**THE NATURE OF THE LOGISTIC FUNCTION AS A NONLINEAR DISCRETE DYNAMICAL SYSTEM**

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**Abstract**

In an attempt to discover the effect of recurrence on the topological dynamics, we came across a nonlinear function/equation whose regime of periodicity amounts to recurrence, thus the logistic function. This research seeks to study the logistic function as to how it really behaves. In the field of dynamics most especially this function in terms of discrete form/equation has been studied. Logistic equation as a model of population growth was first originated by the famous Pierre-FranciosVerhulst. It is a continuous form written as) which depend on time. It can be restructure from the continuous form into a discrete form known as logistic map, written as; . Where, is the state at the discrete time is the control parameter which works within a given intervals. It is a very simple example of nonlinear systems in dynamics. Its true nature/behaviour in changing from one regime to another regime is solely depended on the adjustment or variation of the control parameter . Therefore this research is also about the transitions of this function. For instance, for some parameter values of the logistic map display periodic behavior (period-1 orbits “fixed point”, period-2 orbits and period-n orbits), and for others, it displays chaotic behavior.

**Keywords**: logistic function; nonlinear dynamical system; fixed points; periodic points; bifurcation; chaos; orbits

**Introduction**

The logistic equation or the logistic map as a nonlinear dynamic system has a class of different behaviors. The logistic function is a polynomial mapping of a degree 2 which is a nonlinear equation that behaves in series. The map was popularized in a seminal 1976 paper by the biologist Robert May, in part as a discrete-time demographic model analogous to the logistic equation first created by Pierre-Francois Verhulst. Mathematically, the logistic function/map is written as T () = (1- ), where and k is the value of interest which is a parameter (

The relative simplicity of the logistic map makes it a widely used point of entry into a consideration of the concepts of chaos. The logistic map provides a rich example as to how to explore periodic regions and complex chaotic behavior. For the interest of this research, where the logistic map changes in behavior due to the parameter introduced into the map and allowed to vary continuously in a way that changes in the logistic function can be noticed.

**Preliminaries**

**Definition 1**: orbits (fixed and periodic orbits) let: → the point 0 is *fixed point* for if (0) = 0. The point 0 is a *periodic point* of period for if n(0) = 0 but i(0) 0 for 0 <<. The sequence {0, (0), 2(0) … n(0) …} is called the orbit of 0 under .

**Definition 2: Types of periodic points**

1. A periodic point of period n is *attracting* if < 1
2. The periodic point is *repelling* if >1
3. The point is *neutral* if = 1

*Note:* the prime denote differentiation with respect to

**Theorem 1**:Let: R R be continuous then there exists a periodic point with period-3 in, then all periods exist in f leading to chaos.

**Definition 3**: Let: →, if >0 is a constant such that ɛ>0, then there is satisfying and an integer n, such that.

Where the point 0 is called sensitive point, hence the point x0 has sensitive dependence on the initial condition.

**THE MAIN WORK**

**The logistic map/function**

The logistic map is defined as; where

Let , then = (1), and. By setting the parameter, the logistic equation becomes; = (1) = – 2

**The roots and the maximum values of in the function**

The roots of in the function = – 2

If = 0, then r – 2 = 0

(1) = 0

Implies, = 0 and 1 – = 0

Then, = 0 and = 1

The maximum and minimum point of = – 2

= – 2

= 0 implies that, = o

= 0

= 0

1 – 2 = 0

=

The second derivate or rate of change of the function gives room for the conclusion on the sign, thus () = 2. Hence from the two main features of the logistic function it is clear that it passes through at and and maximum at since it is concave down.

**Note**: and are the first and second derivatives of the logistic function

**The solutions/locations of the logistic function**

The solution of the logistic function occurs when a diagonal line = is been introduced and there is an intersection between the diagonal line and the function as indicated on the diagram below with dash red line. Thus, =

= – 2, then 2 – + = 0

(– + 1) = 0,

= 0 and – – 1 = 0

Hence, = or = 0 are the two main solutions or locations of the logistic function ().

**Determination of a fixed point of the logistic function for period-1 orbit/point**

A period-1 orbits do occur when a function goes through series of iterations with an initial value popping-out as the same value (Mensah, 2016).

From the solutions of the logistic function, that is; 0 = 0 and 0 = which is based on the intersection of the diagonal line and the function.

Then, for

At intersection 0 = 0, (0) = (0) (1-0)

(0) = 0

At intersection 0= , =

=

=

=

=

=

Clearly, both deduction points out that, the point of intersection 0 = 0 and 0 = are the fixed points of the logistic map for the period-1 orbit/point and also serve as solutions.

**Example 1**: let and. Show that is the fixed point of the function.

**Illustration**: if , then,() = =

Hence as the initial point is the fixed point of the function, since it gives back the same point after several iterating, therefor serving as the fixed point for the function

**The nature of the fixed point of the logistic function**

We now show that the nature of the fixed points of the logistic function can be classified under definition 2. That is;

1. Attracting fixed point
2. Repelling fixed point
3. Neutral

**Illustration**: We determine if the fixed points of the logistic function are attracting and repelling fixed points under period-1 orbit base on the definition above.

Let the logistic function

Then, we take the derivative and evaluate the absolute value of the derived function, at = 0 and 0 =

That is, () = – 2

**At the fixed point = 0**

() = – 2,

1. Attracting, < 1

< 1

-1 < – 2(0) < 1

-1 << 1

2. Repelling, >1

>1, then

– 2(0) > 1 or – 2(0) < 1

> 1 or <1

Clearly, [0, 1) is inside the domain of [1, 4], hence =0 is attracting and stable at -1 << 1 for period-1 orbit of the logistic function but repelling at > 1 or <1 since (1, 4]

**At the fixed point 0 =**

() = – 2

1? Attracting, < 1

< 1

-1 < - + 2 < 1

-3 << -1 this implies 1 << 3

2? Repelling, >1

>1, then + 2 > or + 2 < -1

Therefore, < 1 or > 3

**Note**: the fixed point 0 = 0 and 0 = is a neutral fixed points at =1 and = 3. ***Very trivial***

**Example 2**: Algebraic and graphical illustration of the attracting fixed point of the logistic function, when =2.3<3 then the fixed point will be 0.57

Algebraically; taking = 0.10 as an initial point and the control parameter =2.3 < 3. Then for = 2.3 – 2.32 at = 0.10, = () = 0.207, = 2() = 0.378, = 3() = 0.541, = 4() = 0.571, = 5() = 0.563, = 6() = 0.570,= 7() = 0.570

**Example 3:** Illustration of the repelling fixed point through algebraic and graphical representations when =3.5 > 3 with a fixed point 0 = 0.71

Algebraically, taking = 0.10 as an initial point and the control parameter =3.5 > 3. Then for = 3.5 – 3.52 at = 0.10 1 = (0) = 0.315, 2 = 2(1) = 0.755, 3 = 3(2) = 0.647, 4 = 4(3) = 0.799, 5 = 5(4) = 0.561, 6 = 6(5) = 0.862, 7 = 7(6) = 0.417

***The periodic orbits (period-2) and the bifurcation diagram of the logistic map***

From the logistic function = – 2, as a parameter of interest can lie within 0 and 3 i.e. 0<< 3. Taken = 3 or beyond and iterate the function base on this interval it gives birth to an Orbit that alternate between two values (twice the period). The second iterate of the logistic map with the fixed point gives the period2 Orbits (Mensah, 2016)

**Example 4:** Considering the function or map = 3.2 – 3.22 for  
 (0, 1), let 0 = 0.5

By iteration of the function () the following sequence was obtain; at 0 = 0.5, 1 = (0) = 0.80, 2 = 2(1) = 0.51, 3 = 3(2) = 0.80, 4 = 4(3) = 0.51, 5 = 5(4) = 0.80

Clearly, the iteration of the function = 3.2 – 3.22 is a repeat of numbers that alternate between two values. Thus Orb = {0.51, 0.80} for the Orbits for the function = 3.2 – 3.22 with 0 = 0.5 as the initial point. This point 0 is a period2 points for the map

**Attracting and repelling points of logistic map for period- orbits**

A periodic point of period is ***attracting*** if < 1

The periodic point is ***repelling*** if >1

The point is ***neutral*** if = 1

***Note:*** the prime denote differentiation with respect to

It is obvious that by the definition of the periodic point as the iteration of the fixed point in time, thus n () = for instance, (()) =, then it is clear that the conditions for a fixed point to be *attracting fixed point* also hold for *periodic point* for period point. So it is also true for the *periodic point* if a fixed point is a *repelling fixed point*.

**Solutions/locations for the logistic function/map on its second iteration (period-2orbits/points)**

For period-1 point the solution is = 0 and = . Algebraically, we can also find the solutions for period-2 point of the logistic function.

Let = – 2then, for the period-2 point that is the second iteration 2() of the logistic function implies;evaluating 2() = (()),

2() =

=

=

= ………………….. 1.1

But for period-2 point, 2() = ……………………………1.2

Then by equating 1.1 and 1.2

=

0 =

0 =

0 =

This implies, and

Therefore = 0, = and = are the solutions or the fixed points for period-2 point/orbits for the logistic function

But, since our interest is in >0 for to be real. So we setts the discriminant to be greater than or equal to 0.

Thus,

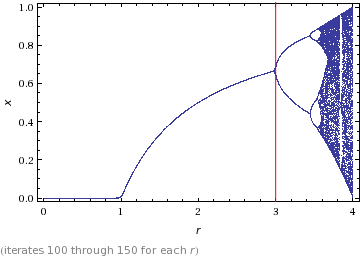
Or

Hence will be our interest for this work at this section since our interest was that >0 for to be real.

**Bifurcation diagram of the logistic function**

At exactly =3 and beyond, the behavior of the logistic map begins to change and it is as a results of the increasing nature of the control parameter, and this bring about bifurcation (splitting), when there is a qualitative change in the long term behavior of the map as the control parameter is varied, we say that the system undergo a bifurcation.(Mensah.2016)

Now by carefully looking at the bifurcation graph below the first bifurcation begins at =3 that is a period-2 periodic points.



**Figure 1.00: Bifurcation diagram of and**

We can also notice in the figure 1.00 below that there is a split (or bifurcation) when >3, this bifurcation represents the number of periods an initial value has when is a certain value/ number.

On this bifurcation of the logistic function as keeps increasing, Mitchell Feigenbaum (in 1978) worked on this process and arrived with the following table. What he discovered is now called the Feigenbaum Constant and is defined by

**Bifurcation table of the logistic function as keeps increasing**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | Bifurcation  (2n-cycle) |  |  |  |
| 1 | 2 | 3 | - | - |
| 2 | 4 | 3.449490 | - | - |
| 3 | 8 | 3.544090 | 0.44949 | 4.7515 |
| 4 | 16 | 3.564407 | 0.09460 | 4.6562 |
| 5 | 32 | 3.568759 | 0.020317 | 4.6684 |
| 6 | 64 | 3.56989 | 0.004352 | 4.6692 |
| 7 | 128 | 3.56993 | 0.001131 | 4.6694 |

It can be seen that the distance between successive bifurcations shrinks by a constant factor. Feigenbaum Constant/ Number can be used to predict subsequent values of where there is a split on the bifurcation diagram.

**Note**: the sequence { } is an infinite series called a period doubling cascade (a period doubling cascade is a sequence of doublings and further doublings of the repeating period, as the parameter is adjusted further and further), where 2n-cycle exist for every positive integer. From Feigenbaum computations the location of numerically appear closer and closer together through successive period doubling bifurcation. (mensah, 2016)

When is slightly higher than 3.54 the function alternate between 8,16,32,64 as in the Table above. Also the lengths  n-1 n-2 of the control parameter distances/gabs producing the same values of alternation reduce speedily.

The ratio  n-1 n-2/ n n-1 between the lengths  n-1 n-2 of two successive such bifurcation distances approaches the Feigenbaum Constant/ Number. And when =4, chaotic behavior of the map occurs.

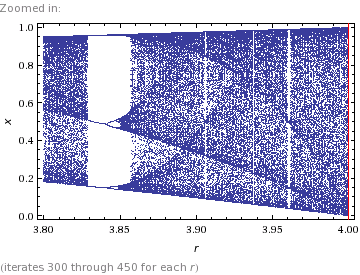
**Chaotic behavior of the logistic function**

The last nature/characteristic of the logistic map = (1), is the chaotic regime. To arrive at this chaotic regime, it has been shown in the various bifurcation diagrams and the Feigenbaum computations that the map moves faster or closer as is been increase.

**Proof of theorem 1: the existence of period-3 as one of the route to chaos**

It is also very clear that for period-3 points, there are some indications of small open spaces which break beyond a certain point, hence periodic leading to chaos.

**Graphical display of the period-3 points of the logistic map (bifurcation diagram)**



**Figure 1.10: Bifurcation diagram for 3.8 << 4.0**

**Illustration**: **Algebraic proof of the period-3 points of the logistic function**

It appears that a period3 point exists when is approximately between 3.83 and 3.84 as shown in **figure 1.10** above. It will be much better and easier if we use algebraic approach.

So Then, by considering the iterations of the logistic function = (1) where =3.83, implies = 3.8 (1). Let 0=0.5

1= () =0.9575, 2= () =0.1559, 3= () =0.5039, 4= () =0.9574, 5= () =0.1561, 6= () =0.5044, 7= () =0.9574, 8= () =0.1561, 9= () =0.5046

Upon iterating the function = 3.8 (1), the sequence we are obtaining are a repeat of numbers that alternate between three values as shown above, thus {0.96, 0.16, 0.50}

This clearly, shows that the logistic map has a periodic point with period 3. Hence period-3 point exists and if period-3 exits then periodic doubling leading to chaos.

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So the route to chaos can be seen through the existence of period-3, the doubling nature of the periodic orbits and all this route are dependents on the strength of the control parameter .

**So the question now is what happens when =4?**

The Approximation of between 3.83 and 3.84, is a period-3 point and begins period doubling after >3.84, hence chaotic at =4?

Then, by considering sensitive dependence on initial conditions as a concept for chaos and setting =4 for the logistic map. The logistic map becomes, = 4 (1- )

**Proof of definition 3**: **sensitive dependence on initial condition at=4**

**Sensitive dependence on initial conditions as a concept for chaos on logistic function/map**

Taking the logistic function = 4 (1- ) and setting 0=0.3333 as the approximation of . Then, *The iteration of the logistic function with initial value and its approximation 0.3333*

When 0=0.3333

1= () =0.8888, 2= () =0.3952, 3= () =0.9561, 4= () =0.1680, 5= () =0.5591, 6= () =0.9860, 7= () =0.0552

When =

1= () =2= () = , 3= () = , 4= () =0.1684, 5= () =0.5602, 6= () =0.9855, 7= () =0.0572

It can be seen in the iterations that as the number of iterations increases the intervals/difference between the outcome values increases. For the chaotic regime we base our argument on the definition and the above iterations.

Then by setting =0.000333. We choose = 0.0001, it can be seen that at 4 the difference is 0.0004 which is more than. And for and 0 to get closer let 0 =0.3333, then 0.0000333 as the difference between and 0 due to the iterations and for =0.000333 and our fixed = 0.0001.

Clearly, implying that and at 7, the resulting difference between the values is 0.0020 which also exceed our fixed = 0.0001.

Thus if implying

Therefor we can easily say that the logistic map/function is sensitive to initial condition.

And since we have sensitive to initial condition for the logistic map/function, we can assume that the logistic map is a chaotic system at =4

Finally we can accept the fact that period-3 lead to chaos since it exits by the algebraic analysis and also through the zooming of the bifurcation diagram of figure 1.10 when lies between 3.83 and 3.84 which are less than equal to 4. And beyond this period-3 subsequent period occurs called the period doubling cascade into chaos. Also at =4 the function is sensitive to initial condition therefore showing chaotic behavior

**Conclusion**

Clearly, the behavior of the logistic function into a period1 points also known as the fixed points was as a results of the control parameter <3 and will only results into periodic point when =3 or beyond. It was found that the period-1 point (fixed points) of the logistic function was attracting and repelling, that is converging and diverging when lies within 0 and 3 and<1 and >3 respectively. On the issue of period2 points of the logistic map, it was shown that the bifurcation diagram (figure 1.00) gives a better and transparent solution than that of the algebraic iteration of the function.That is when it moves from the fixed points/ orbits solution = 0, = to the periodic points which also add another solutions = to the previous solutions.

It can be concluded that for a logistic function, when the parameter keeps increasing the nature or behavior moves from periodicbehaviour to chaotic behaviour, making it unstable. That is an increase in makes the solutions unstable and higher periodic oscillating occurs.

When the cycles keep on becoming unstable, period doubling gives way to a different regime hence chaos then occurs at = 4.

So a route to chaos can be seen through the existence of period-3, the doubling nature of the periodic orbits and the sensitive dependence on its initial condition. All these routes relies on the strength of the controlparameter .

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