FUNCTIONALLY CONVEX SETS AND FUNCTIONALLY CLOSED SETS IN REAL BANACH SPACES

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Abstract. We use of two notions functionally convex (briefly, F–convex) and functionally closed (briefly, F–closed) in functional analysis and obtain more results. We show that if $\{A_\alpha\}_{\alpha\in I}$ is a family F–convex subsets with non empty intersection of a Banach space X, then $\bigcup_{\alpha\in I} A_\alpha$ is F–convex. Moreover, we introduce new definition of notion F–convexiy.

1. Introduction

Convexity is an important tool in many fields of Mathematics, having applications in different areas. Various generalizations of the convexity were given in the literature, including nearly convexity, closely convexity, convexlike, quasiconvex, approximately convex and so on. Furthermore, generalizing of convexity is a difficult task. Several generalizations have appeared to be mere formal extensions of convexity, most of which deal with invexity.

From now on, we suppose that all normed spaces and Banach spaces are real.

Definition 1.1. [6] In a normed space X, we say that $K(\subseteq X)$ is m- functionally convex (briefly, m- F–convex) (for $m \in \mathbb{N}$) if for every bounded linear transformation $T \in B(X, \mathbb{R}^m)$, the subset $T(K)$ of $\mathbb{R}^m$ is convex. A 1- F–convex set is called F–convex. A subset K of X is called permanently F–convex if K is m- F–convex, for all $m \in \mathbb{N}$.

It is easy to see that every convex set is permanently F–convex.

Proposition 1.2. *Every $m+1-$ F–convex set is $m-$ F–convex.*

Proof. For every $T \in B(X, \mathbb{R}^m)$, we define $S : X \longrightarrow \mathbb{R}^{m+1}$ by $S(x) = (Tx, 0)$. Note that, $S \in B(X, \mathbb{R}^{m+1})$ and for every $A \subseteq X$, the set $T(A)$ is convex if and only if $S(A)$ is convex. □

Proposition 1.3. [6] *If T is a bounded linear mapping from a normed space X into a normed space Y, and K is F–convex in X, then $T(K)$ is F–convex in Y.*

Corollary 1.4. [6] *Let A,B be two F–convex subsets of a normed space X and λ be a real number, then*

$$A + B = \{a + b : a \in A, b \in B\}, \quad \lambda A = \{\lambda.a : a \in A\}$$

are F–convex.

2010 *Mathematics Subject Classification.* Primary 26B15, 46B10, 46B22.

Key words and phrases. convex set; F–convex set.

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Proposition 1.5. [6] *Let A and B be F–convex subsets of a linear space X, which have nonempty intersection. Then $A \cup B$ is F–convex.*

Definition 1.6. [6] Let X be a normed space and let $A \subseteq X$. A is functionally closed (briefly, F–closed), if $f(A)$ is closed for all $f \in X^*$.

Note that every compact set is F–closed. Also, every closed subset of real numbers $\mathbb{R}$ is F–closed. In $X = \mathbb{R}^2$, the set $A = \{(x, y) : x, y \geq 0\}$ is (non-compact) F–closed whereas, the set $A = \mathbb{Z} \times \mathbb{Z}$ is closed but it is not F–closed (by taking $f(x, y) = x + \sqrt{2}y$, the set $f(A)$ is not closed in $\mathbb{R}$). By taking $A = \{(x, y) : 1 \leq x^2 + y^2 \leq 4\}$ a nonconvex F–closed and F– convex set is obtained. Also, the set $B = \{(x, y) : x \in [0, \frac{\pi}{2}), y \geq \tan(x)\}$ is a closed convex set which is not F–closed. On the other hand, $A = \{(x, y) : 1 < x^2 + y^2 \leq 4\}$ is a non-compact and F–closed set. The two last examples show that weakly closed(weakly compact) and F–closed sets are different.

Remark 1.7. Note that we can not reduce definition of F–convexity to a basis of X^*, in the sence that a set in X is F–convex whenever its image under elements of a basis is convex. For instance, by taking the Euclidean space $\mathbb{R}^2$ and the set

$$\begin{aligned} A = &\{(0, \alpha) : \alpha \in \mathbb{R} - \mathbb{Q} \cap [-\sqrt{2}, 1]\} \cup \{(\beta, 1) : \beta \in \mathbb{R} - \mathbb{Q} \cap [0, \sqrt{2}]\} \\ &\cup \{(r, -\sqrt{2}) : r \in \mathbb{Q} \cap [0, \sqrt{2}]\} \cup \{(\sqrt{2}, s) : s \in \mathbb{Q} \cap \{[-\sqrt{2}, 1]\} \\ &\cup \{(0, 1), (0, \sqrt{2}), (\sqrt{2}, -\sqrt{2}), (\sqrt{2}, 1)\} \end{aligned}$$

$p_x(x, y) = x$ and $p_y(x, y) = y$, projections on axis, is a base for $X = \mathbb{R}^2$ and $P_x(A) = [0, 1]$ also, $p_y(A) = [-\sqrt{2}, 1]$ but $f(x, y) = x + y$ is an element of X^* and $f(A)$ is not convex.

In [6], we prove the following theorem, which help us to find a big class of F–convex sets.

Theorem 1.8. [6] *Every arcwise connected subset of a normed space X is F–convex.*

Remark 1.9. The converse of the above theorem is not valid. Hence, by taking $S = \{(x, \sin(\frac{1}{x}) : 0 < x \leq 1\}$, the set $\overline{S}$ which is called the sine's curve of topologist is connected and so for any linear functional $f \in (\mathbb{R} \times \mathbb{R})^*$, the set $f(\overline{S})$ is an interval. Thus, $\overline{S}$ is an F–convex set which is not arcwise connected.

2. Main Results

In this section, we show, how construct new subset F–convex one of given ones.

Theorem 2.1. *Let A, B be subsets of Banach space X. If A is F–convex and $A \subset B \subset \overline{A}$ then, B is F–convex.*

Proof. For every $f \in X^*$, we have $f(A) \subseteq f(B) \subseteq f(\overline{A}) \subseteq \overline{f(A)}$. Hence, by assumption, $f(\overline{A})$ is an interval. This completes the proof. $\square$

Remark 2.2. In contrary the case of convex sets, interior of an F–convex set, necessarily is not F–convex. For instance, take $X = \mathbb{R} \times \mathbb{R}$ and let $B = \{(x, y) : x^2 + y^2 \leq 1\}$. Then if A is all elements surrounded by B and $B + \frac{1}{2}$ is F–convex, but the interior of A is not F–convex. Since, by taking f as projection on x-axis we have $f(A^\circ) = (-\frac{1}{2}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2})$, which is not convex.

In the following, for a subset A of a Banach space X, a necessary and sufficient condition for F–convexity is proved.

Theorem 2.3. *Let X be a Banach space, $A \subseteq X$ is F–convex if and only if*

$$co(A) \subseteq \bigcap_{f \in X^*} A + Ker(f).$$

Remark 2.4. Note that in special case $X = \mathbb{R}$, since every nonzero functional is one to one so we have $\bigcap_{f \in X^*} A + Ker(f) = A$. Thus $A \subseteq \mathbb{R}$ is F–convex iff $co(A) \subseteq A$. Also, we have $A \subseteq co(A)$. Then we obtain $A \subseteq \mathbb{R}$ is F–convex iff A is convex.

Let X be a vector space. A hyperplane in X (through $x_0 \in X$) is a set of the form $H = x_0 + Ker(f) \subseteq X$, where f is a non-zero linear functional on X. Equivalently, $H = f^{-1}(\gamma)$, where $\gamma = f(x_0)$. So, we have

$$\bigcap_{f \in X^*} A + Ker(f) = \bigcap_{f \in X^*} \bigcup_{a \in A} a + Ker(f) = \bigcap_{f \in X^*} f^{-1}(f(A)).$$

Hence, $A \subseteq X$ is F–convex if and only if

$$co(A) \subseteq \bigcap_{f \in X^*} f^{-1}(f(A)).$$

Lemma 2.5. [6] *If A is a subset of a Banach space X, then*

$$\bigcap_{f \in X^*} f^{-1}(f(A)) \subseteq \overline{co}(A)$$

Corollary 2.6. [6] *Let A be a F–closed subset of a Banach space X. Then A is F–convex if and only if*

$$\overline{co}(A) = \bigcap_{f \in X^*} f^{-1}(f(A)).$$

Theorem 2.7. *Let $\{A_\alpha\}_{\alpha \in I}$ be collection of F–convex subsets in Banach space X. If $\bigcap_{\alpha \in I} A_\alpha \neq \phi$ then, $\bigcup_{\alpha \in I} A_\alpha$ is F–convex.*

Proof. For each $f \in X^*$ and $\alpha \in I$, we know, $f(A_\alpha)$ is an interval and $\bigcap_{\alpha \in I} f(A_\alpha) \neq \phi$. Thus, $f(\bigcup_{\alpha \in I} A_\alpha) = \bigcup_{\alpha \in I} f(A_\alpha)$ is convex. $\square$

we know that, if $\{A_\alpha\}_{\alpha \in I}$ be a collection of connected subsets in X, A is connected and $A \bigcap A_\alpha \neq \phi$ for all $\alpha \in I$, then $A \bigcup \left(\bigcup_{\alpha \in I} A_\alpha\right)$ is connected. Now, we have the following theorem;

Theorem 2.8. *Let $\{A_\alpha\}_{\alpha \in I}$ be a collection of F–convex subsets in Banach space X. If A is F–convex and $A \bigcap A_\alpha \neq \phi$ for evrey $\alpha \in I$, then $A \bigcup \left(\bigcup_{\alpha \in I} A_\alpha\right)$ is F–convex.*

Proof. For evrey $f \in X^*$ and all $\alpha \in I$, $f(A_\alpha)$ and $f(A)$ are intervals such that $f(A) \cap f(A_\alpha) \neq \phi$. Therefore, $f(A \bigcup \left(\bigcup_{\alpha \in I} A_\alpha\right)) = \bigcup_{\alpha \in I} f(A_\alpha) \bigcup f(A)$ is interval for evrey $f \in X^*$. So, $A \bigcup \left(\bigcup_{\alpha \in I} A_\alpha\right)$ is F–convex. $\square$

We know that, if $\{A_n\}_{n\in\mathbb{N}}$ be a collection of connected subsets in X such that $A_n \cap A_{n+1} \neq \phi$ for all $n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} A_n$ is connected. Now, we have the following theorem;

Theorem 2.9. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a collection of F–convex subsets in Banach space X. If $A_n \cap A_{n+1} \neq \phi$ for evrey $n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} A_n$ is F–convex.*

Proof. For evrey $f \in X^*$ and all $n \in \mathbb{N}$, $f(A_n)$ is interval and $f(A_n) \cap f(A_{n+1}) \neq \phi$. Therefore, $f(\bigcup_{n\in\mathbb{N}} A_n) = \bigcup_{n\in\mathbb{N}} f(A_n)$ is interval for evrey $f \in X^*$. So, $\bigcup_{n\in\mathbb{N}} A_n$ is F–convex. □

Let A be a subset of linear space X. We define an equivalence relation on A as: $x \sim y$ if and only if both lie in a F–convex subset of A. the relation $\sim$ actually is an equivalence relation. For transitivity, note that if $x \sim y$ and $y \sim z$ then there are weakly convex subsets A and B such that $x, y \in A$ and $y, z \in B$. Now, proposition 1.5 asserts that $A \cup B$ is F–convex subset of X and so $x \sim z$.

Definition 2.10. Let A be a subset of linear space X. Let $\dfrac{A}{\sim} = \{A_\alpha\}_{\alpha\in I}$ be the set of all equivalence classes. For each $\alpha \in I, A_\alpha$ is called F–convex component of A.

Theorem 2.11. *Let A be a subset of linear space X. The F–convex components of A are disjoint F–convex subsets of A whose their union is A, such that any non empty F–convex subset of A contains only one of them.*

Proof. Being equivalence classes, the F–convex component of A are disjoint and their union is A. Each F–convex subset of A contains only one of them. For if, A intersects the components A_1, A_2 of A say, in points x_1, x_2 respectively, then $x_1 \sim x_2$. this means $A_1 = A_2$. To show the F–convex component B is F–convex, choose a point x of B. For each $y \in B$, we know that $x_1 \sim x_2$, so there is a F–convex subset A_y containing x, y. By the result just proved $A_y \subset A$. thus, $B = \bigcup_{y\in A} A_y$. Since subsets A_y are F–convex and the point x is in their intersection, by 2.7 B is F–convex. □

Remark 2.12. Let A be a subset of linear space X. A is F–convex if and only if it has one F–convex component.

Theorem 2.13. *Let $(X_i, \|.\|_i)$ be norm linear spaces, then $A_i \subset X_i$ are F–convex if and only if, $\prod_{i=1}^n A_i$ is F–convex in $\prod_{i=1}^n X_i$ equepted by the norm*

$$\|(x_1, x_2, \cdots, x_n)\| = \Big\{ \sum_{i=1}^{n} \|x_i\|_i^2 \Big\}^{\frac{1}{2}}$$

.

Proof. We Know that

$$(\prod_{i=1}^{n} X_i)^* = \oplus_{i=1}^n X_i^*.$$

So, for every $g \in (\prod_{i=1}^n X_i)^*$ there are uniqe $f_i \in X_i^*, i = 1, 2, \cdots, n$ such that, $g = \sum_{i=1}^n f_i$. Now we have

$$g(\prod_{i=1}^{n} A_i) = \sum_{i=1}^{n} f_i(A_i).$$

Since, every A_i is F–convex so, $f_i(A_i)$ and their sum is an interval. Conversly, for every $f_i \in X_i^*$, taking $g = 0 + 0 + \cdots + f_i + \cdots + 0$, we have $f(A_i) = g(\prod_{i=1}^n A_i)$ so, A_i is F–convex. $\square$

Theorem 2.14. *Let Y be a subspace of the norm linear space X. If $A \subset Y$ is F–convex then, A is F–convex in X.*

Proof. Let Y be a subspace of X. There exists subspace $Y^\perp$ of X such that $X = Y \oplus Y^\perp$. Thus, for evrey $f \in X^*$ we have, $f|_Y \in Y^*$. Now, if A is F–convex in Y, Therefore, $f(A) = f|_Y(A) + f(Y^\perp)$. By assumption, $f|_Y(A)$ is F–convex also, since $Y^\perp$ is a subspace, so $Y^\perp$ is F–convex in X. Thus, By using 1.4 $f(A)$ is F–convex in X. $\square$

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