The Lindley Power Series Class of Distributions: Model, Properties and Applications

Gayan Warahena-Liyanage[1] **and Mavis Pararai**[2]

Abstract

The aim of this paper is to propose a new class of lifetime distributions called the Lindley power series (LPS). The distribution properties including survival function, hazard and reverse hazard functions, limiting behavior of the pdf and hazard function, quantile function, moments, distribution of order statistics, mean deviations, Lorenz and Bonferroni curves and Fisher information are presented. The method of maximum likelihood estimation is used to estimate the model parameters of this new class of distributions. The special cases of the LPS distribution including Lindley binomial (LB), Lindley geometric (LG), Lindley Poisson (LP) and Lindley logarithmic (LL) distributions are presented. The Lindley logarithmic (LL) distribution is discussed in detail. A Monte Carlo simulation study is presented to exhibit the performance and accuracy of maximum likelihood estimates of the LL model parameters. Some real data examples are discussed to illustrate the usefulness and applicability of the LL distribution.

Keywords: Lindley power series distribution, Lindley logarithmic distribution, Lindley distribution, Monte-Carlo simulation, Maximum likelihood estimation.

1 Introduction

The modeling of lifetime data has become popular in the area of survival analysis. In survival analysis, we study the lifetime of biological organisms or mechanical systems. In recent years, many distributions have been introduced to model these types of data. The basic concept behind introducing these distributions is that the lifetime of a system with N number of components and a positive continuous random variable, X_i, which denotes the lifetime of the i^{th} component, can be represented by a non-negative random variable Z=min$(X_1, X_2, \ldots, X_N)$ if the components are in a series or

[1]Department of Mathematics, Indiana University of Pennsylvania, Indiana, PA, 15705, USA. e-mail: g.j.warahenaliyanage@iup.edu

[2]Department of Mathematics, Indiana University of Pennsylvania, Indiana, PA, 15705, USA. e-mail: pararaim@iup.edu

Z=max$(X_1, X_2, \ldots, X_N)$ if the components are parallel. Some of the well-known lifetime distributions include the exponential geometric (EG) distribution by Adamidis and Loukas [1], exponential Poisson (EP) distribution by Kus [10], exponential logarithmic (EL) distribution by Tahmasbi and Rezaei [22], Weibull geometric (WG) distribution by Barreto-Souza et al. [2] and Weibull Poisson (WP) distribution by Lu and Shi [12].

Lindley [11] introduced a mixture of exponential and length-biased exponential distributions to illustrate the difference between fiducial and posterior distributions. This mixture is called the Lindley (L) distribution. Ghitany et al. [6] examined and studied the properties and applications of the Lindley distribution in the context of reliability analysis. The cumulative distribution function (cdf) of the Lindley distribution is given by

$$G_L(x;\beta) = 1 - \left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}, \quad x>0, \beta>0, \tag{1.1}$$

and

$$g_L(x;\beta) = \frac{\beta^2}{\beta+1}(1+x)e^{-\beta x}, \quad x>0, \beta>0, \tag{1.2}$$

respectively.

The survival function and hazard function of the Lindley distribution are given by

$$S_L(x;\beta) = \overline{G}_L(x;\beta) = 1 - G_L(x;\beta) = \left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}, \tag{1.3}$$

and

$$h_L(x;\beta) = \frac{g_L(x;\beta)}{\overline{G}_L(x;\beta)} = \frac{\beta^2(1+x)}{(1+\beta+\beta x)}, \tag{1.4}$$

respectively, where $x>0, \alpha>0, \beta>0$.

Using the transformation $Y = X^{\frac{1}{\alpha}}$, Ghitany et al. [5] derived the power Lindley (PL) distribution and its properties. Warahena-Liyanage and Pararai [23] studied the properties of the exponentiated Power Lindley (EPL) distribution which modeled lifetime data better than the Lindley and power Lindley distributions.

Another generalization of EPL distribution called the exponentiated power Lindley Poisson (EPLP) distribution which is obtained by compounding the zero-inflated Poisson distribution and the EPL distribution was derived and studied by Pararai et al. [17].

Noack [16] proposed and studied the power series class of distributions. This class of distributions includes binomial, geometric, logarithmic and Poisson distributions as special cases. However, these distributions may not be useful when a random variable takes the value of zero with high probability i.e. zero-inflated. In such situations, it is more appropriate to consider the distribution which is truncated at zero. More details on these distributions can be found in the context of univariate discrete distributions by Johnson et al. [8].

In recent years, some authors have used the power series class of distributions to develop new distributions. Many of these new distributions have been constructed as a mixture of some well known distributions and the power series class of distributions. Some of the power series distributions include the Weibull-power series (WPS) distributions by Morais and Barreto-Souza [15], generalized exponential-power series (GEPS) distributions by Mahmoudi and Jafari [13], Kumaraswamy-power series (KPS) distributions by Bidram and Nekouhou [3], and Linear failure rate-power series distributions by Mahamoudi and Jafari [14].

We provide three motivations for the new Lindley power series (LPS) class of distributions, which can be applied in some interesting situations as follows:

(i) Due to the stochastic representation $Z{=}\min(X_1, X_2, \ldots, X_N)$ the LPS class of distributions can arise in many industrial applications and biological organisms.

(ii) The LPS class of distributions can be used to model appropriately the time to the first failure of a system of identical components that are in a series.

(iii) The LPS class of distributions exhibit some interesting behaviors with non-monotonic failure rates such as bathtub, upside bathtub and increasing-decreasing-increasing failure rates which are more likely to be encountered in real life situations.

The paper is organized as follows: In section 2, we present some basic information on the Lindley distribution and the power series family of distributions. The general model of the LPS class of distributions is defined and its properties including hazard and reverse hazard functions, limiting behavior of the pdf and hazard function, quantile function, moments, distribution of order statistics, mean deviations, Lorenz and Bonferroni curves and Fisher information are presented. The method of maximum likelihood estimation is used to estimate the model parameters of this new class of distributions. In section 3, we introduce the special cases of the LPS distribution including Lindley binomial (LB) distribution, Lindley geometric (LG) distribution, Lindley Poisson (LP) distribution and Lindley logarithmic (LL) distribution. The Lindley logarithmic (LL) distribution will be explored in detail in section 4. The properties of the LL distribution such as survival function, hazard and reverse hazard functions, quantile function, moments, distribution of order statistics, mean deviations, Lorenz and Bonferroni curves, reliability and entropy measures are also presented. In section 5, we introduce some algorithms for generating random data from LL distribution. A Monte Carlo simulation study is also presented to exhibit the performance and accuracy of maximum likelihood estimates of the LL model parameters. We present some real data examples in section 6 to illustrate the usefulness and applicability of the LL distribution. Section 7 contains the conclusions and suggestions for further study.

2 The General Class and Properties

Let $X_1, X_2, \ldots, X_N$ be independent and identically distributed (iid) random variables from the Lindley (L) distribution whose cumulative distribution function (cdf) and probability density function (pdf) are given by equations (1.1) and (1.2), respectively.

Consider N to be a discrete random variable from a power series distribution (truncated at zero) and whose pdf is given by

$$P(N=n) = \frac{a_n \lambda^n}{C(\lambda)}, \quad n = 1, 2, 3, \ldots, \tag{2.1}$$

where $C(\lambda) = \sum_{n=1}^{\infty} a_n \lambda^n$ and a_n depends on n and $\lambda > 0$. $C(\lambda)$ is finite and its first, second and third derivatives with respect to λ are defined and given by $C'(.), C''(.)$ and $C'''(.)$, respectively. The power series family of distributions includes binomial, Poisson, geometric and logarithmic distributions as defined in Johnson et al. [8]. Table 2.1 represents some useful quantities including a_n, $C(\lambda)$, $C(\lambda)^{-1}$, $C'(\lambda)$, $C''(\lambda)$, and $C'''(\lambda)$ for the Poisson, geometric, logarithmic and binomial (with m being the number of replicas) distributions.

Table 2.1: Useful Quantities for Some Power Series Distributions.

Distribution	$C(\lambda)$	$C'(\lambda)$	$C''(\lambda)$	$C'''(\lambda)$	$C^{-1}(\lambda)$	a_n	Parameter Space
Poisson	$e^{\lambda}-1$	e^{λ}	e^{λ}	e^{λ}	$\log(1+\lambda)$	$(n!)^{-1}$	$(0,\infty)$
Geometric	$\lambda(1-\lambda)^{-1}$	$(1-\lambda)^{-2}$	$2(1-\lambda)^{-3}$	$6(1-\lambda)^{-4}$	$\lambda(1+\lambda)^{-1}$	1	$(0,1)$
Logarithmic	$-\log(1-\lambda)$	$(1-\lambda)^{-1}$	$(1-\lambda)^{-2}$	$2(1-\lambda)^{-3}$	$1-e^{-\lambda}$	n^{-1}	$(0,1)$
Binomial	$(1+\lambda)^m-1$	$\frac{m}{(1+\lambda)^{1-m}}$	$\frac{m(m-1)}{(1+\lambda)^{2-m}}$	$\frac{m(m-1)(m-2)}{(1+\lambda)^{3-m}}$	$(\lambda+1)^{1/m}-1$	$\binom{m}{n}$	$(0,\infty)$

Let $X_{(1)} = \min(X_1, X_2, \ldots, X_N)$. The conditional cdf of $X_{(1)} \mid N = n$ is given by

$$G_{X_{(1)}|N=n}(x) = 1 - [\overline{G}(x)]^n = 1 - \left(\frac{1+\beta+\beta x}{\beta+1}\right)^n e^{-\beta n x}, \quad x > 0, \tag{2.2}$$

where $\overline{G}(.)$ is the survival function in equation (1.3). The cdf of the new LPS class of distributions is the marginal cdf of $X_{(1)}$ which is given by

$$F_{LPS}(x;\beta,\lambda) = 1 - \frac{C\left[\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}\right]}{C(\lambda)}, \tag{2.3}$$

where $x > 0, \beta > 0$ and $\lambda > 0$.

2.1 Density Function

The pdf of a random variable X from a LPS class of distributions whose cdf is in (2.1.7) is given by

$$f_{LPS}(x;\beta,\lambda) = \lambda\beta^2(\beta+1)^{-1}(1+x)e^{-\beta x}\frac{C'\left[\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}\right]}{C(\lambda)}, \tag{2.4}$$

where $x > 0, \beta > 0$, and $\lambda > 0$.

The following propositions discuss the limiting behavior and some other characteristics of the LPS distribution.

Proposition 2.1. *The Lindley distribution is a limiting distribution of the LPS distribution when $\lambda \to 0^+$.*

Proof. Applying $C(\lambda) = \sum\limits_{n=1}^{\infty} a_n \lambda^n$ in (2.3), we obtain

$$\lim_{\lambda\to 0^+} F_{LPS}(x) = 1 - \lim_{\lambda\to 0^+} \frac{\sum\limits_{n=1}^{\infty} a_n\left[\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}\right]^n}{\sum_{n=1}^{\infty} a_n\lambda^n}.$$

By using L'Hôpital's rule, we obtain

$$\begin{aligned}\lim_{\lambda\to 0^+} F_{LPS}(x) &= 1 - \lim_{\lambda\to 0^+} \frac{a_1\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x} + \sum\limits_{n=2}^{\infty} n a_n \lambda^{n-1}\left(\frac{1+\beta+\beta x}{\beta+1}\right)^n e^{-n\beta x}}{a_1 + \sum_{n=2}^{\infty} n a_n\lambda^{n-1}} \\ &= 1 - \left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}.\end{aligned}$$

□

Proposition 2.2. *The density function of LPS class can be expressed as a linear combination of the density of $X_{(1)}$ =min$(X_1, X_2, \ldots, X_n)$.*

Proof. Since $C'(\lambda) = \sum_{n=1}^{\infty} n a_n \lambda^{n-1}$, we have

$$f_{LPS}(x) = \sum_{n=1}^{\infty} P(N=n) g_{X_{(1)}}(x; n),$$

where $g_{X_{(1)}}(x; n)$ is the probability pdf of $X_{(1)}$ =min$(X_1, X_2, \ldots, X_n)$ given by

$$g_{X_{(1)}}(x; n) = \frac{n\beta^2}{(\beta+1)^n}(1+\beta+\beta x)^{n-1}(1+x)e^{-n\beta x}. \tag{2.5}$$

□

Proposition 2.3. *For the pdf of the LPS distribution, we have*

$$\lim_{x \to 0^+} f_{LPS}(x) = \frac{\lambda \beta^2 C'(\lambda)}{(\beta+1)C(\lambda)},$$

and

$$\lim_{x \to \infty} f_{LPS}(x) = 0.$$

2.2 Reverse Hazard and Hazard Functions

The reverse hazard function and the hazard function of the LPS distribution, respectively, are given by

$$\tau_{LPS}(x; \beta, \lambda) = \lambda \beta^2 (\beta+1)^{-1}(1+x)e^{-\beta x} \frac{C'\left[\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}\right]}{C(\lambda) - C\left[\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}\right]}, \tag{2.6}$$

and

$$h_{LPS}(x; \beta, \lambda) = \lambda \beta^2 (\beta+1)^{-1}(1+x)e^{-\beta x} \frac{C'\left[\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}\right]}{C\left[\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}\right]}, \tag{2.7}$$

where $x > 0, \beta > 0$ and $\lambda > 0$.

The following Proposition gives the limiting behavior of the hazard function of the LPS distribution in (2.7).

Proposition 2.4. *For the hazard function of the LPS distribution, we have*

$$\lim_{x \to 0^+} h_{LPS}(x) = \frac{\lambda \beta^2 C'(\lambda)}{(\beta+1)C(\lambda)},$$

and

$$\lim_{x \to \infty} h_{LPS}(x) = \infty.$$

2.3 Quantile Function

In this section, the quantile function of the LPS distribution will be derived. The quantile function, $Q(p)$, defined by $F_{LPS}(Q(p)) = p$ is the root of the equation,

$$1 - \frac{C\left[\lambda\left(\frac{1+\beta+\beta Q(p)}{\beta+1}\right)e^{-\beta Q(p)}\right]}{C(\lambda)} = p, \quad \text{where } 0 < p < 1.$$

Let $Z(p) = -1 - \beta - \beta Q(p)$, then we have

$$1 - \frac{C\left[\lambda\left(\frac{Z(p)}{\beta+1}\right)e^{Z(p)+\beta+1)}\right]}{C(\lambda)} = p.$$

That is,

$$Z(p)e^{Z(p)} = -\frac{(\beta+1)C^{-1}((1-p)C(\lambda))}{\lambda e^{\beta+1}}$$

where $C^{-1}(.)$ is the inverse function of $C(.)$ Then solution for $Z(p)$ is

$$Z(p) = W\left[-\frac{(\beta+1)C^{-1}((1-p)C(\lambda))}{\lambda e^{\beta+1}}\right],$$

for $0 < p < 1$, where $W(.)$ is the negative branch of the Lambert W function. See Corless et.al [4] for history, theory and applications of the Lambert W function.

Consequently, the quantile function of the LPS distribution is given by

$$Q(p) = -1 - \frac{1}{\beta} - \frac{1}{\beta}W\left[-\frac{(\beta+1)C^{-1}((1-p)C(\lambda))}{\lambda e^{\beta+1}}\right], \quad \text{where } 0 < p < 1. \tag{2.8}$$

2.4 Moments and Related Measures

In this section, moments and related measures such as coefficient of variation, skewness and kurtosis of the LPS distribution are presented. In order to find the moments, the following lemma is proved.

Lemma 2.5. *Let*

$$L_1(\beta, n, p) = \int_0^\infty x^p(1+x)(1+\beta+\beta x)^{n-1}e^{-n\beta x}dx,$$

then

$$L_1(\beta, n, p) = \sum_{i=0}^{n-1}\sum_{j=0}^{i+1}\binom{n-1}{i}\binom{i+1}{j}\frac{\Gamma(j+p+1)}{n^{j+p+1}\beta^{p+j-i+1}}.$$

Proof. Using the series expansion for $|z| < 1$ and $a > 0$,

$$(1-z)^{a-1} = \sum_{i=0}^{\infty} \binom{a-1}{i} (-1)^i z^i. \tag{2.9}$$

Using $z = \beta(1+x)$, we have,

$$\begin{aligned} L_1(\beta, n, p) &= \int_0^{\infty} x^p (1+x)(1+\beta(1+x))^{n-1} e^{-n\beta x} dx \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{i+1} \binom{n-1}{i} \binom{i+1}{j} \beta^i \int_0^{\infty} x^{j+p} e^{-n\beta x} dx. \end{aligned}$$

Consequently,

$$L_1(\beta, n, p) = \sum_{i=0}^{n-1} \sum_{j=0}^{i+1} \binom{n-1}{i} \binom{i+1}{j} \frac{\Gamma(j+p+1)}{n^{j+p+1} \beta^{p+j-i+1}}. \tag{2.10}$$

□

The r^{th} moment of a random variable X from the LPS distribution, say μ'_r, is given by

$$\mu'_r = E(X^r) = \int_0^{\infty} x^r f_{LPS}(x) dx.$$

Using the fact in Proposition (2.2), we have

$$\begin{aligned} E(X^r) &= \sum_{n=1}^{\infty} P(N=n) E(X_{(1)}^r) = \sum_{n=1}^{\infty} P(N=n) \int_0^{\infty} x^r g_{X_{(1)}}(x; n) dx \\ &= \sum_{n=1}^{\infty} P(N=n) \frac{n\beta^2}{(\beta+1)^n} \int_0^{\infty} x^r (1+x)(1+\beta+\beta x)^{n-1} e^{-n\beta x} dx. \end{aligned}$$

Using Lemma (2.5), it follows that

$$E(X^r) = \sum_{n=1}^{\infty} P(N=n) \frac{n\beta^2 L_1(\beta, n, r)}{(\beta+1)^n}.$$

Consequently, the r^{th} moment of the LPS distribution is given by

$$\mu'_r = E(X^r) = \sum_{n=1}^{\infty} \frac{a_n \lambda^n}{C(\lambda)} \frac{n\beta^2 L_1(\beta, n, r)}{(\beta+1)^n}. \tag{2.11}$$

The mean (μ), variance(σ^2), coefficient of variation (CV), coefficient of skewness (CS), and coefficient of kurtosis (CK) are given by

$$\mu = E(X) = \sum_{n=1}^{\infty} \frac{a_n \lambda^n}{C(\lambda)} \frac{n\beta^2 L_1(\beta, n, 1)}{(\beta+1)^n}, \tag{2.12}$$

$$\sigma^2 = \mu_2' - \mu^2, \tag{2.13}$$

$$CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu_2' - \mu^2}}{\mu} = \sqrt{\frac{\mu_2'}{\mu^2} - 1}, \tag{2.14}$$

$$CS = \frac{E\left[(X-\mu)^3\right]}{\left[E(X-\mu)^2\right]^{3/2}} = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{(\mu_2' - \mu^2)^{3/2}}, \tag{2.15}$$

and

$$CK = \frac{E\left[(X-\mu)^4\right]}{\left[E(X-\mu)^2\right]^2} = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{(\mu_2' - \mu^2)^2}, \tag{2.16}$$

2.5 Conditional Moments

For lifetime models, it is useful to study the conditional moments which are defined as $E(X^r \mid X > x)$. The following lemma is introduced to evaluate conditional moments.

Lemma 2.6. *Let*

$$L_2(\beta, n, p, t) = \int_t^{\infty} x^p (1+x)(1+\beta+\beta x)^{n-1} e^{-n\beta x} dx,$$

then

$$L_2(\beta, n, p, t) = \sum_{i=0}^{n-1} \sum_{j=0}^{i+1} \binom{n-1}{i} \binom{i+1}{j} \frac{\Gamma(j+p+1, n\beta t)}{n^{j+p+1} \beta^{p+j-i+1}}.$$

Proof. Applying the series expansion in Equation (2.9) with $z = \beta(1+x)$ we obtain,

$$\begin{aligned} L_2(\beta, n, p, t) &= \int_0^{\infty} x^p (1+x)(1+\beta(1+x))^{n-1} e^{-n\beta x} dx \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{i+1} \binom{n-1}{i} \binom{i+1}{j} \beta^i \int_t^{\infty} x^{j+p} e^{-n\beta x} dx. \end{aligned}$$

Consequently,

$$L_2(\beta, n, p, t) = \sum_{i=0}^{n-1} \sum_{j=0}^{i+1} \binom{n-1}{i} \binom{i+1}{j} \frac{\Gamma(j+p+1, n\beta t)}{n^{j+p+1} \beta^{p+j-i+1}}. \tag{2.17}$$

□

Using Proposition (2.2) and Lemma (2.6), the r^{th} conditional moment of the LPS distribution is given by

$$\begin{aligned} E(X^r \mid X > x) &= \sum_{n=1}^{\infty} P(N=n) \frac{n\beta^2}{(\beta+1)^n} \frac{L_2(\beta, n, r, x)}{1 - F_{LPS}(x)} \\ &= \sum_{n=1}^{\infty} \frac{a_n \lambda^n}{C\left(\lambda \left(\frac{1+\beta+\beta x}{\beta+1}\right) e^{-\beta x}\right)} \frac{n\beta^2 L_2(\beta, n, r, x)}{(\beta+1)^n}. \end{aligned} \tag{2.18}$$

2.6 Moment Generating Function

The moment generating function (MGF) of the LPS distribution is defined by

$$M_X(t) = E(e^{tX}).$$

By considering the fact in Proposition (2.2), we obtain

$$M_X(t) = \sum_{n=1}^{\infty} P(N=n) M_{X_{(1)}}(t) = \sum_{n=1}^{\infty} P(N=n) \int_0^{\infty} e^{tx} g_{X_1}(x; n) dx. \tag{2.19}$$

Using the series expansion $e^{tx} = \sum_{k=0}^{\infty} \frac{t^k x^k}{k!}$ and Lemma (2.5) in equation (2.19), the moment generating function (MGF) of the LPS distribution is given by

$$M_X(t) = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{a_n \lambda^n}{C(\lambda)} \frac{n\beta^2 t^k L_1(\beta, n, k)}{(\beta+1)^n k!}. \tag{2.20}$$

2.7 Distribution of Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample from the LPS distribution and suppose $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ denote the corresponding order statistics. The pdf of the k^{th} order statistic is given by

$$f_{k:n}(x) \frac{n!}{(k-1)!(n-k)!} f(x) [F(x)]^{k-1} [1 - F(x)]^{n-k}, \tag{2.21}$$

where $F(.)$ and $f(.)$ are the cdf and pdf of LPS distributions given in equations (2.3) and (2.4), respectively. Using the series expansion in equation (2.9), we can re-write Equation (2.21) as follows:

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l f(x)[F(x)]^{k+l-1}. \tag{2.22}$$

Note that,

$$f(x)[F(x)]^{k+l-1} = \frac{1}{k+l} \frac{d}{dx} [F(x)]^{k+l}.$$

The corresponding cdf of $f_{k:n}(x)$, denoted by $F_{k:n}(x)$ can be obtained as

$$\begin{aligned} F_{k:n}(x) &= \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \frac{\binom{n-k}{l}(-1)^l}{k+l} [F(x)]^{k+l} \\ &= \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \frac{\binom{n-k}{l}(-1)^l}{k+l} \left[1 - \frac{C\left(\lambda \left(\frac{1+\beta+\beta x}{\beta+1}\right) e^{-\beta x}\right)}{C(\lambda)} \right]^{k+l} \\ &= \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \frac{\binom{n-k}{l}(-1)^l}{k+l} F_G(x; \beta, \lambda, k+l), \end{aligned}$$

where G follows an exponentiated LPS (ELPS) distribution with parameters β, λ and $k+l$. Thus, the cdf of the k^{th} order statistic can be expressed as a linear combination of the cdf of the ELPS distribution with parameters β, λ and $k+l$.

2.8 Mean Deviations

In this section, the mean deviation about the mean and the mean deviation about the median of the LPS distribution are presented. The amount of scatter in a population can be measured by the totality of deviations from the mean and median. The mean deviation about the mean, say $D(\mu)$, and the mean deviation about the median, say $D(M)$, are defined as

$$D(\mu) = \int_0^\infty | x - \mu | f_{LPS}(x)dx, \quad D(M) = \int_0^\infty | x - M | f_{LPS}(x)dx,$$

respectively, where $\mu = E(X)$ and $M = Median(X) = F^{-1}(1/2)$ is the median of F_{LPS}. The measures $D(\mu)$ and $D(M)$ can be evaluated using the following relationships:

$$D(\mu) = 2\mu F_{LPS}(\mu) - 2\mu + 2 \int_\mu^\infty x f_{LPS}(x), \tag{2.23}$$

and

$$D(M) = -\mu + 2\int_{M}^{\infty} x f_{LPS}(x). \tag{2.24}$$

Using Lemma (2.6), we obtain

$$D(\mu) = 2\mu F_{LPS}(\mu) - 2\mu + 2\sum_{n=1}^{\infty} \frac{a_n\lambda^n}{C(\lambda)} \frac{n\beta^2 L_2(\beta, n, 1, \mu)}{(\beta+1)^n},$$

and

$$D(M) = -\mu + 2\sum_{n=1}^{\infty} \frac{a_n\lambda^n}{C(\lambda)} \frac{n\beta^2 L_2(\beta, n, 1, M)}{(\beta+1)^n}.$$

2.9 Lorenz and Bonferroni Curves

In this section, the Lorenz and Bonferroni curves for the LPS distribution are presented. The Lorenz and Bonferroni curves have a number of applications in different fields such as medicine, income and poverty, reliability and insurance. The Lorenz and Bonferroni curves are given by

$$L(F_{LPS}(x)) = \frac{\int_0^x t f_{LPS}(t)dt}{E(X)}, \quad \text{and} \quad B(F_{LPS}(x)) = \frac{L(F_{LPS}(x))}{F_{LPS}(X)},$$

or

$$L(p) = \frac{1}{\mu}\int_0^q x f_{LPS}(x)dx, \quad \text{and} \quad B(p) = \frac{1}{p\mu}\int_0^q x f_{LPS}(x)dx, \tag{2.25}$$

respectively, where $q = F_{LPS}^{-1}(p)$. Applying Lemma (2.6) in (2.25), we obtain

$$L(p) = \frac{1}{\mu}\left(\mu - \sum_{n=1}^{\infty} \frac{a_n\lambda^n}{C(\lambda)} \frac{n\beta^2 L_2(\beta, n, 1, q)}{(\beta+1)^n}\right), \tag{2.26}$$

and

$$B(p) = \frac{1}{p\mu}\left[\mu - \sum_{n=1}^{\infty} \frac{a_n\lambda^n}{C(\lambda)} \frac{n\beta^2 L_2(\beta, n, 1, q)}{(\beta+1)^n}\right]. \tag{2.27}$$

2.10 Maximum Likelihood Estimation

In this section, the maximum likelihood method used in estimating the parameters β and λ is presented. Let $x_1, x_2, \ldots, x_n$ be n observations of a random sample from LPS distribution and $\mathbf{\Theta} = (\beta, \lambda)^T$ be the unknown parameter vector. From Equation (2.4), the log-likelihood function of LPS distribution is given by

$$\begin{aligned} l_n(\beta,\lambda) &= n\log(\lambda) + 2n\log(\beta) - n\log(\beta+1) - n\log(C(\lambda)) + \sum_{i=1}^{n}\log(1+x_i) \\ &\quad -\beta\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\log\left\{C'\left[\lambda\left(\frac{1+\beta+\beta x_i}{\beta+1}\right)e^{-\beta x_i}\right]\right\}. \end{aligned} \tag{2.28}$$

The associated score function is $U_n(\mathbf{\Theta}) = \left(\frac{\partial l_n}{\partial\beta}, \frac{\partial l_n}{\partial\lambda}\right)^T$, where $\frac{\partial l_n}{\partial\beta}$ and $\frac{\partial l_n}{\partial\lambda}$ are the partial derivatives of the log-likelihood function given by

$$\begin{aligned} \frac{\partial l_n}{\partial\beta} &= \frac{2n}{\beta} - \frac{n}{\beta+1} - \sum_{i=1}^{n}x_i - \sum_{i=1}^{n}\frac{\lambda x_i e^{-\beta x_i}[(1+\beta+\beta x_i)(\beta+1)-1]}{(\beta+1)^2} \\ &\quad \times\frac{C''\left[\lambda\left(\frac{1+\beta+\beta x_i}{\beta+1}\right)e^{-\beta x_i}\right]}{C'\left[\lambda\left(\frac{1+\beta+\beta x_i}{\beta+1}\right)e^{-\beta x_i}\right]}, \end{aligned} \tag{2.29}$$

and

$$\frac{\partial l_n}{\partial\lambda} = \frac{n}{\lambda} - \frac{nC'^{(\lambda)}}{C(\lambda)} + \sum_{i=1}^{n}\frac{(1+\beta+\beta x_i)e^{-\beta x_i}}{\beta+1}\frac{C''\left[\lambda\left(\frac{1+\beta+\beta x_i}{\beta+1}\right)e^{-\beta x_i}\right]}{C'\left[\lambda\left(\frac{1+\beta+\beta x_i}{\beta+1}\right)e^{-\beta x_i}\right]}, \tag{2.30}$$

respectively. The maximum likelihood estimates of $\mathbf{\Theta}$ can be obtained by solving the non-linear system $U_n(\mathbf{\Theta}) = 0$. Since the equations (2.29) and (2.30) are not in closed form, the solutions can be found by using a numerical method such as the Newton-Raphson procedure.

2.11 Fisher Information Matrix and Asymptotic Confidence Intervals

In this section, we present a measure for the amount of information. This information can be used for interval estimation and hypothesis testing for the model parameters β and λ.

Let X be a random variable with the LPS pdf $f_{LPS}(\cdot;\mathbf{\Theta})$, where $\mathbf{\Theta} = (\theta_1, \theta_2)^T = (\beta, \lambda)^T$. The Fisher information matrix (FIM) is the 2×2 symmetric matrix given by

$$\mathbf{I}(\mathbf{\Theta}) = \begin{bmatrix} \mathrm{I}_{\beta\beta} & \mathrm{I}_{\beta\lambda} \\ \mathrm{I}_{\lambda\beta} & \mathrm{I}_{\lambda\lambda} \end{bmatrix},$$

where elements $\mathbf{I}_{ij}(\mathbf{\Theta}) = -E_{\mathbf{\Theta}}\left[\frac{\partial^2 \log(f_{LPS}(X;\mathbf{\Theta}))}{\partial\theta_i\partial\theta_j}\right]$. The elements of the FIM can be obtained by considering the second order partial derivatives of (2.28). These elements can be numerically obtained by MATLAB or MAPLE software. The total FIM $\mathbf{I}_n(\mathbf{\Theta})$ can be approximated by

$$\mathbf{J}_n(\hat{\mathbf{\Theta}}) \approx \left[-\frac{\partial^2 \log L}{\partial\theta_i\partial\theta_j}\bigg|_{\mathbf{\Theta}=\hat{\mathbf{\Theta}}}\right]_{2\times 2} \tag{2.31}$$

For real data, the matrix given in Equation (2.31) is obtained after the convergence of the Newton-Raphson procedure in MATLAB or R software. Let $\hat{\mathbf{\Theta}} = (\hat{\beta}, \hat{\lambda})$ be the maximum likelihood estimate of $\mathbf{\Theta} = (\beta, \lambda)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\mathbf{\Theta}} - \mathbf{\Theta}) \xrightarrow{d} N_2(\underline{\mathbf{0}}, \mathbf{I}^{-1}(\mathbf{\Theta}))$, where $\mathbf{I}(\mathbf{\Theta})$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $\mathbf{I}(\mathbf{\Theta})$ is replaced by the observed information matrix evaluated at $\hat{\mathbf{\Theta}}$, that is $\mathbf{J}(\hat{\mathbf{\Theta}})$. The multivariate normal distribution with mean vector $\underline{0} = (0,0)^T$ and covariance matrix $\mathbf{I^{-1}}(\mathbf{\Theta})$ can be used to construct confidence intervals for the model parameters. The approximate $100(1-\eta)\%$ two-sided confidence intervals for β and λ are given by

$$\hat{\beta} \pm Z_{\eta/2}\sqrt{\mathbf{I}^{-1}_{\beta\beta}(\hat{\mathbf{\Theta}})}, \quad \text{and} \quad \hat{\lambda} \pm Z_{\eta/2}\sqrt{\mathbf{I}^{-1}_{\lambda\lambda}(\hat{\mathbf{\Theta}})},$$

respectively, where $\mathbf{I}^{-1}_{\beta\beta}(\hat{\mathbf{\Theta}})$, and $\mathbf{I}^{-1}_{\lambda\lambda}(\hat{\mathbf{\Theta}})$ are diagonal elements of $\mathbf{I}_n^{-1}(\hat{\mathbf{\Theta}}) = (n\mathbf{I}\hat{\mathbf{\Theta}}))^{-1}$ and $Z_{\eta/2}$ is the upper $(\eta/2)^{th}$ percentile of a standard normal distribution.

3 Special Cases of LPS Class of Distributions

In this section, we study some special cases of the LPS class of distributions including Lindley binomial (LB) distribution, Lindley geometric (LG) distribution, Lindley Poisson (LP) distribution, and Lindley logarithmic (LL) distribution. The LG and LP distributions were introduced and studied as two separate studies by Zakerzadeh and Mahmoudi [24], and Gui et al. [7], respectively. We present an overview of the LB, LG and LP distributions as special cases of the LPS class and discuss the LL distribution in detail in the following sections.

3.1 Lindley Binomial Distribution

The binomial distribution (truncated at zero) is a special case of the class of power series distributions with $a_n = \binom{m}{n}$ and $C(\lambda) = (1+\lambda)^m - 1$, where

$m(n \leqslant m)$ is the number of replicas. From Equation (2.3), the cdf of the Lindley binomial (LB) distribution is given by

$$F_{LB}(x;\beta,\lambda,m) = 1 - \frac{\left[1+\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}\right]^m - 1}{(1+\lambda)^m - 1}, \quad x>0, \beta>0, \lambda>0, \tag{3.1}$$

where m is a positive integer.

The associated pdf is given by

$$f_{LB}(x;\beta,\lambda,m) = \frac{m\lambda\beta^2(1+x)e^{-\beta x}\left[1+\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}\right]^{m-1}}{(\beta+1)[(1+\lambda)^m - 1]}, \tag{3.2}$$

where $x > 0, \beta > 0, \lambda > 0$, and m is a positive integer.

3.2 Lindley Geometric Distribution

The Lindley geometric (LG) distribution was introduced and studied by Zakerzadeh and Mahmoudi [24]. The cdf and pdf of the LG distribution are presented to show that it is a special case of LPS class. Using Equation (2.3) with $a_n = 1$ and $C(\lambda) = \lambda(1-\lambda)^{-1}$, the cdf of the LG distribution is given by

$$F_LG(x;\beta,\lambda) = \frac{1-\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}}{1-\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}}, \quad x>0, \beta>0, 0<\lambda<1. \tag{3.3}$$

The pdf of the LG distribution is given by

$$f_{LG}(x;\beta,\lambda) = \frac{\beta^2(1-\lambda)(1+x)e^{-\beta x}}{(\beta+1)\left[1-\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}\right]^2}, \quad x>0, \beta>0, 0<\lambda<1. \tag{3.4}$$

The LG distribution was fitted to two real data sets. The first data set consists of waiting times (in minutes) before service of 100 bank customers. The second data set consisted of vinyl chloride data obtained from clean upgradient monitoring wells in mg/L. For each data set the LG distribution proved to be superior to all the other models it was compared to.

3.3 Lindley Poisson Distribution

The Lindley Poisson (LP) distribution was introduced and studied by Gui et al. [7]. Using the stochastic representation $X_{(n)} = \max(X_1, X_2, \ldots, X_N)$, Pararai et al. [17] introduced the exponentiated power Lindley Poisson (EPLP) distribution for which the LP distribution is a submodel. The cdf and pdf of the LP distribution are presented to show that it is a special case of LPS class. Using Equation (2.3) with $a_n = 1/n!$ and $C(\lambda) = e^\lambda - 1$, where $\lambda > 0$, the cdf of the LP distribution is given by

$$F_{LP}(x;\beta,\lambda) = \frac{e^\lambda - e^{\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}}}{e^\lambda - 1}, \quad x > 0, \beta > 0, \lambda > 0. \tag{3.5}$$

The LP distribution was fitted to two real data sets. The first data set consisted of time intervals of the successive earthquakes taken from University of Bosphoros, Kandilli Observatory and Earthquake Research Institute-National Earthquake Monitoring Center. The second data set consists of survival times of guinea pigs injected with different doses of tubercle bacilli. For each data set the LP distribution proved to be superior to all the other models it was compared to.

The corresponding pdf is given by

$$f_{LP}(x;\beta,\lambda) = \frac{\lambda\beta^2(1+x)e^{\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x} - \beta x}}{(\beta+1)(e^\lambda - 1)}, \quad x > 0, \beta > 0, \lambda > 0. \tag{3.6}$$

4 Lindley Logarithmic Distribution

In this section, the Lindley logarithmic (LL) distribution is introduced as a special case of LPS class of distributions and will be studied in detail. The properties of the LL distribution are derived from the properties of the general class. The entropy measures and the reliability of the LL distribution are also presented.

4.1 The Model

The LL distribution is defined by using the cdf of LPS distribution given in Equation (2.3) with $a_n = 1/n$, and $C(\lambda) = -\log(1-\lambda)$, where $0 < \lambda < 1$. The cdf and pdf of LL distribution are given by

$$F_{LL}(x;\beta,\lambda) = 1 - \frac{\log\left(1 - \lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}\right)}{\log(1-\lambda)}, \tag{4.1}$$

and

$$f_{LL}(x;\beta,\lambda) = \frac{\lambda\beta^2(1+x)e^{-\beta x}}{(\beta+1)\log(1-\lambda)\left[\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x} - 1\right]}, \tag{4.2}$$

respectively, where $x > 0, \beta > 0, 0 < \lambda < 1$. Figure 4.1 represents the plots for the pdf of the LL distribution for several combinations of the parameters β

and λ. The plots indicate that the LL distribution can be decreasing or right skewed. The LL distribution has a positive asymmetry.

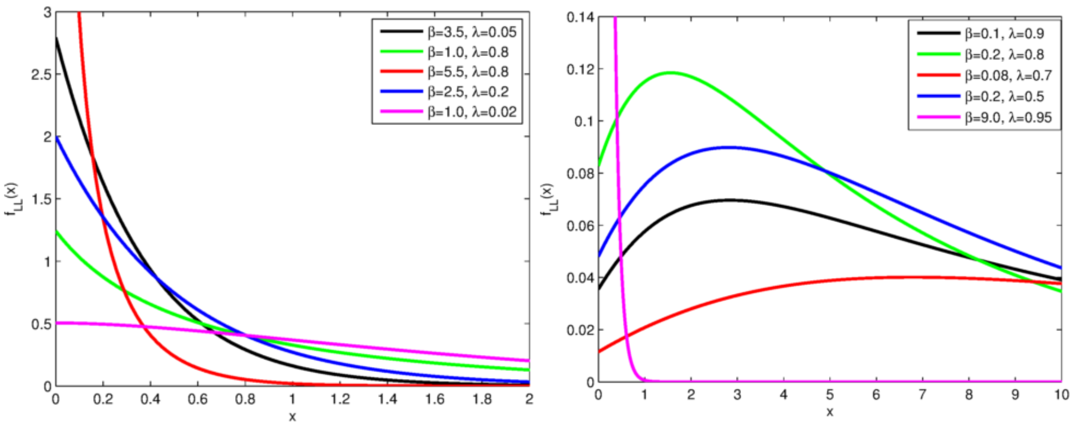

Figure 4.1: Plots of the pdf for different values of β and λ

4.2 Quantile Function

By substituting $C^{-1}(\lambda) = 1 - e^{-\lambda}$ in Equation (2.8), the quantile function for the LL distribution is given as

$$X_q = -1 - \frac{1}{\beta} - \frac{1}{\beta} W\left(-\frac{(\beta+1)(1-e^{(1-p)\log(1-\lambda)})}{\lambda e^{\beta+1}}\right), \quad 0 < p < 1, \tag{4.3}$$

where $W(.)$ is the negative branch of the Lambert W function.

4.3 Reverse Hazard and Hazard Functions

The reverse hazard function and hazard function of the LL distribution are respectively given as follows:
from Equation (2.6)

$$\tau_{LL}(x;\beta,\lambda) = \lambda\beta^2(\beta+1)^{-1}(1+x)e^{-\beta x} \frac{\left[1-\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}\right]^{-1}}{\log\left[1-\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}\right]-\log(1-\lambda)}, \tag{4.4}$$

and from Equation (2.7)

$$h_{LL}(x;\beta,\lambda) = \lambda\beta^2(\beta+1)^{-1}(1+x)e^{-\beta x} \frac{\left[\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}-1\right]^{-1}}{\log\left(1-\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}\right)}, \tag{4.5}$$

where $x > 0, \beta > 0$ and $0 < \lambda < 1$. Plots of the hazard function of the LL distribution for several combinations of the parameters β and λ are given in Figure 4.2. The plots for the hazard function of LL distribution exhibit different shapes including monotonically increasing, monotonically decreasing, bathtub, upside down bathtub and increasing-decreasing-increasing shapes. These interesting shapes of the hazard function imply that LL distribution is suitable for monotonic and non-monotonic hazard behaviors which are more likely to be encountered in real life situations.

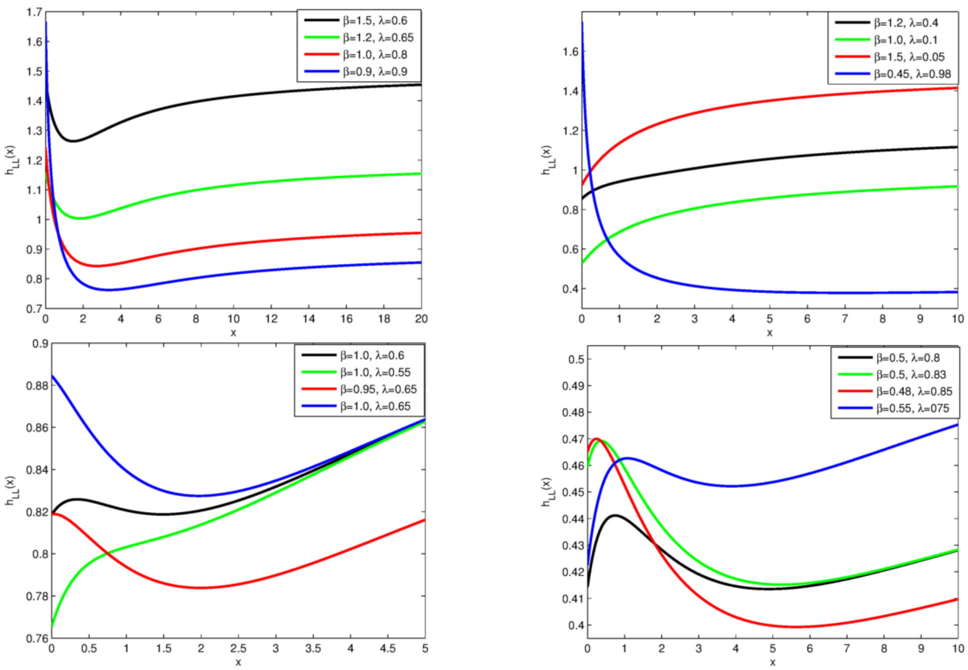

Figure 4.2: Plots of the hazard function for different values of β and λ

4.4 Moments, Conditional Moments and Moment Generating Function

The r^{th} moment of a random variable X from the LL distribution, say μ'_r is given by

$$\mu'_r = E(X^r) = -\sum_{n=1}^{\infty} \frac{\lambda^n \beta^2 L_1(\beta, n, r)}{\log(1-\lambda)(\beta+1)^n}. \tag{4.6}$$

Table 4.1 lists the first six moments, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) of the LL distribution for some selected values of the parameters β and λ. These values can be determined numerically using R and MATLAB. See appendix.

Table 4.1: Moments of the LL Distribution for Some Selected Values of Parameters β and λ

μ'_s	$\beta=1.5,\lambda=0.2$	$\beta=2.5,\lambda=0.8$	$\beta=3.0,\lambda=0.1$	$\beta=5.0,\lambda=0.95$	$\beta=0.9,\lambda=0.9$
μ'_1	0.88532	0.34703	0.40608	0.11284	0.99769
μ'_2	1.47751	0.28771	0.32071	0.04024	2.39911
μ'_3	3.56137	0.37932	0.37171	0.02469	9.11713
μ'_4	11.14074	0.67679	0.56450	0.02106	46.57626
μ'_5	42.68053	1.50879	1.05659	0.02273	295.69180
μ'_6	193.08020	4.01367	2.34584	0.02949	2230.43400
SD	0.83290	0.40900	0.39472	0.16586	1.18479
CV	0.94079	1.17856	0.97202	1.46994	1.18754
CS	1.77394	2.38785	1.86890	3.05540	2.35858
CK	7.55190	11.24393	8.09289	16.52638	10.93549

From Equation (2.5) and Equation (2.20), the r^{th} conditional moment and the moment generating function (mgf) of the LL distribution are respectively given by

$$E(X^r \mid X > x) = -\sum_{n=1}^{\infty} \frac{\lambda^n \beta^2 L_2(\beta, n, r, x)}{\log\left[1-\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}\right](\beta+1)^n}, \tag{4.7}$$

and

$$M_X(t) = -\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \frac{\lambda^n \beta^2 t^k L_1(\beta, n, k)}{\log(1-\lambda)(\beta+1)^n k!}. \tag{4.8}$$

4.5 Distribution of Order Statistics

By substituting the pdf and cdf of the LL distribution into Equation (2.22) and the series expansion in (2.9), the pdf of the k^{th} order statistic is given by

$$\begin{aligned} f_{k:n}(x) &= \frac{n!}{(k-1)!(n-k)!}\sum_{l=0}^{n-k}\sum_{m=0}^{k+l-1}\binom{n-k}{l}\binom{k+l-1}{m}\frac{(-1)^{l+m}\lambda\beta^2(1+x)e^{-\beta x}}{(\beta+1)[\log(1-\lambda)]^{m+1}} \\ &\quad \times\left[\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}-1\right]^{-1}\left[\log\left\{1-\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}\right\}\right]^m. \end{aligned}$$

The corresponding cdf of $f_{k:n}(x)$, denoted by $F_{k:n}(x)$ can be obtained by using Equation (2.23) and it is given by

$$\begin{aligned} F_{k:n}(x) &= \frac{n!}{(k-1)!(n-k)!}\sum_{l=0}^{n-k}\frac{\binom{n-k}{l}(-1)^l}{k+l}\left[1-\frac{\log\left\{1-\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}\right\}}{\log(1-\lambda)}\right]^{k+l} \\ &= \frac{n!}{(k-1)!(n-k)!}\sum_{l=0}^{n-k}\frac{\binom{n-k}{l}(-1)^l}{k+l}F_G(x;\beta,\lambda,k+l), \end{aligned}$$

where G follows an exponentiated Lindley logarithmic (ELL) distribution with parameters β, λ and $k + l$. Thus, the cdf of the k^{th} order statistic can be expressed as a linear combination of the cdf of the ELL distribution with parameters β, λ and $k + l$.

4.6 Measures of Uncertainty

In this section, the measures of uncertainty including Shannon [20],[21] entropy and Rényi [19] entropy of the LL distribution are presented. The concept of entropy plays an important role in information theory. The entropy of a random variable is defined using its probability distribution and can be considered as a good measure of randomness or uncertainty.

4.6.1 Shannon Entropy

The Shannon entropy of the LL distribution is defined by $H[f_{LL}] = E[-\log(f_{LL}(X;\beta,\lambda))]$. Thus, we have

$$\begin{aligned} H[f_{LL}] &= E\left[-\log\left\{\frac{-\lambda\beta^2(1+X)e^{-\beta X}}{(\beta+1)\log(1-\lambda)}\left(1-\lambda\left(\frac{1+\beta+\beta X}{\beta+1}\right)e^{-\beta X}\right)^{-1}\right\}\right] \\ &= -\log\left(\frac{-\lambda\beta^2}{(\beta+1)\log(1-\lambda)}\right) - E[\log(1+X)] + \beta E[X] \\ &\quad + E\left[\log\left\{1-\lambda\left(\frac{1+\beta+\beta X}{\beta+1}\right)e^{-\beta X}\right\}\right]. \end{aligned} \tag{4.9}$$

We evaluate $E[\log(1+X)]$, $E[X]$, and $E\left[\log\left(1-\lambda\left(\frac{1+\beta+\beta X}{\beta+1}\right)e^{-\beta X}\right)\right]$ separately . Consider the series expansion

$$\log(1+z) = \sum_{q=1}^{\infty}\frac{(-1)^{q+1}z^q}{q} \quad \text{for } \mid z \mid < 1. \tag{4.10}$$

Using the series expansion in Equation (4.10) with $z = X$ and Equation (4.6), we have

$$E[\log(1+X)] = \sum_{q=1}^{\infty}\frac{(-1)^{q+1}}{q}E[X^q] = \sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{q+1}\lambda^n\beta^2L_1(\beta,n,q)}{q\log(1-\lambda)(\beta+1)^n}. \tag{4.11}$$

From Equation (4.6), we have

$$E[X] = \sum_{n=1}^{\infty}\frac{\lambda^n\beta^2L_1(\beta,n,1)}{\log(1-\lambda)(\beta+1)^n}. \tag{4.12}$$

Consider $E\left[\log\left(1-\lambda\left(\frac{1+\beta+\beta X}{\beta+1}\right)e^{-\beta X}\right)\right]$ and let $V=\left[1-\lambda\left(\frac{1+\beta+\beta X}{\beta+1}\right)e^{-\beta X}\right]$. Note that $\log(V)=\log[1+(V-1)]$. Using Equation (4.10) with $z=(V-1)$, we obtain

$$\log(V)=\sum_{p=1}^{\infty}\frac{(-1)^{p+1}(V-1)^p}{p}=-\sum_{p=1}^{\infty}\frac{(1-V)^p}{p}.$$

Using the series expansion in Equation (2.9), we have

$$\begin{aligned}E[\log(V)] &= \sum_{p=1}^{\infty}\sum_{r=0}^{p}\binom{p}{r}\frac{(-1)^{r+1}E[V^r]}{p}\\ &= \sum_{p=1}^{\infty}\sum_{r=0}^{p}\binom{p}{r}\frac{(-1)^{r+1}}{p}E\left[\left(1-\lambda\left(\frac{1+\beta+\beta X}{\beta+1}\right)e^{-\beta X}\right)^r\right]\\ &= \sum_{p=1}^{\infty}\sum_{r=0}^{p}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t}\binom{p}{r}\binom{r}{s}\binom{s}{t}\binom{t}{u}\frac{(-1)^{r+s+1}\lambda^s\beta^t}{p(\beta+1)^s}E\left[X^u e^{-\beta s X}\right].\end{aligned} \tag{4.13}$$

Note that $E\left[X^u e^{-\beta s X}\right]=\sum_{v=0}^{\infty}\frac{(-1)^v\beta^v s^v E[X^{v+u}]}{v!}$. Thus, we have

$$E[\log(V)]=\sum_{p=1}^{\infty}\sum_{r=0}^{p}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t}\sum_{v=0}^{\infty}\binom{p}{r}\binom{r}{s}\binom{s}{t}\binom{t}{u}\frac{(-1)^{r+s+v+1}\lambda^s s^v\beta^{t+v}}{pv!(\beta+1)^s}E[X^{u+v}].$$

Using Equation (4.6), we obtain

$$\begin{aligned}E[\log(V)] &= \sum_{p=1}^{\infty}\sum_{r=0}^{p}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t}\sum_{v=0}^{\infty}\sum_{n=1}^{\infty}\binom{p}{r}\binom{r}{s}\binom{s}{t}\binom{t}{u}\\ &\quad\times\frac{(-1)^{r+s+v+1}\lambda^{s+n}s^v\beta^{t+v+2}L_1(\beta,n,u+v)}{pv!\log(1-\lambda)(\beta+1)^{s+n}}.\end{aligned} \tag{4.14}$$

Substituting equations (4.11), (4.12), and (4.14) in Equation (4.9), we have

$$\begin{aligned}H[f_{LL}] &= -\log\left(\frac{-\lambda\beta^2}{(\beta+1)\log(1-\lambda)}\right)-\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{q+1}\lambda^n\beta^2L_1(\beta,n,q)}{q\log(1-\lambda)(\beta+1)^n}\\ &\quad+\sum_{n=1}^{\infty}\frac{\lambda^n\beta^3L_1(\beta,n,1)}{\log(1-\lambda)(\beta+1)^n}+\sum_{p=1}^{\infty}\sum_{r=0}^{p}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t}\sum_{v=0}^{\infty}\sum_{n=1}^{\infty}\binom{p}{r}\binom{r}{s}\binom{s}{t}\binom{t}{u}\\ &\quad\times\frac{(-1)^{r+s+v+1}\lambda^{s+n}s^v\beta^{t+v+2}L_1(\beta,n,u+v)}{pv!\log(1-\lambda)(\beta+1)^{s+n}}.\end{aligned} \tag{4.15}$$

4.6.2 Rényi Entropy

Rényi entropy is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v)=\frac{1}{1-v}\log\left(\int_0^\infty f_{LL}^v(x;\beta,\lambda)dx\right);v\neq 1,v>0. \tag{4.16}$$

Note that Rényi entropy tends to Shannon entropy as $v\to 1$.

$$\begin{aligned}I_R(v) &= \frac{1}{1-v}\log\left(\int_0^\infty\left[\frac{-\lambda\beta^2(1+x)e^{-\beta x}}{(\beta+1)\log(1-\lambda)}\left\{1-\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}\right\}^{-1}\right]^v dx\right)\\ &= \frac{1}{1-v}\log\left\{\left[\frac{-\lambda\beta^2}{(\beta+1)\log(1-\lambda)}\right]^v\int_0^\infty(1+x)^v e^{-\beta v x}\right.\\ &\quad\left.\times\left[1-\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}\right]^{-v}dx\right\}.\end{aligned}$$

Consider the evaluation of the integral part, that is

$$\int_0^\infty(1+x)^v e^{-\beta v x}\left[1-\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}\right]^{-v}dx.$$

Consider the series expansion

$$(1-z)^{-v}=\sum_{k=0}^\infty\binom{k+v-1}{k}z^k, \tag{4.17}$$

where $\mid z\mid<1$ and $k>0$. Applying series expansions in equations (4.10) and (2.9) we obtain

$$\begin{aligned}&\int_0^\infty(1+x)^v e^{-\beta v x}\left[1-\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}\right]^{-v}dx\\ &= \sum_{k=0}^\infty\sum_{l=0}^k\sum_{m=0}^\infty\binom{k+v-1}{k}\binom{k}{l}\binom{l+v}{m}\frac{\lambda^k\beta^l}{(\beta+1)^k}\int_0^\infty x^m e^{-\beta(k+v)x}dx.\end{aligned}$$

Consider the evaluation of the integral part $\int_0^\infty x^m e^{-\beta(k+v)x}dx$. Let $u=\beta(k+v)x$, then $x=u/(\beta(k+v))$ and $dx=du/(\beta(k+v))$. Using the definition of the complete gamma function, we obtain

$$\int_0^\infty x^m e^{-\beta(k+v)x}dx=\frac{1}{\beta^{m+1}(k+v)^{m+1}}\int_0^\infty u^m e^{-u}du\frac{\Gamma(m+1)}{\beta^{m+1}(k+v)^{m+1}}.$$

We thus have

$$\begin{aligned}
&\int_0^\infty (1+x)^v e^{-\beta v x}\left[1-\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}\right]^{-v} dx \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{\infty}\binom{k+v-1}{k}\binom{k}{l}\binom{l+v}{m}\frac{\lambda^k}{\beta^{m-l+1}(\beta+1)^k(k+v)^{m+1}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
I_R(v) &= \frac{1}{1-v}\log\left[\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{\infty}\binom{k+v-1}{k}\binom{k}{l}\binom{l+v}{m}\right. \\
&\quad \left.\times\frac{-\lambda^{v+k}\beta^{l+2v-m-1}\Gamma(m+1)}{\log(1-\lambda)(\beta+1)^{k+v}(k+v)^{m+1}}\right]. \qquad (4.18)
\end{aligned}$$

4.7 Reliability

The reliability of a component which has a random strength X_1 that is subjected to a random stress X_2 can be measured by the stress-strength parameter $R = P(X_1 > X_2)$. If $(X_2 > X_1)$. When the stress applied to it exceeds the strength, this results in the component failing. Reliability is widely applicable in areas where the lifetime of a component is of importance.

Let X_1 $LL(x;\beta_1,\lambda_1)$ and X_2 $LL(x;\beta_2,\lambda_2)$ be independent random variables. Then the stress-strength parameter is defined as

$$\begin{aligned}
R &= P(X_1 > X_2) = \int_0^\infty f_{X_1}(x;\beta_1,\lambda_1)F_{X_2}(x;\beta_2,\lambda_2)dx \\
&= 1-\int_0^\infty \frac{\lambda_1\beta_1^2(1+x)e^{-\beta_1 x}\log\left(1-\lambda_2\left(\frac{1+\beta_2+\beta_2 x}{\beta_2+1}\right)e^{-\beta_2 x}\right)}{(\beta_1+1)\log(1-\lambda_1)\log(1-\lambda_2)} \\
&\quad \times\left[\lambda_1\left(\frac{1+\beta_1+\beta_1 x}{\beta_1+1}\right)e^{-\beta_1 x}-1\right]^{-1} dx,
\end{aligned}$$

We will consider the evaluations of $\log\left[1-\lambda_2\left(\frac{1+\beta_2+\beta_2 x}{\beta_2+1}\right)e^{-\beta_2 x}\right]$ and $\left[\lambda_1\left(\frac{1+\beta_1+\beta_1 x}{\beta_1+1}\right)e^{-\beta_1 x}-1\right]^{-1}$, separately. Following a similar procedure that we used to derive Equation (4.13), we obtain

$$\log\left[1-\lambda_2\left(\frac{1+\beta_2+\beta_2 x}{\beta_2+1}\right)e^{-\beta_2 x}\right] = \sum_{p=1}^{\infty}\sum_{r=0}^{p}\sum_{s=0}^{r}\sum_{t=0}^{s}\binom{p}{r}\binom{r}{s}\binom{s}{t} \times\frac{(-1)^{r+s+1}\lambda_2^s\beta_2^t}{p(\beta_2+1)^s}(1+x)^t e^{-\beta_2 s x}. \tag{4.19}$$

Using series expansions in (4.17) and (2.9), we have

$$\left[\lambda_1\left(\frac{1+\beta_1+\beta_1 x}{\beta_1+1}\right)e^{-\beta_1 x}-1\right]^{-1} = -\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{\lambda_1^k\beta_1^l}{(\beta_1+1)^k}(1+x)^l e^{-\beta_1 k x}. \tag{4.20}$$

By substituting equations (4.19) and (4.20) in Equation (4.19), we obtain

$$R = 1-\frac{\lambda_1\beta_1^2(\beta_1+1)^{-1}}{\log(1-\lambda_1)\log(1-\lambda_2)}\sum_{p=1}^{\infty}\sum_{r=0}^{p}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{q=0}^{t+l+1}\binom{p}{r}\binom{r}{s}\binom{s}{t} \times\binom{t+l+1}{q}\frac{(-1)^{r+s}\lambda_1^k\beta_1^l\lambda_2^s\beta_2^t}{p(\beta_2+1)^s(\beta_1+1)^k}\int_0^{\infty}x^q e^{-\beta_1(k+1)x}e^{-\beta_2 s x}dx.$$

Note that, $e^{-\beta_2 s x}=\sum_{w=0}^{\infty}\frac{(-1)^w\beta_2^w s^w x^w}{w!}$. Thus,

$$R = 1-\frac{\lambda_1\beta_1^2(\beta_1+1)^{-1}}{\log(1-\lambda_1)\log(1-\lambda_2)}\sum_{p=1}^{\infty}\sum_{r=0}^{p}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{q=0}^{t+l+1}\sum_{w=0}^{\infty}\binom{p}{r}\binom{r}{s}\binom{s}{t} \times\binom{t+l+1}{q}\frac{(-1)^{r+s+w}\lambda_1^k\beta_1^l\lambda_2^s\beta_2^{t+w}s^w}{pw!(\beta_2+1)^s(\beta_1+1)^k}\int_0^{\infty}x^{q+w}e^{-\beta_1(k+1)x}dx.$$

Consider the evaluation of the integral part $\int_0^{\infty}x^{q+w}e^{-\beta_1(k+1)x}dx$.Let $u=\beta_1(k+1)x$, then $x=u/(\beta_1(k+1))$ and $dx=du/(\beta_1(k+1))$. Using the definition of the complete gamma function, we get

$$\int_0^{\infty}x^{q+w}e^{-\beta_1(k+1)x}dx=\frac{\Gamma(q+w+1)}{[\beta_1(k+1)]^{q+w+1}}.$$

Consequently,

$$R = 1-\frac{\lambda_1\beta_1^2(\beta_1+1)^{-1}}{\log(1-\lambda_1)\log(1-\lambda_2)}\sum_{p=1}^{\infty}\sum_{r=0}^{p}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{q=0}^{t+l+1}\sum_{w=0}^{\infty}\binom{p}{r}\binom{r}{s}\binom{s}{t} \times\binom{t+l+1}{q}\frac{(-1)^{r+s+w}\lambda_1^k\beta_1^l\lambda_2^s\beta_2^{t+w}s^w\Gamma(q+w+1)}{pw!(\beta_2+1)^s(\beta_1+1)^k[\beta_1(k+1)]^{q+w+1}}.$$

4.8 Mean Deviations, Lorenz and Bonfferoni Curves

In this section, the mean deviations, Lorenz and Bonferroni curves of the LL distribution are presented.

The mean deviation about the mean, $D(\mu)$ and the mean deviation about the median, $D(M)$ of the LL distribution are respectively given as follows:
From Equation (2.23),

$$D(\mu) = 2\mu F_{LPS}(\mu) - 2\mu - 2\sum_{n=1}^{\infty} \frac{\lambda^n \beta^2 L_2(\beta, n, 1, \mu)}{\log(1-\lambda)(\beta+1)^n}, \tag{4.21}$$

and from Equation (2.24),

$$D(M) = -\mu - 2\sum_{n=1}^{\infty} \frac{\lambda^n \beta^2 L_2(\beta, n, 1, M)}{\log(1-\lambda)(\beta+1)^n}, \tag{4.22}$$

where $\mu = E(X)$ and $M = Median(X) = F_{LL}^{-1}(1/2)$.

The Lorenz and Bonferroni curves for the LL distribution are respectively given as follows:
From Equation (2.26),

$$L(p) = \frac{1}{\mu}\left(\mu + \sum_{n=1}^{\infty} \frac{\lambda^n \beta^2 L_2(\beta, n, 1, q)}{\log(1-\lambda)(\beta+1)^n}\right), \tag{4.23}$$

and from Equation (2.27),

$$B(p) = \frac{1}{p\mu}\left(\mu + \sum_{n=1}^{\infty} \frac{\lambda^n \beta^2 L_2(\beta, n, 1, q)}{\log(1-\lambda)(\beta+1)^n}\right), \tag{4.24}$$

where $q = F_{LL}^{-1}(p)$.

4.9 Maximum Likelihood Estimation

Let $x_1, x_2, \ldots, x_n$ be n observations of a random sample from LL distribution and $\mathbf{\Theta} = (\beta, \lambda)^T$ be the unknown parameter vector. From equation (4.2), the log-likelihood function of LL distribution is given by

$$\begin{aligned} l_n(\beta, \lambda) &= n\log(\lambda) + 2n\log(\beta) - n\log(\beta+1) - n\log[-\log(1-\lambda)] \\ &\quad + \sum_{i=1}^{n} \log(1+x_i) - \beta\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \log\left[V(x_i)\right], \end{aligned} \tag{4.25}$$

where $V(x_i) = \left[1 - \lambda\left(\frac{1+\beta+\beta x_i}{\beta+1}\right)e^{-\beta x_i}\right]$.
The partial derivatives of $V(x_i)$ with respect to the parameters β and λ are given by

$$\frac{\partial V(x_i)}{\partial \beta} = \lambda x e^{-\beta x_i}\left[\left(\frac{1+\beta+\beta x_i}{\beta+1}\right) - \frac{1}{(\beta+1)^2}\right], \tag{4.26}$$

and

$$\frac{\partial V(x_i)}{\partial \lambda} = -\left(\frac{1+\beta+\beta x_i}{\beta+1}\right) e^{-\beta x_i}. \tag{4.27}$$

The associated score function is $U_n(\mathbf{\Theta}) = \left(\frac{\partial l_n}{\partial \beta}, \frac{\partial l_n}{\partial \lambda}\right)^T$, where $\frac{\partial l_n}{\partial \beta}$ and $\frac{\partial l_n}{\partial \lambda}$ are the partial derivatives of the log-likelihood function given by

$$\frac{\partial l_n}{\partial \beta} = \frac{2n}{\beta} - \frac{n}{\beta+1} - \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{\partial V(x_i)/\partial \beta}{V(x_i)}, \tag{4.28}$$

and

$$\frac{\partial l_n}{\partial \lambda} = \frac{n}{\lambda} - \frac{n}{(1-\lambda)\log(1-\lambda)} - \sum_{i=1}^{n} \frac{\partial V(x_i)/\partial \lambda}{V(x_i)}, \tag{4.29}$$

respectively, where $\frac{\partial V(x_i)}{\partial \beta}$ and $\frac{\partial V(x_i)}{\partial \lambda}$ are given in equations (4.26) and (4.27), respectively.

The maximum likelihood estimates of $\mathbf{\Theta}$ can be obtained by solving the non-linear system $U_n(\mathbf{\Theta}) = 0$. Since the equations (4.28) and (4.29) are not in closed form, the solutions can be found by using a numerical method such as the Newton-Raphson procedure. The Fisher information matrix (FIM) of LL distribution is the 2×2 symmetric matrix given by

$$\mathbf{I}(\mathbf{\Theta}) = \begin{bmatrix} \mathrm{I}_{\beta\beta} & \mathrm{I}_{\beta\lambda} \\ \mathrm{I}_{\lambda\beta} & \mathrm{I}_{\lambda\lambda} \end{bmatrix},$$

where elements $\mathbf{I}_{ij}(\mathbf{\Theta}) = -E_{\mathbf{\Theta}}\left[\frac{\partial^2 \log(f_{LL}(X;\mathbf{\Theta}))}{\partial\theta_i\partial\theta_j}\right]$. The elements of the FIM can be obtained by considering the second order partial derivatives of (4.25). The second order partial derivatives of (4.25) with respect to the parameters β and λ are given by

$$\frac{\partial^2 l_n}{\partial \beta^2} = \frac{-2n}{\beta^2} + \frac{n}{(\beta+1)^2} - \sum_{i=1}^{n} \left\{\frac{V(x_i)[\partial^2 V(x_i)/\partial\beta^2] - [\partial V(x_i)/\partial\beta]^2}{[V(x_i)]^2}\right\}, \tag{4.30}$$

$$\frac{\partial^2 l_n}{\partial \beta \partial \lambda} = \frac{\partial^2 l_n}{\partial \lambda \partial \beta} = x e^{-\beta x_i}\left[\left(\frac{1+\beta+\beta x_i}{\beta+1}\right) - \frac{1}{(\beta+1)^2}\right], \tag{4.31}$$

and

$$\frac{\partial^2 l_n}{\partial \lambda^2} = \frac{-n}{\lambda^2} - \frac{n}{[(1-\lambda)\log(1-\lambda)]^2} + \sum_{i=1}^{n} \left[\frac{\partial V(x_i)/\partial \lambda}{V(x_i)}\right]^2, \tag{4.32}$$

where $\frac{\partial^2 V(x_i)}{\partial \beta^2} = \lambda x e^{-\beta x}\left[\frac{2x}{(\beta+1)^2} - \frac{x(1+\beta+\beta x)}{\beta+1} + \frac{2}{(\beta+1)^3}\right]$, $\frac{\partial V(x_i)}{\partial \beta}$ and $\frac{\partial V(x_i)}{\partial \lambda}$ are given in equations (4.26) and (4.27), respectively. These elements can be numerically obtained by MATLAB or MAPLE software.

5 Generation Algorithms and Monte Carlo Simulation Study

In this section, the algorithms for generating random data from LL distribution are given. A simulation study was also conducted to check the performance and accuracy of maximum likelihood estimates of the LL model parameters.

5.1 Generation Algorithms

In this subsection, we present two different algorithms that can be used to generate random data from LL distribution.

(a) The first algorithm is developed by taking the mixture form of the Lindley (L) distribution. The density function of the Lindley distribution can be defined as a two-component mixture of an exponential distribution with scale β, and a gamma distribution with shape 2 and scale β, using a mixing proportion $p = \beta/(\beta + 1)$. That is,

$$\begin{aligned} f_L(x) &= \frac{\beta^2}{\beta+1}(1+x)e^{-\beta x} \\ &= pf_1(x) + (1-p)f_2(x), \end{aligned}$$

where $p = \beta/(\beta + 1)$, $f_1(x) = \beta e^{-\beta x}$ and $f_2(x) = \beta^2 x e^{-\beta x}$ for $\beta > 0$ and $x > 0$.

Algorithm I (Mixture form of the Lindley distribution)

1. Generate $L_i \sim$ logarithmic (λ), $i = 1, \ldots, n$.
2. Generate $U_{i,j} \sim$ Uniform $(0, 1)$, $j = 1, \ldots, L_i$.
3. Generate $E_{i,j} \sim$ Exponential (β), $j = 1, \ldots, L_i$.
4. Generate $G_{i,j} \sim$ Gamma $(2, \beta)$, $j = 1, \ldots, L_i$.
5. If $U_{i,j} \leqslant p = \frac{\beta}{\beta+1}$, then set $X_{i,j} = E_{i,j}$, otherwise set $X_{i,j} = G_{i,j}$, $j = 1, \ldots, L_i$.
6. Set Y_i=min$(X_{i,1}, \ldots, X_{i,L_i})$, $i = 1, \ldots, n$.

(b) The second algorithm is based on generating random data from the inverse CDF in Equation (4.3) of the LL distribution.

Algorithm II(Inverse CDF)

1. Generate $U_i \sim$ Uniform $(0, 1)$, $i = 1, \ldots, n$.

2. Set
   $X_i = -1 - \frac{1}{\beta} - \frac{1}{\beta} W\left(-\frac{(\beta+1)(1-e^{(1-U_i)\log(1-\lambda)})}{\lambda e^{\beta+1}}\right)$, for $i = 1, \ldots, n$,
   where $W(.)$ is the negative branch of the Lambert W function.

5.2 Monte Carlo Simulation Study

In this subsection, we study the performance and accuracy of maximum likelihood estimates of the LL model parameters by conducting various simulations for different combinations of 7 sample sizes with two sets of parameter values. Algorithm II was used to generate random data from the LL distribution. The simulation study was repeated $N = 5,000$ times each with samples of size $n = 25, 50, 75, 100, 200, 400, 600$ combined with parameter values $I : \beta = 0.1, \lambda = 0.7$ and $II : \beta = 2.5, \lambda = 0.7$. Four quantities were computed in this simulation study:(i) Average bias of the MLE $\hat{\vartheta}$ of the parameter $\vartheta = \beta, \lambda : \frac{1}{N}\sum_{i=1}^{N}(\hat{\vartheta} - \vartheta)$;$(ii)$ Root mean squared error (RMSE) of the MLE $\hat{\vartheta}$ of the parameter $\vartheta = \beta, \lambda : [\frac{1}{N}\sum_{i=1}^{N}(\hat{\vartheta} - \vartheta)^2]^{0.5}$;$(iii)$ Coverage probability (CP) of 95% confidence intervals of the parameter $\vartheta = \beta, \lambda$;$(iv)$ Average width (AW) of 95% confidence intervals of the parameter $\vartheta = \beta, \lambda$.

Table 5.1 presents the Average Bias, RMSE, CP and AW values of the parameters β and λ for different sample sizes. According to the results, it can be concluded that as the sample size n increases, the RMSEs decrease toward zero. We also observe that for all the parameters, the biases decrease as the sample size n increases. The results show that the coverage probabilities of the confidence intervals are quite close to the nominal level of 95% and that the average confidence widths decrease as the sample size increases. Consequently, the MLE's and their asymptotic results can be used for estimating and constructing confidence intervals even for reasonably small sample sizes.

Table 5.1: Monte Carlo Simulation Results: Average Bias, RMSE, CP and AW

		I				II			
Parameter	n	Average Bias	RMSE	CP	AW	Average Bias	RMSE	CP	AW
β	25	0.00468	0.02262	0.98080	0.13859	0.16756	0.71375	0.98720	3.63908
	50	0.00289	0.01807	0.98080	0.09807	0.10053	0.52504	0.98220	2.50945
	75	0.00262	0.01610	0.98000	0.08024	0.09089	0.45331	0.98220	2.04865
	100	0.00192	0.01478	0.97880	0.06945	0.07244	0.40718	0.97460	1.75856
	200	0.00184	0.01205	0.96960	0.04924	0.04158	0.31182	0.96360	1.22790
	400	0.00101	0.00885	0.95740	0.03451	0.02955	0.22128	0.95560	0.86329
	600	0.00063	0.00720	0.95400	0.02804	0.02160	0.18137	0.95240	0.70262
λ	25	-0.10488	0.28690	0.95260	2.26965	-0.08595	0.26145	0.98160	1.98164
	50	-0.08667	0.26303	0.95020	1.61974	-0.07816	0.24433	0.93800	1.38358
	75	-0.08384	0.24889	0.93420	1.33696	-0.07781	0.23321	0.93560	1.14433
	100	-0.06867	0.23041	0.92120	1.13093	-0.06477	0.21019	0.92480	0.94998
	200	-0.05515	0.19298	0.92560	0.77673	-0.04021	0.16403	0.92660	0.62195
	400	-0.03259	0.13717	0.93500	0.50956	-0.02591	0.11198	0.94780	0.41789
	600	-0.02087	0.10701	0.94380	0.39892	-0.01637	0.08733	0.94860	0.32971

6 Applications

In this section, we present examples to illustrate the flexibility and superiority of the LL distribution in modeling real data. We fit the density functions of LL distribution in Equation (4.1) and its sub-model the Lindley (L) distribution in Equation (1.2).

We also compare the LL distribution with the exponential logarithmic (EL) distribution by Tahmasbi and Rezaei [22], the exponential geometric (EG) by Adamidis and Loukas [1], the exponential Poisson (EP) distribution by Kus [10], the Weibull (W) distribution and the Gamma (G) distribution. The density functions EL, EG, EP, W and G distributions are respectively given by

$$f_{EL}(x;\beta,\lambda) = \frac{\beta(1-\lambda)e^{-\beta x}}{\log(\lambda)[(1-\lambda)e^{-\beta x}-1]}, \quad x>0, \beta>0, 0<\lambda<1, \tag{6.1}$$

$$f_{EG}(x;\beta,\lambda) = \frac{\beta(1-\lambda)e^{-\beta x}}{(1-\lambda e^{-\beta x})^2}, \quad x>0, \beta>0, 0<\lambda<1, \tag{6.2}$$

$$f_{EP}(x;\beta,\lambda) = \frac{\lambda\beta e^{-\lambda-\beta x+\lambda e^{-\beta x}}}{1-e^{-\lambda}}, \quad x>0, \beta>0, \lambda>0, \tag{6.3}$$

$$f_{W}(x;\beta,\lambda) = \lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}}, \quad x>0, \beta>0, \lambda>0, \tag{6.4}$$

and

$$f_{G}(x;\beta,\lambda) = \frac{\beta^{\lambda}x^{\lambda-1}e^{-\beta x}}{\Gamma(\lambda)}, \quad x>0, \beta>0, \lambda>0. \tag{6.5}$$

For each data set, the estimates of the parameters of the distributions, standard errors (in parentheses), Akaike Information Criterion ($AIC = 2p - 2\log(L)$), Corrected Akaike Information Criterion ($AICC = AIC + \frac{2p(p+1)}{n-p-1}$), Bayesian Information Criterion ($BIC = p\log(n) - 2\log(L)$), where $L = L(\hat{\Theta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are obtained. Moreover, the goodness-of-fit statistics: Cramer von Mises (W^*), Anderson-Darling (A^*), and sum of squares (SS) from the probability plots are also presented.

When comparing models, the model with the smallest AIC is considered to be the best fit model for a given data set. However, when n is small or the number of parameters is large, the chance of selecting a model with many parameters as the best model will be increased using AIC. In such situations, it is strongly recommended to use AICC to select the best model. Note that when n is too large, the AICC converges to AIC. When selecting the best model for a given data set based on the values of SS, the model with the smallest SS is considered as the best fit model.

Plots of the fitted densities, the histogram of the data and probability plots are given for each example. For the probability plot, we plotted $F_{LL}(x_{(j)}; \hat{\beta}, \hat{\lambda}$ against $\dfrac{j - 0.375}{n + 0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares

$$SS = \sum_{j=1}^{n} \left[F_{LL}(x_{(j)}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2.$$

The goodness-of-fit statistics W^* and A^*, are also presented in the tables. These statistics can be used to verify which distribution provides the best fit to the data. In general, the smaller the values of W^* and A^*, the better the fit.

Since the Lindley distribution is a sub-model of the LL distribution, we use the likelihood ratio (LR) test to compare the fits of the two distributions. For example, to test $\beta = 1$, the LR statistic is $\omega^* = 2[\ln(L(\hat{\beta}, \hat{\lambda})) - \ln(L(1, \tilde{\lambda}))]$, where $\hat{\beta}$ and $\hat{\lambda}$ are the unrestricted estimates, and $\tilde{\lambda}$ is the restricted estimate. The LR test rejects the null hypothesis if $\omega^* > \chi^2_{\epsilon}$, where χ^2_{ϵ} denote the upper $100\epsilon\%$ point of the χ^2 distribution with 1 degree of freedom.

6.1 Breastfeeding Data

The first data set is a subset of a breastfeeding study from the National Longitudinal Survey of Youth. The complete data set is given by Klein and Moeschberger [9]. The data set consists of the times to weaning for 927 children of white-race mothers who chose to breast feed their children. The duration of the breast feeding was measured in weeks.

Estimates of the parameters of LL distribution (standard error in parentheses), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC) are given in Table 6.1 for breastfeeding data. The sum of squares (SS) and the goodness-of-fit statistics W^* and A^* are also given.

Table 6.1: Estimates of Models for Breastfeeding Data

	Estimates		Statistics						
Distribution	$\hat{\beta}$	$\hat{\lambda}$	$-2\log L$	AIC	AICC	BIC	SS	W^*	A^*
LL	0.05741	0.9804	6979.1 q	6983.1	6983.1	6992.7	0.44734	0.54892	4.45339
	(0.0034)	(0.0043)							
EL	0.05162	0.4886	7002.9	7006.9	7007.0	7016.6	0.50385	0.78992	5.71575
	(0.0033)	(0.0988)							
EG	0.05055	0.3237	7002.7	7006.7	7006.7	7016.4	0.49585	0.78489	5.69381
	(0.0038)	(0.0793)							
EP	0.005374	12.6047	7006.1	7010.1	7010.1	7019.8	0.68138	0.90504	6.43375
	(0.0021)	(4.4768)							
W	0.9610	0.07017	7012.2	7016.2	7016.2	7025.8	0.81253	0.95875	6.68162
	(0.0241)	(0.0059)							
L	0.1172	-	7272.9	7274.9	7274.9	7279.7	10.33162	1.40222	9.34569
	(0.0027)	-							
G	0.06078	0.9832	7014.6	7018.6	7018.6	7028.3	1.11402	0.99753	6.90138
	(0.0032)	(0.0402)							

Note. Standard errors are in parentheses.

Plots of the fitted densities and histogram, observed probability versus predicted probability for the breastfeeding data data are given in Figure 6.1.

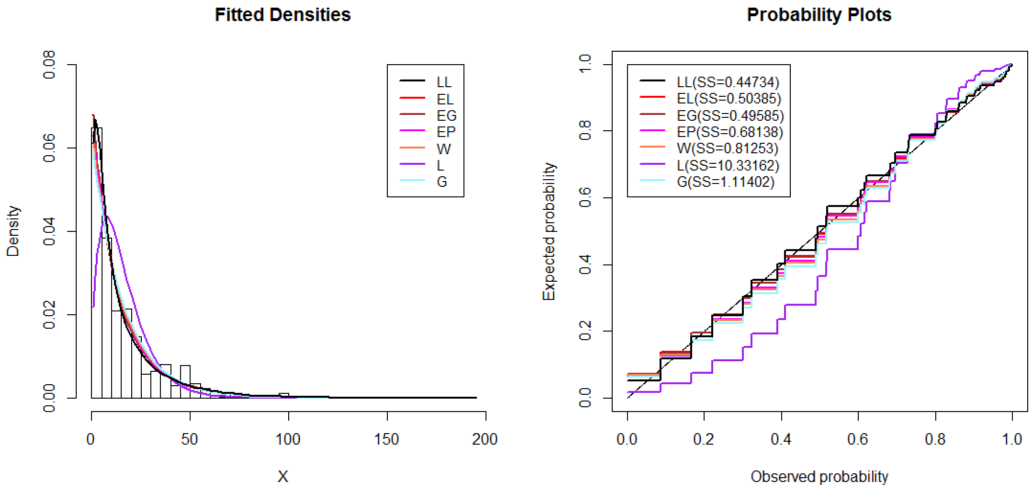

Figure 6.1: Histogram, fitted density and probability plots for breastfeeding data

For the hypothesis H_0: L against H_a: LL the LR test statistic is 293.8 (p-value=7.389×10^{-66}). Since the p-value is small, we reject H_0. We conclude that there is a significant difference between L and LL distributions. The values of the statistics AIC, AICC, and BIC show that the LL distribution performs better than other distributions it is compared to for the breastfeeding data. The value of the sum of squares (SS=0.44734) from the probability plots in Figure 6.1 is smallest for the LL distribution. Consequently, there is clear evidence, based on the goodness-of-fit statistics W^* and A^* that the LL distribution provides the best fit for the breastfeeding data.

The asymptotic covariance matrix of the MLEs of the LL model parameters, which is the inverse of the observed Fisher information matrix $\mathbf{I}_n^{-1}(\hat{\mathbf{\Theta}})$ is given by:

$$\begin{bmatrix} 0.000011 & -0.0001 \\ -0.0001 & 0.000019 \end{bmatrix},$$

and the 95% confidence intervals for the model parameters β and λ are given by 0.05741 ± 0.0065 and 0.9804 ± 0.0085 respectively.

6.2 Plasma Concentrations Data

The second data set consists of real data taken from the R base package. It is located in the Indometh object. The data set contains 66 observations of plasma concentrations of indomethicin (mcg/ml).

Estimates of the parameters of LL distribution (standard error in parentheses), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC) are given in Table 6.2 for plasma concentrations data. The sum of squares (SS) and the goodness-of-fit statistics W^* and A^* are also given.

Table 6.2: Estimates of Models for Plasma Concentration Data

	Estimates		Statistics						
Distribution	$\hat{\beta}$	$\hat{\lambda}$	$-2\log L$	AIC	AICC	BIC	SS	W*	A*
LL	1.6572 (0.3325)	0.7596 (0.1758)	60.9	64.9	65.1	69.3	0.15874	0.23068	1.48439
EL	1.3254 (0.3171)	0.3754 (0.2711)	61.1	65.1	65.3	69.5	0.15925	0.23235	1.49235
EG	1.2949 (0.3691)	0.4080 (0.2598)	61.2	65.2	65.4	69.5	0.16284	0.23402	1.50117
EP	1.3178 (0.4003)	0.9791 (0.9636)	61.4	65.4	65.6	69.7	0.17784	0.23868	1.52920
W	0.9546 (0.0904)	1.6857 (0.2078)	62.5	66.5	66.7	70.9	0.22627	0.25085	1.60805
L	2.2152 (0.2208)	- -	64.3	66.3	66.3	68.5	0.36964	0.26346	1.68826
G	1.6513 (0.3257)	0.9773 (0.1495)	62.7	66.7	66.9	71.1	0.26865	0.25365	1.62600

Note. Standard errors are in parentheses.

Plots of the fitted densities and histogram, observed probability versus predicted probability for the plasma concentrations data are given in Figure 6.2.

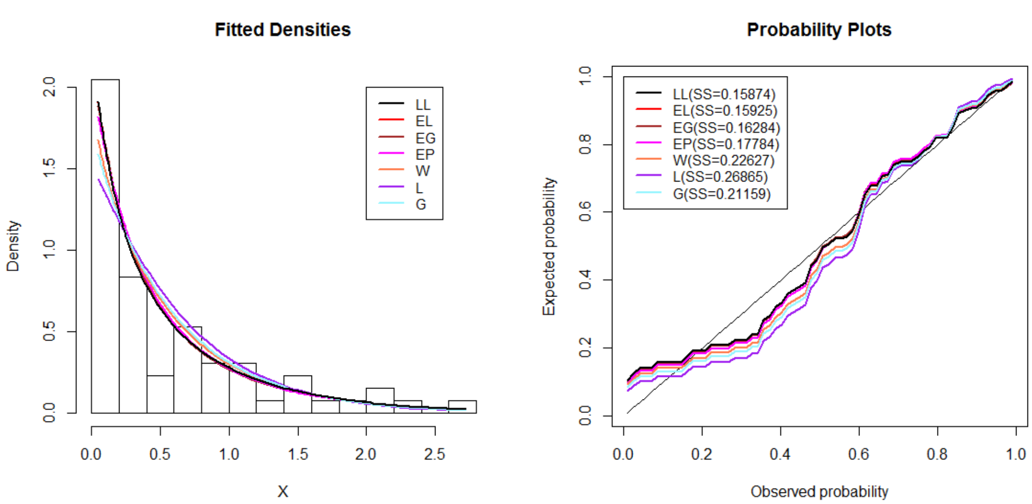

Figure 6.2: Histogram, fitted density and probability plots for plasma concentrations data

For the hypothesis H_0: L against H_a: LL the LR test statistic is 3.4 (p-value = 0.065196). We conclude that there is no significant differences between L and LL distributions at the 5% level. However, the result is significant at 10%. The value of BIC also shows that the L distribution is a better fit, whereas the goodness of fit-statistics W^* and A^* show that the LL distribution is better than its sub-model Lindley (L) distribution and all other distributions that were fitted to the data. The value of the sum of squares (SS=0.15874) from the probability plots in Figure 6.2 is smallest for the LL distribution. Consequently, there is clear evidence, based on the goodness-of-fit statistics W^* and A^* that the LL distribution provides the best fit for the plasma concentration data.

The asymptotic covariance matrix of the MLEs of the LL model parameters, which is the inverse of the observed Fisher information matrix $\mathbf{I}_n^{-1}(\hat{\mathbf{\Theta}})$ is given by:

$$\begin{bmatrix} 0.1106 & -0.04822 \\ -0.04822 & 0.03092 \end{bmatrix},$$

and the 95% confidence intervals for the model parameters β and λ are given by 1.6146 ± 0.6518 and 0.7374 ± 0.3446 respectively.

6.3 Air Conditioning System Data

The third data set consists of the number of successive failures for the air conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes. This data set was used by Proschan [18].
Estimates of the parameters of LL distribution (standard error in parentheses), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC) are given in Table 6.3 for the air conditioning system data. The sum of squares (SS) and the goodness-of-fit statistics W^* and A^* are also given.

Table 6.3: Estimates of Models for Air Conditioning System Data

	Estimates		Statistics						
Distribution	$\hat{\beta}$	$\hat{\lambda}$	$-2\log L$	AIC	AICC	BIC	SS	W*	A*
LL	0.008897 (0.0012)	0.9932 (0.0034)	2063.9	2067.9	2068.0	2074.4	0.02601	0.02192	0.16651
EL	0.008211 (0.0012)	0.3331 (0.1395)	2070.0	2074.0	2074.1	2080.5	0.07521	0.09774	0.62895
EG	0.007663 (0.0014)	0.4824 (0.1388)	2069.5	2073.5	2073.6	2080.0	0.05991	0.08444	0.55095
EP	0.007113 (0.0018)	1.5746 (0.8074)	2069.3	2073.3	2073.4	2079.8	0.05333	0.08398	0.54977
W	0.9109 (0.0505	0.01698 (0.0044)	2073.5	2077.5	2077.6	2084.0	0.14302	0.15940	0.99691
L	0.02149 (0.0011)	- -	2165.3	2167.3	2167.3	2170.5	3.15895	0.22965	1.43466
G	0.009826 (0.0012)	0.9047 (0.0815)	2075.2	2079.2	2079.3	2085.7	0.23089	0.19000	1.17913

Note. Standard errors are in parentheses.

Plots of the fitted densities and histogram, observed probability versus predicted probability for the air conditioning system data are given in Figure 6.3.

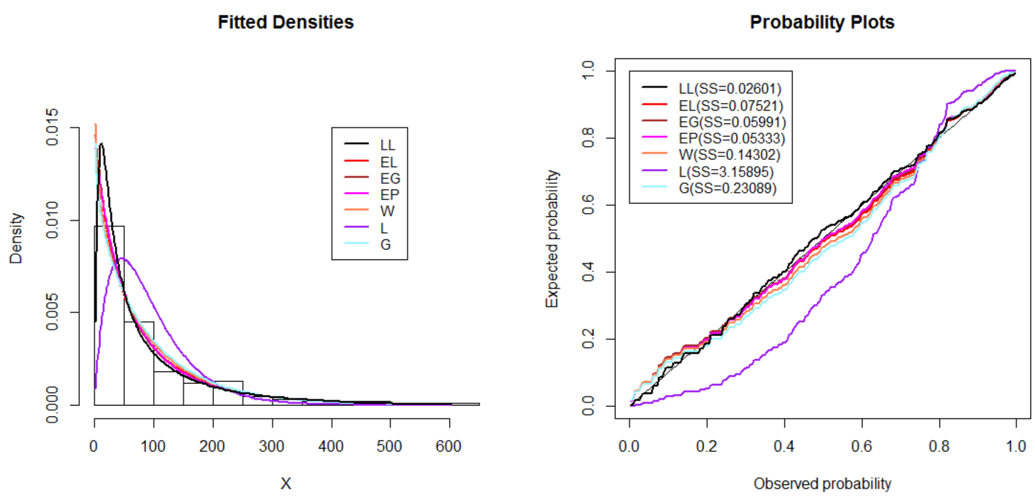

Figure 6.3: Histogram, fitted density and probability plots for air conditioning system data

For the hypothesis H_0: L against H_a: LL the LR test statistic is 101.4 (p-value=7.5163 $\times 10^{-24}$). Since the p-value is small, we reject the null hypothesis. We can conclude that there is significant difference between the L and LL distributions. The values of statistics AIC, AICC, and BIC show that the LL distribution performs the best for the air conditioning data. The value of the sum of squares (SS=0.02601) from the probability plots in Figure 6.3 is smallest for the LL distribution. Consequently, there is clear evidence, based on the goodness-of-fit statistics W^* and A^* that the LL distribution provides the best fit for the air conditioning system data.

The asymptotic covariance matrix of the MLEs of the LL model parameters, which is the inverse of the observed Fisher information matrix $\mathbf{I}_n^{-1}(\hat{\boldsymbol{\Theta}})$ is given by:

$$\begin{bmatrix} 1.459 \times 10^{-6} & -3.29 \times 10^{-6} \\ -3.29 \times 10^{-6} & 1.1 \times 10^{-5} \end{bmatrix},$$

and the 95% confidence intervals for the model parameters β and λ are given by 0.008897 ± 0.00236 and 0.9932 ± 0.0065 respectively.

7 Concluding Remarks

We propose a new class of lifetime distributions called the Lindley power series (LPS) class of distributions. This class of distributions is a generalization of the Lindley (L) distribution and is obtained by compounding the Lindley (L) distribution and the power series class of distributions. The properties of the LPS distribution including reverse hazard function, hazard function, quantile function, moments, distribution of order statistics, mean deviations, Lorenz and Bonferroni curves and maximum likelihood estimates are presented. We introduce four special cases of the LPS distribution called Lindley binomial (LB) distribution, Lindley geometric (LG) distribution, Lindley Poisson (LB) distribution and Lindley logarithmic (LL) distribution. In addition to the properties of the general class, measures of uncertainty, and reliability for the LL distribution are obtained. The hazard function of the LL distribution has different shapes including monotonically increasing, monotonically decreasing, bathtub, upside down bathtub and increasing-decreasing-increasing shapes. We present a simulation study to exhibit the performance and accuracy of maximum likelihood estimates of the LL model parameters. Real data applications are also presented to illustrate the usefulness and applicability of the LL distribution.

References

[1] Adamidis, K., & Loukas, S. (1998). A lifetime distribution with decreasing failure rate. *Statistics & Probability Letters*, *39*(1), 35-42.

[2] Barreto-Souza, W., de Morais, A. L., & Cordeiro, G. M. (2011). The Weibull-geometric distribution. *Journal of Statistical Computation and Simulation*, *81*(5), 645-657.

[3] Bidram, H., & Nekoukhou, V. (2013). Double bounded Kumaraswamy-power series class of distributions. *SORT-Statistics and Operations Research Transactions*, *1*(2), 211-230.

[4] Corless, R. M., Gonnet, G. H., Hare, D. E., Jeffrey, D. J., & Knuth, D. E. (1996). On the LambertW function. *Advances in Computational Mathematics*, *5*(1), 329-359.

[5] Ghitany, M. E., Al-Mutairi, D. K., Balakrishnan, N., & Al-Enezi, L. J. (2013). Power Lindley distribution and associated inference. *Computational Statistics & Data Analysis*, *64*, 20-33.

[6] Ghitany, M. E., Atieh, B., & Nadarajah, S. (2008). Lindley distribution and its application. *Mathematics & Computers in Simulation*, *78*(4), 493-506.

[7] Gui, W., Zhang, S., & Lu, X. (2014). The Lindley-Poisson distribution in lifetime analysis and its properties. *Journal of Mathematics & Statistics*, 1063.

[8] Johnson, N. L., Kemp, A. W., & Kotz, S. (2005). *Univariate discrete distributions*. John Wiley & Sons.

[9] Klein, J. P., & Moeschberger, M. L. (2003). *Survival analysis: techniques for censored and truncated data*. Springer Science & Business Media.

[10] Kus, C. (2007). A new lifetime distribution. *Computational Statistics & Data Analysis*, *51*(9), 4497-4509.

[11] Lindley, D. V. (1958). Fiducial distributions and Bayes' theorem. *Journal of the Royal Statistical Society. Series B (Methodological)*, 102-107.

[12] Lu, W., & Shi, D. (2012). A new compounding life distribution: the Weibull Poisson distribution. *Journal of Applied Statistics*, *39*(1), 21-38.

[13] Mahmoudi, E., & Jafari, A. A. (2012). Generalized exponential power series distributions. *Computational Statistics & Data Analysis*, *56*(12), 4047-4066.

[14] Mahmoudi, E., & Jafari, A. A. (2014). The Compound Class of Linear Failure Rate-Power Series Distributions: Model, Properties and Applications. arXiv preprint arXiv:1402.5282.

[15] Morais, A. L., & Barreto-Souza, W. (2011). A compound class of Weibull and power series distributions. *Computational Statistics & Data Analysis*, *55*(3), 1410-1425.

[16] Noack, A. (1950). A class of random variables with discrete distributions. *The Annals of Mathematical Statistics*, 127-132.

[17] Pararai, M., Warahena-Liyanage, G. & Oluyede, B.O. (2015). Exponentiated Power Lindley Poisson Distribution: Properties and Applications. Submitted to *Communications in Statistics-Theory and Methods*.

[18] Proschan, F. (1963). Theoretical explanation of observed decreasing failure rate. *Technometrics*, *5*(3), 375-383.

[19] Rényi, A. L. F. R. P. E. D. (1961). On measures of entropy and information. In *Fourth Berkeley Symposium on Mathematical Statistics & Probability*, *1*, 547-561.

[20] Shannon, E. A. (1948a). A Mathematical Theory of Communication. *The Bell System Technical Journal*, *27*(10), 461-464.

[21] Shannon, E. A. (1948b). A Mathematical Theory of Communication. *The Bell System Technical Journal*, *27*(10), 379-423.

[22] Tahmasbi, R., & Rezaei, S. (2008). A two-parameter lifetime distribution with decreasing failure rate. *Computational Statistics & Data Analysis*, *52*(8), 3889-3901.

[23] Warahena-Liyanage, G., & Pararai, M. (2014). A Generalized Power Lindley Distribution with Applications. *Asian Journal of Mathematics and Applications*, *2014*, Article ID ama0169, 1-23.

[24] Zakerzadeh, H., & Mahmoudi, E. (2012). A new two parameter lifetime distribution: model and properties. arXiv preprint arXiv:1204.4248.

R Algorithms

In this appendix, we present R codes to compute cdf, pdf, moments, reliability, Rényi entropy, mean deviations, maximum likelihood estimates and variance-covariance matrix for the LL distribution.

```
#Define the cdf of LL distribution
F_cdf=function(x,beta,lambda){
aa1=(1+beta+beta*x)/(1+beta)
aa2=exp(-beta*x)
aa3=log(1-lambda)
aa4=log(1-lambda*aa1*aa2)
y=1-aa4/aa3
return(y)
}
```

```
#Define the pdf of LL distribution
f_pdf=function(x,beta,lambda){
aa1=(1+beta+beta*x)/(1+beta)
aa2=exp(-beta*x)
aa3=lambda*(beta**2)*(1+x)*aa2
aa4=(1+beta)*log(1-lambda)*(lambda*aa1*aa2-1)
y=aa3/aa4
return(y)
}

#Define the moments of LL distribution
moment=function(beta,lambda,r){
f=function(x,beta,lambda,r)
{(x^r)*(f_pdf(x,beta,lambda))}
y=integrate(f,lower=0,upper=Inf,subdivisions=100,
beta=beta,lambda=lambda,r=r)
return(y)
}

#Define the reliability of LL distribution
reliability=function(beta1,lambda1,beta2,lambda2){
f=function(x,beta1,lambda1,beta2,lambda2)
{f_pdf(x,beta1,lambda1)*(F_cdf(x,beta2,lambda2))}
y=integrate(f,lower=0,upper=Inf,subdivisions=100,
beta1=beta1,lambda1=lambda1,
beta2=beta2,lambda2=lambda2)
return(y)
}

#Define Mean Deviation about the mean of LL distribution
DU=function(beta,lambda){
mu=moment(beta,lambda,1)$ value
f=function(x,beta,lambda){(abs(x-mu)*f_pdf(x,beta,lambda)}
y=integrate(f,lower=0,upper=Inf,subdivisions=100
,beta=beta,lambda=lambda)
return(y)
}

#Define Mean Deviation about the median of LL distribution
DM=function(beta,lambda){
M=median(c(X)) #X is the data set
f=function(x,beta,lambda){(abs(x-M)*f_pdf(x,beta,lambda)}
y=integrate(f,lower=0,upper=Inf,subdivisions=100
,beta=beta,lambda=lambda)
```

```
return(y)
}

Define the Renyi entropy of LL distribution
t=function(beta,lambda,v){
f=function(x,beta,lambda,v)
{(f_pdf(x,beta,lambda))^(v)}
y=integrate(f,lower=0,upper=Inf,subdivisions=100
,beta=beta,lambda=lambda,v=v)$ value
return(y)
}
Renyi=function(beta,lambda,v){
y=log(t(beta,lambda,v))/(1-v)
return(y)
}

#Calculate the maximum likelihood estimators
#of LL distribution
library('bbmle')
xvec<-c(X) #X is the data set
ln<-function(beta,lambda){
-sum(log((lambda*(beta**2)*(1+x)*exp(-beta*x))
/(1+beta)*log(1-lambda)*(lambda*((1+beta+beta*x)
/(1+beta))*exp(-beta*x)-1)))
}
mle.results1<-mle2(fn1,start=list(beta=beta
,lambda=lambda),hessian.opt=TRUE)
summary(mle.results1)

# Variance-covariance matrx of LL distribution
vcov(mle.results1)
```