due to non-zero correlation between stochastic discount rates and cash-flows, the iterative application of Equations (6) and (11) by backward induction is not straightforward. It simplifies under deterministic discount rates. As mentioned in the Introduction, in many practical applications stochastic valuation of risky cash-flows is treated within some approximating scheme in which deterministic discount rates are used. In the next section we provide a general model to evaluate risky cash-flows generated by levered investment projects (or by levered firms) through deterministic discounting. The proposed approach is not an approximating scheme, it properly accounts for the risk of the cash-flows and produces exact results as in the stochastic discounting method.

3 Deterministic discounting of risky cash-flows

In this Section we propose a model to value risky cash-flows using deterministic discount rates. Three are the main results that will be presented and discussed.

The first one provides a formal relationship between deterministic discount rates and stochastic discount rates. It will be shown that deterministic rates can be expressed as weighted averages of stochastic discount rates. A further representation of deterministic rates is given in terms of cash-flows without involving stochastic rates. Such relations allow to account for the level of risk of the cash-flow in a proper way.

As a second result, we derive a generalization of the Proposition II of the Modigliani-Miller theorem. Taking into account the risk of the levered cash-flow, it provides a relationship between deterministic unlevered rates and deterministic levered rates, debt rates and tax shield rates.

The third result regards the definition of the deterministic Weighted Average Cost of Capital (dWACC). General relationships between dWACC and deterministic unlevered rates and between dWACC and deterministic levered rates will be derived.

Let us define the *deterministic* cost of capital, k_t^X, by the following recursive relation:

$$\mathbb{E}_0[V_t^X] = \frac{\mathbb{E}_0[F_{t+1}^X + V_{t+1}^X]}{1 + k_t^X}, \tag{14}$$

where X stands for U (unlevered), D (debt), TS (tax shield) and the symbol $\mathbb{E}_0$ denotes the conditional expectation operator under the information available at the present time (or time 0).

Representing deterministic discount rates

One of the main results of the paper is to provide a characterization of deterministic discount rates, k_t^X, in terms of a weighted averages of stochastic discount rates r_t^X. Namely, the following representation holds:

$$k_t^X = \mathbb{E}_0[r_t^X p_t^X], \tag{15}$$

where p_t^X is a $t-$measurable random variable that can be expresses as:

$$p_t^X = \frac{V_t}{\mathbb{E}_0[V_t]} = \frac{\mathbb{E}_t^*[W_{t+1,m}^X]}{\mathbb{E}_0\left[\mathbb{E}_t^*[W_{t+1,m}^X]\right]}, \tag{16}$$

in which

$$W_{t+1,m}^X = \sum_{k=t+1}^{m} \frac{F_k^X}{(1+r^f)^{k-t}} \tag{17}$$

is the risk-free discounted $X-$flow. Under a risk-neutral measure, the symbol $\mathbb{E}_t^*$ denotes the conditional expectation with respect to such probability measure. However, we will show that Equations (15)-(17) hold also in multi-factor models as APT or in the case of CAPM. In such cases the symbol $\mathbb{E}_t^*$ denotes the conditional expectation with respect to well defined measures. A detailed proof of the above results is presented in Appendix B. We notice that the random variable p_t^X satisfies the normalization condition $\mathbb{E}_0[p_t^X] = 1$ at any time t. Of course, if r_t^X is non stochastic, we get $r_t^X = k_t^X$.

A further representation of deterministic rates can be specified in terms of the risk-free discounted $X-$flow without involving stochastic rates. In fact, the following relation holds:

$$k_t^X = r^f + \mu_t^X, \tag{18}$$

where the risk premium μ_t^X is given by:

$$\mu_t^X = (1+r^f)\frac{\mathbb{E}_0\left[\mathbb{E}^*_{t+1}[W^X_{t+1,m}] - \mathbb{E}^*_t[W^X_{t+1,m}]\right]}{\mathbb{E}_0\left[\mathbb{E}^*_t[W^X_{t+1,m}]\right]}. \tag{19}$$

The proof is provided in Appendix B. The above equation provides an exact formula to account for the risk of cash-flows in a proper way. Although it looks quite complicated, in some practical applications it simplifies. For example, if the stochastic $X-$flow is constant in time, i.e. $F_t^X = F^X, \quad t = 1, 2, \cdots, m$, we get:

$$k_t^X = r^f + \frac{r^f}{\left[1-(1+r^f)^{-(m-t)}\right]}\left(\frac{\mathbb{E}_0[F^X]}{\mathbb{E}^*_0[F^X]} - 1\right). \tag{20}$$

Although the $X-$flow does not depend on time, the discount rate shows an explicit time dependence. The limit $m \to \infty$ is also interesting. In this case the discount rate does not depend on time and is given by:

$$k^X = r^f \frac{\mathbb{E}_0[F^X]}{\mathbb{E}^*_0[F^X]}. \tag{21}$$

In the case of constant growing flows at a rate g, we obtain in the limit $m \to \infty$:

$$k^X = g + (r^f - g)\frac{\mathbb{E}_0[F^X]}{\mathbb{E}^*_0[F^X]}. \tag{22}$$

For more complicated situations, Monte Carlo simulations can be used.

Under deterministic discounting, the iterative application of Equation (14) is straightforward and the expected value at time t of a risky cash-flow can be expressed by,

$$\mathbb{E}_0[V_t^X] = \frac{\mathbb{E}_0[F^X_{t+1}]}{1+k_t^X} + \frac{\mathbb{E}_0[F^X_{t+2}]}{(1+k_t^X)(1+k_{t+1}^X)} + \cdots + \frac{\mathbb{E}_0[F^X_m]}{(1+k_t^X)\cdots(1+k^X_{m-1})}. \tag{23}$$

Generalizing the Proposition II of the Modigliani-Miller Theorem

Deterministic equity rates are related to (deterministic) unlevered rates, debt rates and tax shields rates by the linear combination:

$$k_t^S = k_t^U + (k_t^U - k_t^D)\frac{\mathbb{E}_0[V_t^D]}{\mathbb{E}_0[V_t^S]} - (k_t^U - k_t^{TS})\frac{\mathbb{E}_0[V_t^{TS}]}{\mathbb{E}_0[V_t^S]}. \tag{24}$$

Appendix B contains a detailed proof of the above result.

Within this approach, Equation (24) provides a generalization of Proposition II of the Modigliani-Miller theorem. It accounts for the risk of the levered cash-flow. The entity of the risk depends on several variables. Among the others, the ratio between the expected value of the outstanding debt and the expected equity value is the most important one. Since $k_t^U \geq k_t^D$, deterministic equity rates show a non decreasing behavior as this ratio rises. Also, the ratio between the expected value of the tax shield and the expected equity value plays an important role in determining the risk of the levered cash-flow. Depending on the sign of the difference between the cost of debt and the cost of the tax shield, the third term in Equation (24) may increase or decrease the risk of the levered cash-flow. Section 4 provides a two-period numerical example in which the cost of debt is greater than the tax shield rate at time $t = 1$ but it is lower at time $t = 0$.

The deterministic Weighted Average Cost of Capital (dWACC)

A deterministic Weighted Average Cost of Capital (dWACC), k_t^W, can be also introduced. It is defined by the following recursive relation:

$$\mathbb{E}_0[V_t] = \frac{\mathbb{E}_0[F_{t+1}^U + V_{t+1}]}{1 + k_t^W}. \tag{25}$$

Iterative applications of the above equation provide the expected value of a leveraged firm (or a leverage project) at time t in term of the present value of the future unlevered cash-flow,

$$\mathbb{E}_0[V_t] = \frac{\mathbb{E}_0[F_{t+1}^U]}{1 + k_t^W} + \frac{\mathbb{E}_0[F_{t+2}^U]}{(1 + k_t^W)(1 + k_{t+1}^W)} + \cdots + \frac{\mathbb{E}_0[F_m^U]}{(1 + k_t^W) \cdots (1 + k_{m-1}^W)}. \tag{26}$$

Deterministic WACC rates are related to (deterministic) unlevered rates, debt rates and tax shield rates by the linear combination:

$$k_t^W = k_t^U - (k_t^U - k_t^{TS}) \frac{\mathbb{E}_0[V_t^{TS}]}{\mathbb{E}_0[V_t]} - \frac{\mathbb{E}_0[F_{t+1}^{TS}]}{\mathbb{E}_0[V_t]}. \tag{27}$$

Finally, dWACC rates can be also expressed in terms of equity rates and debt rates,

$$k_t^W = \frac{\mathbb{E}_0[V_t^S]}{\mathbb{E}_0[V_t]} k_t^S + \frac{\mathbb{E}_0[V_t^D]}{\mathbb{E}_0[V_t]} k_t^D - \frac{\mathbb{E}_0[F_{t+1}^{TS}]}{\mathbb{E}_0[V_t]}. \tag{28}$$

Appendix B contains a detailed proof of Equations (27) and (28) .

It is to be noted that, at the end of the debt term, unlevered rates, equity rates and dWACC rates must coincide. This can be easily viewed from Equations (24) and (27) implying at any time t after the debt term,

$$k_t^E = k_t^W = k_t^U. \tag{29}$$

To illustrate the deterministic discounting model, a numerical example on the valuation of a two-period investment project is provided in the next Section. The relevant cash-flows are defined on a stochastic lattice to simulate levered project in presence of default risky debt. The analysis is performed by using both, the stochastic discounting approach and the deterministic discounting model.

4 A numerical example

Let us suppose a firm produces the unlevered cash-flow described in the two-period lattice represented in the left panel of Figure (1). Actually, the firm has a risky debt with a market value $V_0^D = 100$ to be repaid by two periodic payments whose nominal values are respectively 60 and 75.5. The debt is, therefore, issued at a nominal rate of about 21.92%. The debt structure is depicted in the right panel of Figure (1).

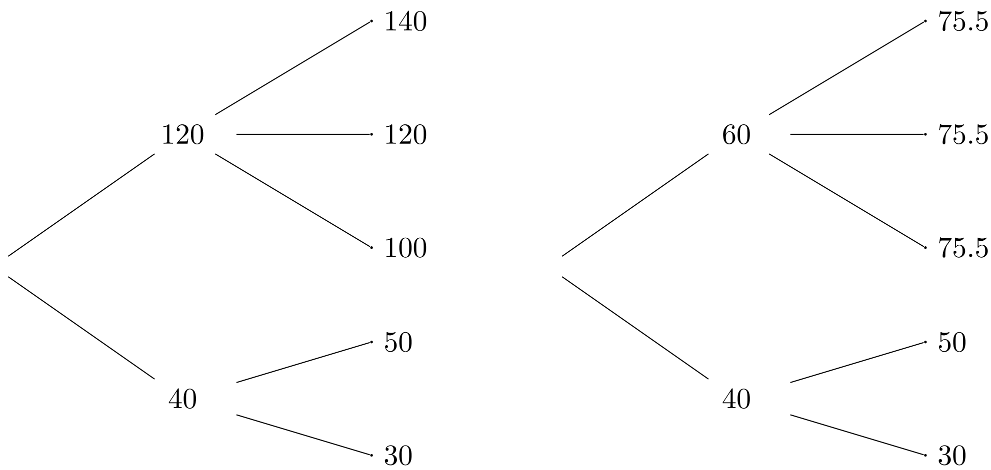

Figure 1: The unlevered cash-flow (left panel); the debt payment structure (right panel).

The tax shield structure is reported in the left panel of Figure (2): it is obtained multiplying the nominal interest rate for the outstanding debt and then for the corporate tax rate ($T_c = 30\%$) with the exception of the default event in which we state that the tax shield is set equal to zero (Koziol, 2014).

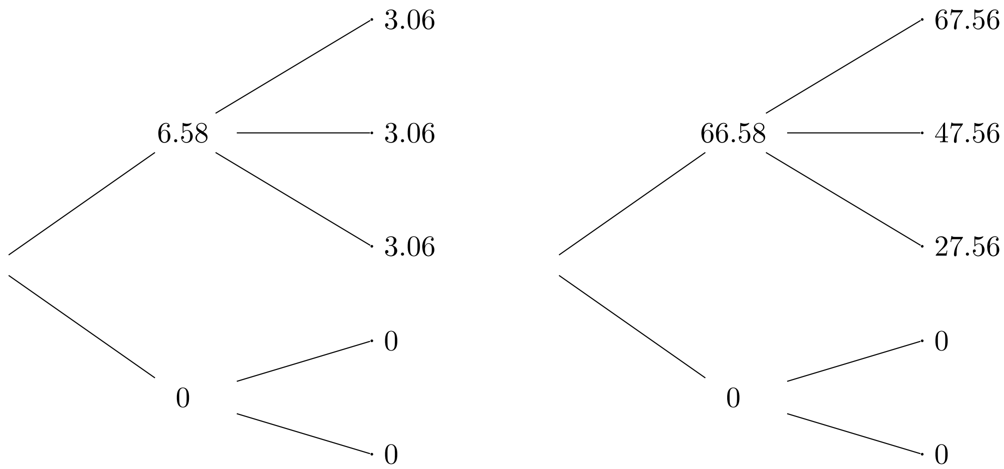

Figure 2: The tax shield structure (left panel); the levered cash-flow (right panel).

The probabilistic structure of the example is depicted in Figure (3) in which we reported the conditional risk-neutral probabilities (left panel) and the conditional natural probabilities (right panel). The risk-free rate is assumed to be $r^f = 5\%$.

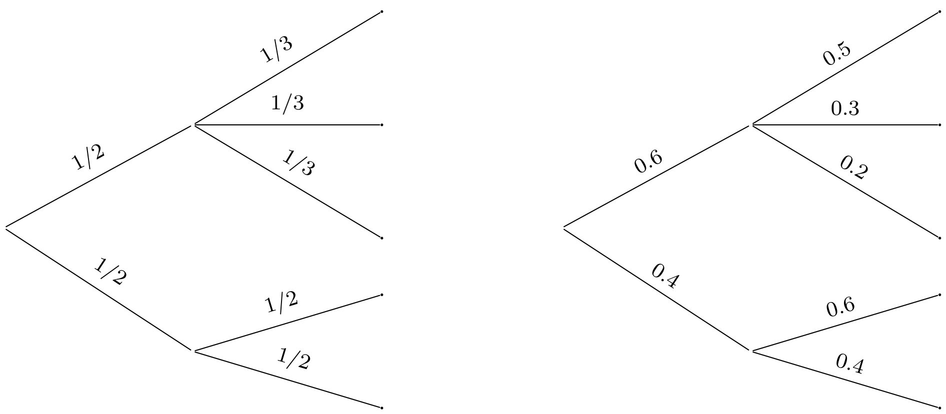

Figure 3: Conditional risk-neutral probabilities (left panel) and the conditional natural probabilities (right panel).

Although risk-neutral valuation has been used to determine present values of unlevered and levered cash-flows, as well as present values of the debt and of the tax shield, it is not a necessary approach. It guarantees that the valuation is arbitrage-free. Alternatively, we could determine the present value of cash-flows by defining the cost of capital by using, for example, CAPM or APT and discounting the relevant cash-flows at the appropriate stochastic rate as discussed in Section 1. The cost of capital is determined at each node of the lattice at time $t = 1$ (u and d) and at time $t = 0$ applying Equation (8) and the obtained results are described in the left panel of Figure (4). Levered rates are computed applying Equation (9). The weighted average cost of capital can be calculated using Equation (12) or Equation (13). Present values of cash-flows are calculated at time $t = 0$ and at time $t = 1$ in each node and they are reported in the right panel of Figure (4).

t	0	1 u	1 d
r_t^U	15.50%	10.25%	10.25%
r_t^{TS}	26.00%	5.00%	5.00%
r_t^D	10.38%	5.00%	10.25%
r_t^S	26.00%	18.25%	5.00%
r_t^W	13.24%	7.50%	10.25%

t	0	1 u	1 d
V_t^U	148.75	114.28	38.09
V_t^{TS}	4.52	2.91	0
V_t^D	100.00	71.90	38.09
V_t^S	53.27	45.30	0
V_t	153.27	117.20	38.09

Figure 4: The cost of capital (left panel); the present value structure (right panel).

Let us notice that the variability of the tax shield discount rate is very pronounced. The default event causes a "jump" in the discount rate at time $t = 0$ ($r_0^{TS} = 26\%$). Such a value greatly differs from the value of the unlevered rate at time $t = 0$ ($r_0^U = 15.5\%$ thus showing that the risk of the tax shield may be very different from the risk of the firm's assets (Harris and Pringle, 1985).

Now, we will show that the above valuation can be accomplished within the context of a deterministic discounting model. According to the obtained results, deterministic discount rates are calculated and present values of cash-flows are determined. In particular k_t^X for $t = 0, 1$ and $X = U, TS, D$ can be calculated using Equation (15); k_t^S using Equation (24); k_t^W using indifferently Equation (27) or Equation (28). Results are presented in the right panel of Figure (5). Averaged present values $\mathbb{E}_0[V_t^X]$ for $t = 1, 2$ and $X = U, TS, D, S$ are determined according to Equation (23); averaged present value $\mathbb{E}_0[V_t]$ for $t = 0, 1$ are determined according to Equation (26). Results are presented in the left panel of Figure (5).

t	0	1
k_t^U	15.50%	10.25%
k_t^{TS}	26.00%	5.00%
k_t^D	10.38%	6.37%
k_t^S	26.00%	18.25%
k_t^W	13.24%	7.99%

t	0	1
$E_0[V_t^U]$	148.75	83.81
$E_0[V_t^{TS}]$	4.52	1.75
$E_0[V_t^D]$	100.00	58.38
$E_0[V_t^S]$	53.27	27.18
$E_0[V_t]$	153.27	85.56

Figure 5: Deterministic discount rates (left panel); the averaged present value structure (right panel).

5 Concluding remarks

The developed analysis aimed at putting in some evidence the possibility to determine present values of risky cash-flows through deterministic discounting. One of the main result of the paper is to provide an analytic characterization of deterministic rates in terms of weighted averages of stochastic discount rates. A further representation of deterministic rates has been provided in terms of cash-flows without involving stochastic rates. Such relations allow to account for the level of risk of the cash-flow in a proper way. Although they looks quite complicated, it has been shown that in some practical applications such relationships simplify. For more complicated situations, Monte Carlo simulations can be used. Under deterministic discounting, multi-period calculations of present values simplifies and in this sense the model presented in this paper can represent a first step toward a bridge between stochastic models for capital budgeting and more traditional capital budgeting techniques based on discounted cash-flow analysis.

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A Appendix

This Appendix contains detailed profs of the main results stated in Section 2.

Proof of Equation (9)

Let us rewrite Equations (4), (2), (1) and (5) in the following way:

$$(1 + r_t^S)V_t^S = \mathbb{E}_t[F_{t+1}^S + V_{t+1}^S],$$

$$(1 + r_t^D)V_t^D = \mathbb{E}_t[F_{t+1}^D + V_{t+1}^D],$$

$$(1 + r_t^U)V_t^U = \mathbb{E}_t[F_{t+1}^U + V_{t+1}^U],$$

$$(1 + r_t^{TS})V_t^{TS} = \mathbb{E}_t[F_{t+1}^{TS} + V_{t+1}^{TS}].$$

Summing the first and the second equation and then subtracting the third and the fourth one, after some algebraic manipulations (in which the Modigliani-Miller theorem and Equation (3) have been used) we easily get:

$$r_t^S V_t^S + r_t^D V_t^D - r_t^U V_t^U - r_t^{TS} V_t^{TS} = 0.$$

Since $V_t^U = V_t - V_t^{TS}$, the above equation can be rewritten as follows:

$$r_t^E = r_t^U + (r_t^U - r_t^D)\frac{V_t^D}{V_t^S} - (r_t^U - r_t^{TS})\frac{V_t^{TS}}{V_t^S}. \tag{A.1}$$

Proof of Equation (12)

Let us rewrite Equations (11), (1) and (5) in the following way:

$$(1 + r_t^W)V_t = \mathbb{E}_t[F_{t+1}^U + V_{t+1}],$$

$$(1 + r_t^U)V_t^U = \mathbb{E}_t[F_{t+1}^U + V_{t+1}^U],$$

$$(1 + r_t^{TS})V_t^{TS} = \mathbb{E}_t[F_{t+1}^{TS} + V_{t+1}^{TS}].$$

Subtracting from the first equation the second and the third one, after some algebraic manipulations (in which Proposition I of the Modigliani-Miller theorem has been used) we easily get:

$$r_t^W V_t - r_t^U V_t^U - r_t^{TS} V_t^{TS} + \mathbb{E}_t[F_{t+1}^{TS}] = 0.$$

Since $V_t^U = V_t - V_t^{TS}$, the above equation can be rewritten as follows:

$$r_t^W = r_t^U - (r_t^U - r_t^{TS})\frac{V_t^{TS}}{V_t} - \frac{\mathbb{E}_t[F_{t+1}^{TS}]}{V_t}. \tag{A.2}$$

Proof of Equation (13)

Let us rewrite Equations (11), (4) and (2) in the following way:

$$(1 + r_t^W)V_t = \mathbb{E}_t[F_{t+1}^U + V_{t+1}],$$

$$(1 + r_t^S)V_t^S = \mathbb{E}_t[F_{t+1}^S + V_{t+1}^S],$$

$$(1 + r_t^D)V_t^D = \mathbb{E}_t[F_{t+1}^D + V_{t+1}^D].$$

Subtracting from the first equation the second and the third one, after some algebraic manipulations (in which the Proposition I of the Modigliani-Miller theorem and equation (3) have been used) we easily get:

$$r_t^W V_t - r_t^S V_t^S - r_t^D V_t^D + \mathbb{E}_t[F_{t+1}^{TS}] = 0.$$

The above equation can be rewritten, therefore, as follows:

$$r_t^W = \frac{V_t^S}{V_t} r_t^S + \frac{V_t^D}{V_t} r_t^D - \frac{\mathbb{E}_t[F_{t+1}^{TS}]}{V_t}. \tag{A.3}$$

B Appendix

This Appendix contains detailed profs of the main results stated in Section 3.

Proof of Equation (15)

Let us rewrite Equations (6) for the $X-$flow in the following way:

$$(1 + r_t^X)V_t^X = \mathbb{E}_t[F_{t+1}^X + V_{t+1}^X],$$

Taking expectation under the natural probability with respect to the information available at time 0, we get:

$$\mathbb{E}_0[(1 + r_t^X)V_t^X] = \mathbb{E}_0[F_{t+1}^X + V_{t+1}^X] = (1 + k_t^X)\mathbb{E}_0[V_t^X], \tag{B.1}$$

in which Equation (14) has been used. The above equation can be cast in the following form:

$$k_t^X = \mathbb{E}_0[r_t^X p_t^X], \tag{B.2}$$

where p_t^X is a $t-$measurable random variable given by:

$$p_t^X = \frac{V_t}{\mathbb{E}_0[V_t]}. \tag{B.3}$$

Under a risk-neutral probability measure the value at time t, V_t^X, of the $X-$flow can be expressed as:

$$V_t^X = \frac{\mathbb{E}_t^*[F_{t+1}^X + V_{t+1}^X]}{1 + r^f}, \tag{B.4}$$

where $\mathbb{E}_t^*$ denotes the conditional expectation operator under the risk-neutral probability distribution. The previous recursive equation can be expressed in the following equivalent form:

$$V_t^X = \mathbb{E}_t^*[W_{t+1,m}^X], \tag{B.5}$$

where

$$W_{t+1,m}^X = \sum_{k=t+1}^{m} \frac{F_k^X}{(1 + r^f)^{k-t}} \tag{B.6}$$

is the risk-free discounted $X-$flow. In such a case p_t^X can be expressed as follows:

$$p_t^X = \frac{\mathbb{E}_t^*[Z_{t+1,m}^X]}{\mathbb{E}_0\left[\mathbb{E}_t^*[Z_{t+1,m}^X]\right]}. \tag{B.7}$$

Proof of Equation (18)

Let use rewrite Equation (14) in the following way:

$$1 + k_t^X = \frac{\mathbb{E}_0[F_{t+1}^X + V_{t+1}^X]}{\mathbb{E}_0[V_t^X]} = \frac{\mathbb{E}_0\left[F_{t+1}^X + \mathbb{E}_{t+1}^*[W_{t+2,m}^X]\right]}{\mathbb{E}_0\left[\mathbb{E}_t^*[W_{t+1,m}^X]\right]},$$

in which Equation (B.5) has been used. After some algebraic manipulations, the above expression can be cast as:

$$k_t^X = (1 + r^f)\frac{\mathbb{E}_0\left[\mathbb{E}_{t+1}^*[W_{t+1,m}^X]\right]}{\mathbb{E}_0\left[\mathbb{E}_t^*[W_{t+1,m}^X]\right]} - 1,$$

from which

$$k_t^X = r^f + \mu_t^X, \tag{B.8}$$

where μ_t^X is given by:

$$\mu_t^X = (1 + r^f)\frac{\mathbb{E}_0\left[\mathbb{E}_{t+1}^*[W_{t+1,m}^X] - \mathbb{E}_t^*[W_{t+1,m}^X]\right]}{\mathbb{E}_0\left[\mathbb{E}_t^*[W_{t+1,m}^X]\right]}. \tag{B.9}$$

Deterministic discounting in a multi-period APT and CAPM

We will show that representation (B.2) works also in a multi-period APT as well as in a multi-period CAPM. To prove this let us pose $r_t^X = E_t[\tilde{r}_{t+1}^X]$ where

$$\tilde{r}_{t+1}^X = \frac{F_{t+1}^X + V_{t+1}^X}{V_t^X} - 1. \tag{B.10}$$

In a multi-period APT with n risky factors, the cost of capital, r_t^X, can be expressed as

$$r_t^X = r^f + \sum_{j=1}^{n} \beta_{jt}^X \left(\mathbb{E}_t[\tilde{r}_{t+1}^j] - r^f\right),$$

where

$$\beta_{jt}^X = \frac{cov_t(\tilde{r}_{t+1}^X, \tilde{r}_{t+1}^j)}{\sigma_t^2(\tilde{r}_{t+1}^j)},$$

and $\tilde{r}_{t+1}^j$ is the return of the $j-$th factor portfolio on the time interval $[t, t+1]$. By using Equation (B.10), β_{jt}^X can be cast in the following useful form:

$$\beta_{jt}^X = \frac{1}{V_t^X} \frac{cov_t(F_{t+1}^X + V_{t+1}^X, \tilde{r}_{t+1}^j)}{\sigma_t^2(\tilde{r}_{t+1}^j)}. \tag{B.11}$$

Since,

$$V_t^X = \frac{\mathbb{E}_t[F_{t+1}^X + V_{t+1}^X]}{1 + r_t^X} = \frac{\mathbb{E}_t[F_{t+1}^X + V_{t+1}^X]}{r^f + \sum_{j=1}^{n} \beta_{jt}^X \left(\mathbb{E}_t[\tilde{r}_{t+1}^j] - r^f\right)},$$

after some algebraic manipulations in which Equation (B.11) has been used, we get:

$$V_t^X = \frac{\mathbb{E}_t[F_{t+1}^X + V_{t+1}^X] - \sum_{j=1}^{n} \lambda_{jt} cov_t(F_{t+1}^X + V_{t+1}^X, \tilde{r}_{t+1}^j)}{1 + r^f}, \tag{B.12}$$

where

$$\lambda_{jt} = \frac{\mathbb{E}_t[\tilde{r}_{t+1}^j] - r^f}{\sigma_t^2(\tilde{r}_{t+1}^j)}. \tag{B.13}$$

Equation (B.12) can be cast therefore in the following form:

$$V_t^X = \frac{\mathbb{E}_t\left[Z_{t+1}\left(F_{t+1}^X + V_{t+1}^X\right)\right]}{1 + r^f} = \frac{\mathbb{E}_t^*\left[F_{t+1}^X + V_{t+1}^X\right]}{1 + r^f},$$

where

$$Z_{t+1} = 1 - \sum_{j=1}^{n} \lambda_{jt}\left(\tilde{r}_{t+1}^j - \mathbb{E}_t[\tilde{r}_{t+1}^j]\right) \tag{B.14}$$

is the "Radon-Nikodym" derivative of the new measure (Neftci, 2000). Following the same line of reasoning of the previous part, Equation (15) and Equation (18) can be recovered. The CAPM representation can be obtained as a particular case in which the only risk factor is the market portfolio.

Proof of Equation (24)

Let us explicitly rewrite Equation (14) in the four different specifications as follows:

$$(1+k_t^S)\mathbb{E}_0[V_t^S] = \mathbb{E}_0[F_{t+1}^S + V_{t+1}^S],$$

$$(1+k_t^D)\mathbb{E}_0[V_t^D] = \mathbb{E}_0[F_{t+1}^D + D_{t+1}],$$

$$(1+k_t^U)\mathbb{E}_0[V_t^U] = \mathbb{E}_0[F_{t+1}^U + V_{t+1}^U],$$

$$(1+k_t^{TS})\mathbb{E}_0[V_t^{TS}] = \mathbb{E}_0[F_{t+1}^{TS} + V_{t+1}^{TS}].$$

Summing the first and the second equation and then subtracting the third and the fourth one, after some algebraic manipulations (in which the Proposition I of the Modigliani-Miller theorem and Equation (3) have been used) we easily get:

$$k_t^S\mathbb{E}_0[V_t^S] + k_t^D\mathbb{E}_0[V_t^D] - k_t^U\mathbb{E}_0[V_t^U] - k_t^{TS}\mathbb{E}_0[V_t^{TS}] = 0.$$

Since $V_t^U = V_t - V_t^{TS}$, the above equation can be rewritten in the following way:

$$k_t^E = k_t^U + (k_t^U - k_t^D)\frac{\mathbb{E}_0[V_t^D]}{\mathbb{E}_0[V_t^S]} - (k_t^U - k_t^{TS})\frac{\mathbb{E}_0[V_t^{TS}]}{\mathbb{E}_0[V_t^S]}. \tag{B.15}$$

Proof of Equation (27)

Let us consider the following set of recursive equations:

$$(1+k_t^W)\mathbb{E}_0[V_t] = \mathbb{E}_0[F_{t+1}^U + V_{t+1}],$$

$$(1+k_t^U)\mathbb{E}_0[V_t^U] = \mathbb{E}_0[F_{t+1}^U + V_{t+1}^U],$$

$$(1+k_t^{TS})\mathbb{E}_0[V_t^{TS}] = \mathbb{E}_0[F_{t+1}^{TS} + V_{t+1}^{TS}].$$

Subtracting from the first equation the second and the third one, after some algebraic manipulations (in which Proposition I of the Modigliani-Miller theorem has been used) we easily get:

$$k_t^W\mathbb{E}_0[V_t] - k_t^U\mathbb{E}_0[V_t^U] - k_t^{TS}\mathbb{E}_0[V_t^{TS}] + \mathbb{E}_0[F_{t+1}^{TS}] = 0.$$

Since $V_t^U = V_t - V_t^{TS}$, the above equation can be rewritten as follows:

$$k_t^W = k_t^U - (k_t^U - k_t^{TS}) \frac{\mathbb{E}_0[V_t^{TS}]}{\mathbb{E}_0[V_t]} - \frac{\mathbb{E}_0[F_{t+1}^{TS}]}{\mathbb{E}_0[V_t]}. \tag{B.16}$$

Proof of Equation (28)

Let us consider the following set of recursive equations:

$$(1 + k_t^W)\mathbb{E}_0[V_t] = \mathbb{E}_0[F_{t+1}^U + V_{t+1}],$$

$$(1 + k_t^S)\mathbb{E}_0[V_t^S] = \mathbb{E}_0[F_{t+1}^S + V_{t+1}^S],$$

$$(1 + k_t^D)\mathbb{E}_0[V_t^D] = \mathbb{E}_0[F_{t+1}^D + V_{t+1}^D].$$

Subtracting from the first equation the second and the third one, after some algebraic manipulations (in which Proposition I of the Modigliani-Miller theorem and Equation (3) have been used) we easily get:

$$k_t^W \mathbb{E}_0[V_t] - k_t^S \mathbb{E}_0[V_t^S] - k_t^D \mathbb{E}_0[V_t^D] + \mathbb{E}_0[F_{t+1}^{TS}] = 0.$$

The above equation can be rewritten, therefore, as follows:

$$k_t^W = \frac{\mathbb{E}_0[V_t^S]}{\mathbb{E}_0[V_t]} k_t^S + \frac{\mathbb{E}_0[V_t^D]}{\mathbb{E}_0[V_t]} k_t^D - \frac{\mathbb{E}_0[F_{t+1}^{TS}]}{\mathbb{E}_0[V_t]}. \tag{B.17}$$