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Improved version of:

From Sierpiński's conjecture to Legendre's

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Abstract

This improved version of the original article published in vol.4, no.2, 2014, 65-77 of this journal is published here in full to close a gap in the proof of Sierpiński's conjecture of section 3.5. It includes a full rewriting of sections 3.3 to 3.5 as well as other minor changes.

Legendre's conjecture (supposedly 1808) states that:

There is always at least one prime number between two consecutive squares N^2 and $(N+1)^2$ for any integer $N > 0$.

In the present article, an elementary proof of this conjecture is given by creating and solving a D conjecture that includes Oppermann's conjecture (1882), Sierpiński's S conjecture (1958) as well as Legendre's.

Finally, as consequences, $p_{m+1} - p_m = O(\sqrt{p_m})$, Andrica's (1986) and Brocard's (1904) conjectures are proved.

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1 Introduction

The unsolved Oppermann's conjecture (1882) [1] states that:

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For any integer $N > 1$ one has : $\pi(N^2 + N) > \pi(N^2) > \pi(N^2 - N)$, where $\pi(x)$ is the prime counting function.
The unsolved Sierpiński's S conjecture (1958) [2] states that:

For any integer $N > 1$, there is always at least one prime number in each line of a $N \times N$ matrix filled up from left to right and from bottom to top with the N^2 integers from 1 to N^2.

Let's then write Sierpiński's S(N) matrix:

Table 1. Sierpiński's matrix S(N)

N	...	...	...	Low	Opp.	Conj.	...	...	N^2
N-1	...	...	...	...	...	...	...	...	$N^2 - N$
...	...	...	...	...	...	...	...	...	...
...	...	...	...	...	Sierp.	Conj.	...	...	...
...	...	...	...	...	...	...	...	...	...
4	3N+1	...	...	...	...	...	...	...	4N
3	2N+1	...	...	...	...	...	...	...	3N
2	N+1	...	...	...	...	...	...	...	2N
1	1	2	3	...	...	...	...	...	N
L/C	C1	C2	...	...	...	...	...	C(N-1)	C(N)

In this Table 1, with the definitions of the conjectures given above, one can see that all the lines of the matrix S(N) correspond to Sierpiński's conjecture and that the top line only corresponds to the lower part of Oppermann's conjecture. Considering only the integers $N > 1$, one can see that for $N = 2$ et $N = 3$, Sierpiński's matrices are:

Table 2. Sierpiński's matrix S(2)

3	4
1	2

Table 3. Sierpiński's matrix S(3)

7	8	9
4	5	6
1	2	3

and one can check that Sierpiński's conjecture is verified by these two matrices. But they do not show any kind of recurrence.

2 Preliminary Notes

Now, let's define a new matrix that we name D(N) and in which we introduce some recurrence with the help of the recurrence relation:

$$(N + 1)^2 = N^2 + (2N + 1)$$

In order to do that, we simply add to Sierpiński's S(N) matrix, two lines upwards with the 2N numbers immediately greater than N^2 and then, one column rightwards filled up with zeros except the number $(N + 1)^2$ at its top, as follows:

Table 4. Matrix D(N) (from column 1 to column N+1)

N+2	A	...	...	...	...	...	B	$(N+1)^2$
N+1	C	High	Opp.	Conj.	...	D	N^2+N	0
N	E	Low	Opp.	Conj.	F	N^2-1	N^2	0
N-1	...	...	...	...	...	...	N^2-N	0
...	...	...	...	...	...	...	...	0
k	...	...	Sierp.	Conj.	...	...	kN	0
...	...	...	...	...	...	...	...	0
4	3N+1	...	...	...	...	...	4N	0
3	2N+1	...	...	...	...	...	3N	0
2	N+1	...	...	...	...	...	2N	0
1	1	2	...	...	...	...	N	0
L/C	1	2	...	...	...	...	N	N+1

As lines L=N+1 and L=N+2 correspond to Legendre's conjecture, this Table 4 ties three independent conjectures into one real matrix. With this model, matrix D(N+1) is:

Table 5. Matrix D(N+1)

N+3	...	Low	Opp.	Conj.	...	Legend.	Conj.	$(N+2)^2$
N+2	...	High	Opp.	Conj.	...	Legend.	Conj.	0
N+1	A	Low	Opp.	Conj.	...	B	$(N+1)^2$	0
N	N^2	C	High	Opp.	Conj.	D	N^2+N	0
N-1	...	...	E	Low	Opp.	F	N^2-1	0
...	...	...	...	...	...	...	...	0
k	...	...	...	...	...	...	k(N+1)	0
...	...	...	...	...	...	...	...	0
4	3N+4	...	...	...	...	...	4(N+1)	0
3	2N+3	...	...	...	...	...	3(N+1)	0
2	N+2	...	...	...	...	...	2(N+1)	0
1	1	2	...	...	...	...	N+1	0
L/C	1	2	...	...	...	...	N+1	N+2

Using Table 4 to replace numbers up to $(N+1)^2$ of Table 5, this last one can be written :

Table 6. New matrix D(N+1)

N+4	...	Low	Opp.	...	...	Legend.	Conj.	$(N+2)^2$
N+3	...	High	Opp.	...	...	Legend.	Conj.	0
N+2	A	Low	Opp.	...	...	B	$(N+1)^2$	0
N+1	C	High	Opp.	...	D	N^2+N	0	0
N	E	Low	Opp.	F	N^2-1	N^2	0	↑
N-1	...	...	...	...	...	N^2-N	0	0
...	...	...	...	...	...	...	0	Sierp.'s
k	...	...	...	...	...	kN	0	matrix
...	...	...	...	...	...	...	0	S(N)
4	3N+1	...	...	...	...	4N	0	0
3	2N+1	...	...	...	...	3N	0	0
2	N+1	...	...	...	...	2N	0	0
1	1	2	...	...	...	N	0	↓
L/C	1	2	...	...	...	N	N+1	N+2

Let's notice that line 1 always contains the prime number 2 when $N > 1$. Now, let's modify Sierpiński's conjecture in order to create conjecture D which states that:

For any integer $N > 1$, there is always at least one prime number in each line

of a matrix D(N), filled in according to the model defined by Table 4.
We will now prove the three tied conjectures by induction.

3 Main Result : Proof by induction

3.1 For N=2 and N=3

One can easily check that conjecture D is verified when N=2 and 3:

Table 7. Matrix D(2)

7	8	9
5	6	0
3	4	0
1	2	0

Table 8. Matrix D(3)

13	14	15	16
10	11	12	0
7	8	9	0
4	5	6	0
1	2	3	0

and one can check that these two matrices show a beginning of recurrence that we will use in the next step.

3.2 From N to N+1

Now, we suppose that conjecture D is verified for a value $N > 3$ and we will prove that it is still true for N+1.

3.3 Extension of matrix D(N+1) of Table 6

By using recurrently the principle used to transform matrix D(N+1) of Table 5 into the new matrix D(N+1) of Table 6, we get at the end of the process, from any $N > 2$ down to $N = 1$:

Table 9. Matrix D(N+1) of Table 6 recurrently transformed

...	Low Opp.	for $(N+2)^2$	...	Leg.	Conj.	...	$(N+2)^2$
...	High Opp.	for $(N+1)^2$	...	Leg.	Conj.	...	0
A	Low Opp.	for $(N+1)^2$	...	...	B	$(N+1)^2$	0
C	High Opp.	for N^2	...	D	N^2+N	0	0
E	Low Opp.	for N^2	F	...	N^2	...	0
...	High Opp.	for $(N-1)^2$	...	...	0	0	0
...	Low Opp.	for $(N-1)^2$	...	$(N-1)^2$	0	0	0
...	High Opp.	for $(N-2)^2$	...	0	0	0	0
...	Low Opp.	...	...	...	...	...	...
...	High Opp.	...	...	...	...	...	...
21	22	23	24	5^2	L Opp.	for 5^2	0
17	18	19	20	0	H Opp.	for 4^2	0
13	14	15	4^2	0	L Opp.	for 4^2	0
10	11	12	0	0	H Opp.	for 3^2	0
7	8	3^2	0	0	L Opp.	for 3^2	0
5	6	0	0	0	H Opp.	for 2^2	0
3	2^2	0	0	0	L Opp.	for 2^2	0
2	0	0	0	0	H Opp.	for 1^2	0
1^2	0	0	0	0	0	0	0
C1	C2	C3	C4	...	C(N)	C(N+1)	C(N+2)

Let's notice that all squares n^2 are located at column C=n and line L = $2n-1$.

3.4 Conditional proof of Oppermann's conjecture

Proof. If we suppose that conjecture D is true for matrix D(N), it means that all Oppermann's, Sierpiński's and Legendre's conjectures are true for N, and particularly, it means that both lines N and N+1 of matrix D(N) of Table 4 contain at least one prime number.

As line parts CD and EF of these two lines of matrix D(N) of Table 4 are parts of Oppermann's conjecture for N and become respectively parts of lines N and N+1 of matrix D(N+1) of Table 6, these lines also contain at least one prime number. Oppermann's conjecture, which is already verified for N=2 and N=3 in matrices D(2) and D(3) of Tables 7 and 8, is therefore proved for D(N+1), conditionally to the validity of conjecture D for matrix D(N). □

Noticing that Oppermann's conjecture (just conditionally proved), applies to lines N to N+4 of matrix D(N+1) of Table 6, these five lines therefore contain, still conditionally, at least one prime number.

3.5 Conditional proof of Sierpiński's conjecture

3.5.1 Proof by the global reverse process

Proof. Still because Oppermann's conjecture is conditionally proved, it also applies to all the lines of matrix D(N+1) of Table 9 except line 1. This proves that all lines except line 1 in Table 9, contain at least one prime number.
Now, we will do the reverse operation that we did to get Table 9 from Table 6, operation that was exactly to expand the N lines of the N×N matrix of Sierpiński into 2N-1 lines for which we have just shown that each of them contains at least one prime number except line 1. We can therefore say that this reverse operation consists, ignoring zeroes, to force the *first N-1* lines of Table 9, each of them containing at least one prime number except line 1, into the N lines *from line N to line 2N-1* of Table 9, each of them also containing at least one prime number. At the end of this process, the N-2 prime numbers of the *first N-1* lines of Table 9 have been forced into the N lines *from N to 2N-1* of Table 9, lines that, at the start of the reverse process, already contained at least one prime number.

Therefore, according to the pigeonhole principle, all lines of the regenerated matrix of Sierpiński in Table 6 contain at least one prime number. This proves Sierpiński's conjecture, conditionally to the validity of conjecture D for matrix D(N). □

3.5.2 A deeper analysis of the reverse process

In the original article, the last proof does not define clearly the status of each line of Sierpiński's matrix and so, has been criticized. This is the reason why a deeper analysis of the reverse process is given here, that gives the exact status of each line and thus validates this last proof.

At the beginning of the reverse process, that is to say when Table 9 is just obtained, all the lines of Table 9 are lines corresponding to the lower or upper conditions of Oppermann's conjecture and so, all the lines including numbers from 2 to N^2 contain at least a prime number. To make it simple, we will name these lines (zeroes and square numbers excluded): *Oppermann's lines.*

At the end of the reverse process, that is to say when the N×N matrix of Sierpiński has been regenerated in Table 6, all lines of this N×N matrix have the fixed length N and we will name them: *Sierpinski's lines.*

We can then identify Oppermann's lines by accolades in this N×N Sierpiński's matrix S(N) of Table 6 for D(N+1), which we extract into Table 10 that follows:

Table 10. Sierpiński's matrix S(N) in D(N+1) with N=10

N	{91	92	93	94	95	96	97	98	99}	$\underline{N^2}$	
N-1	$\underline{(N-1)^2}$	{82	83	84	85	86	87	88	89	90}	n_2
8	71	72}	{73	74	75	76	77	78	79	80}	
7	61	62	63}	$\underline{(N-2)^2}$	{65	66	67	68	69	70	n_3
6	51	52	53	54	55	56}	{57	58	59	60	
5	41	42}	{43	44	45	46	47	48}	$\underline{7^2}$	{50	
4	{31	32	33	34	35}	$\underline{6^2}$	{37	38	39	40	
3	{21	22	23	24}	$\underline{5^2}$	{26	27	28	29	30}	n_1
2	11	12}	{13	14	15}	$\underline{4^2}$	{17	18	19	20}	
1	$\underline{1^2}$	{2}	{3}	$\underline{2^2}$	{5	6}	{7	8}	$\underline{3^2}$	{N	
L/C	1	2	3	4	5	6	7	8	9	N	

3.5.3 A first set of n_1 Sierpiński's lines

In Table 10 (with N=10) which is indeed Sierpiński's matrix S(N) of Table 6, we notice that the square numbers $(N-1)^2$(=81), $(N-2)^2$(=64) and $(N-3)^2$(=49) of general equation:

$$z = (N - \sqrt{C})^2$$

where z is integer only when the column number C is a square number, are located on a parabola of horizontal axis that is described by the equation:

$$L = (N + 1) - 2\sqrt{C}$$

where C is the column number and L is the line number or:

$$C = ((N + 1) - L)/2)^2$$

This means that in this Table 10, the biggest of these square numbers (here $z = 49$) *allowing that the Oppermann's line preceeding it* (of length $\sqrt{z}-1 = 6$): $\{43, 44, 45, 46, 47, 48\}$ *can be fully inserted on the left of this square in the corresponding Sierpinski's line*, is the greatest square near the intersection of the arc of parabola and the last column on the right of Sierpinski's matrix. The position of this intersection is defined by the condition:

$$C = N$$

which gives, putting this value into the definition of the arc of parabola defined above:

$$L = (N + 1) - 2[N^{1/2}] = 11 - 6 = 5$$

where the square brackets [x] will indicate, from now on, the integer value of x.

So, in Table 10 and from the condition $C = N$, lines from L=1 to L=(N+1) -2[$N^{1/2}$](=5) always contain at least one full Oppermann's line and as Oppermann's conjecture has been conditionally proved in section 3.4, all these lines contain at least one prime number.

Hence, for Table 10, a first number of lines containing at least one prime number is:

$$n_1 = (N + 1) - 2[N^{1/2}]$$

3.5.4 A second set of n_2 Sierpiński's lines

Now, beginning from top of Table 10, we see that:

- line N=10 has remained unchanged during the whole reverse process.
- line N-1=9 has been shifted one number on the right without losing any of its numbers.

As Oppermann's conjecture has been conditionally proved in section 3.4, these two lines also contain at least one prime number.

Hence, for Table 10, a second number of lines containing at least one prime number is:

$$n_2 = 2$$

3.5.5 The third and last set of n_3 Sierpiński's lines

The n_3=N-(n_1+n_2)(=3) remaining lines, from L_1=(N+1) -2[$N^{1/2}$]+1 (=6) to L_2=N-2 (=8), are divided in two parts during the reverse process and it is more difficult to formulate a status, but this remains possible. Indeed:
1) In Table 9 and for N=10, for numbers up to N^2(=100), the number of Oppermann's lines is:

$$2N\text{-}1 = 19$$

The number of full Oppermann's lines from number 1 to number (L_1-1)N = N((N+1) -2[$N^{1/2}$]) (= 50) is that which stops at a square $z = n^2$ equal or immediately less than this value, which gives:

$$n = [\sqrt{z}] = \left[\sqrt{N((N+1) - 2[N^{1/2}])}\right] = 7$$

and the Oppermann's line in table 9 where z(=49) can be located is therefore:

$$2n - 1 = 2\left[\sqrt{N((N+1) - 2[N^{1/2}])}\right] - 1 = 2 \times 7 - 1 = 13$$

2) In Table 10 for N=10, the number of Sierpiński's lines is : N = 10.
The number of Sierpiński's lines for which it is already known that they contain at least one prime number is:

- for numbers from 1 to N((N+1) − 2[$N^{1/2}$]) (=50) :
  n_1 = ((N+1) -2[$N^{1/2}$]) (= 5)
- for lines N=10 and N-1=9 : $n_2 = 2$
- Total : $n_1 + n_2$ = (N+1) − 2[$N^{1/2}$] + 2 (= 7)

The number of remaining Sierpiński's whole lines to study for numbers from N((N+1) − 2[$N^{1/2}$]) (=50) to $(N\text{-}1)^2 - 1$ (=80) is thus:

$$n_3 = N - (n_1 + n_2) = N - ((N + 1) - 2[N^{1/2}]) + 2) \ (= 10{-}7 = 3)$$

3) But, in Table 9:

- from $49 = n^2$ located at line $2n - 1 = 13$,
- to $80 = (N - 1)^2 - 1$ located at line $(2N - 1) - 2 = 2N - 3 = 17$,

there are exactly $n_4 = (2N-3)-(2n-1)$ (=4) *Oppermann's lines* (neglecting the square 81=(N-1)^2). Therefore, there is always:

$$n_4 = (2N-3)-(2n-1) \text{ (=4) Oppermann's lines of Table 9}$$

to share out in:

$$n_3 = N-((N+1)-2[N^{1/2}])+2 \text{ (=3) Sierpiński's lines in Table 10.}$$

So, the difference d between *the number of Oppermann's lines in Table 9 located between N((N+1) − 2[N*$^{1/2}$*]) (=50) and (N-1)*$^2 - 1$ *(=80)* and *the number of Sierpiński's lines in Table 10 for the same interval*, is always:

$$d = n_4 - n_3 = (2N-3)-(2n-1)-\{N-((N+1)-2[N^{1/2}])+2)\}$$
$$d = (2N-3)-2n+1-\{-1+2[N^{1/2}]-2\}$$
$$d = (2N-3)-2n+1+1-2[N^{1/2}]+2$$
$$d = (2N+1)-2n-2[N^{1/2}]$$
$$\text{and, as } n = \left[\sqrt{N((N+1)-2[N^{1/2}])}\right]:$$
$$d = (2N+1)-2\left[\sqrt{N((N+1)-2[N^{1/2}])}\right]-2[N^{1/2}]$$

expression that, for any N, always (weirdly) gives:

$$d = 1$$

Remark. This can be easily verified by considering the continuous function d' of same form as d and obtained by replacing any square bracket (for integer values) by a simple parenthesis of same direction, as shown below:

$$d = (2N+1)-2[(N((N+1)-2[N^{1/2}]))^{1/2}]-2[N^{1/2}]$$
$$d' = (2N+1)-2(N((N+1)-2(N^{1/2})))^{1/2})-2(N^{1/2})$$
$$d' = (2N+1)-2(N(N^{1/2}-1)^2)^{1/2}-2N^{1/2}$$
$$d' = (2N+1)-2N^{1/2}(N^{1/2}-1)-2N^{1/2}$$
$$d' = (2N+1)-(2N-2N^{1/2})-2N^{1/2}$$
$$d' = 1$$

So, for $N > 3$ there is always n_3 Sierpiński's lines of constant length N in Table 10 having to be shared by $n_4 = n_3 + 1$ Oppermann's lines. This evokes inevitably the pigeonhole principle of Dirichlet (1834) which states that if n pigeons have to share m pigeonholes with $n > m$, all pigeonholes will get one pigeon and one or more pigeonholes will contain more than one pigeon.

To be allowed to apply this principle in our case, we have to clarify what are the equivalents of the pigeonholes and what are the equivalents of the pigeons.

For the m pigeonholes, it's easy to find that their equivalents must be the n_3 fixed length Sierpiński's lines of Table 10, in which numbers range from N((N+1) - 2[$N^{1/2}$]) + 1 (=51) to $(N-1)^2 - 1$ (=80).

Then, for the n pigeons, it would be logical to choose as equivalents, the $n_4 = n_3 + 1$ Oppermann's lines of Table 9. But as these lines are split in two parts in the reverse process, they are disqualified to be the equivalents of pigeons. The solution here is to choose as equivalent of a pigeon, the smallest prime number contained in each Oppermann's line. These new pigeons, whose existence is proved as Oppermann's conjecture has been proved earlier in 3.4, conditionally to the validity of conjecture D for matrix D(N), are in same number as Oppermann's lines and are not split in the reverse process.

Now, having clarified the situation in our case, the pigeonhole principle can be used, which proves that, for $N > 3$, the n_3 (=3) Sierpiński's lines in Table 10 and/or Table 6, in which numbers range from N((N+1) $-$ 2[$N^{1/2}$]) + 1 (= 51) to $(N-1)^2 - 1$ (= 80), all contain at least one prime number.

As this is already the case for the $n_1 + n_2$ (=7) lines identified earlier, we can now conclude that *all Sierpiński's lines* of S(N) in Table 10 or in table 6 for D(N+1) contain at least one prime number for a chosen $N > 3$.

Proof. Finally, as all lines of S(N) in D(N+1) of Table 6 contain at least one prime number when it is supposed true for S(N) in D(N) and as we have seen in section 3.1 that it is also true for N=2 and N=3, all this proves Sierpiński's conjecture, conditionally to the validity of conjecture D for matrix D(N). □

3.6 Unconditional proof of conjecture D (2014)

Proof. As we have seen that:

• conditionally to the validity of conjecture D for matrix D(N), lines *N to N*+4 of the new matrix D(N+1) of Table 6 contain at least one prime number according to the conditionally proved Oppermann's conjecture of section 3.4,

• and that lines *1 to N* of the new matrix D(N+1) of Table 6 contain at least one prime number according to the conditionally proved Sierpiński's conjecture of section 3.5,

• one can therefore conclude that *all lines* of the new matrix D(N+1) of Table 6 contain at least one prime number when it is supposed true for D(N).

As in section 3.1 we have seen that conjecture D is also verified for N=2 and 3, it is therefore unconditionally proved for any N. □

3.7 Unconditional proofs of Oppermann's (1882) and Sierpiński's (1958) conjectures

Proof. Oppermann's and Sierpiński's conjectures which were only proved conditionally to the validity of conjecture D for matrix D(N) are now unconditionally proved, as conjecture D has been unconditionally proved in section 3.6. □

3.8 Unconditional proof of Legendre's conjecture (1808)

Proof. Finally, as Oppermann's conjecture has been unconditionally proved in section 3.7, Legendre's conjecture is also unconditionally proved. □

This result can even be improved: as Legendre's conjecture concerns two Oppermann's lines, it is then proved that:
For N>0, there are always at least two prime numbers between N^2 and $(N+1)^2$.

4 Consequences

The above four proved conjectures allow other proofs of conjectures. Three of these are given hereafter.

4.1 Conjecture on gaps $d_m = p_{m+1} - p_m = O(\sqrt{p_m})$

Proof. As Oppermann's and Sierpiński's conjectures have been proved, we can therefore say that in lines 1 to N+1 of the following extended matrix S(N), there is always at least one prime number:

Table 11. Extended Sierpiński's matrix S(N)

N+1	$N^2 + 1$	...	High	Opp.	Conj.	for N^2	...	p_{m+1}	(N+1)N
N	p_m	...	Low	Opp.	Conj.	for N^2	...	...	N^2
N-1	...	...	...	...	...	...	...	...	(N-1)N
N-2	...	...	...	...	...	...	...	...	(N-2)N
...	...	...	...	...	...	...	...	...	...
k	...	...	...	Sierp.	Matrix	...	...	...	...
...	...	...	...	...	...	...	...	...	...
4	3N+1	...	...	...	...	...	...	4N-1	4N
3	2N+1	...	...	...	...	...	...	...	3N
2	N+1	...	...	...	...	...	...	...	2N
1	1	...	...	...		...	...	...	N
L/C	1	2	3	4	...	...	...	N-1	N

As all numbers of the column on the right are composite, for any couple of lines N and N+1, the maximum possible distance d_m between two consecutive prime numbers p_m and p_{m+1} is:

$$d_m = p_{m+1} - p_m \leq (kN - 1) - ((k-2)N + 1) = 2N - 2 \tag{1}$$

Verifying this on the couples of lines (3, 4) and (N, N+1), we get respectively:

$$d_m = p_{m+1} - p_m \leq (4N - 1) - (2N + 1) = 2N - 2$$
$$d_m = p_{m+1} - p_m \leq ((N+1)N - 1) - ((N-1)N + 1) = 2N - 2$$

As for lines N and N+1, Oppermann's proved conjecture implies that:

$$(N-1)^2 < N^2 - N < p_m < N^2 < p_{m+1} < N^2 + N < (N+1)^2$$

one also has, taking only the positive square roots:

$$(N-1) < \sqrt{p_m} < N < \sqrt{p_{m+1}} < (N+1) \tag{2}$$

which shows that:

$$\sqrt{p_{m+1}} - \sqrt{p_m} < (N+1) - (N-1) = 2 \tag{3}$$

But as (2) contains:

$$N < \sqrt{p_{m+1}} \tag{4}$$

and as (3) can also be written:

$$\sqrt{p_{m+1}} < \sqrt{p_m} + 2 \tag{5}$$

from relations (1), (4) and (5) applied in that order, we get:

$$d_m = p_{m+1} - p_m \leq 2N - 2 < 2\sqrt{p_{m+1}} - 2 < 2(\sqrt{p_m} + 2) - 2$$

or :

$$d_m = p_{m+1} - p_m < 2\sqrt{p_m} + 2$$

which proves the limit searched for by Hoheisel [3] and others since 1930:

$$d_m = p_{m+1} - p_m = O(\sqrt{p_m})$$

where O( ) is the big O of Landau's notation. □

4.2 Andrica's conjecture (1986)

Proof. This conjecture [4] states that for any $m > 0$:

$$\sqrt{p_{m+1}} - \sqrt{p_m} < 1$$

Proof - As $p_{m+1} - p_m > 0$, with relation (3) we have:

$$0 < \sqrt{p_{m+1}} - \sqrt{p_m} < 2$$

which gives, by division by $\sqrt{p_m}$:

$$0 < \frac{\sqrt{p_{m+1}} - \sqrt{p_m}}{\sqrt{p_m}} < \frac{2}{\sqrt{p_m}} \tag{6}$$

Since Euclid it is known that there are infinitely many increasing primes. This implies that when m tends to infinity, we have:

$$\frac{2}{\sqrt{p_m}} \to 0$$

which, put together with relation (6), gives :

$$0 < \frac{\sqrt{p_{m+1}} - \sqrt{p_m}}{\sqrt{p_m}} < \frac{2}{\sqrt{p_m}} \to 0$$

which, in turn, implies that when m tends to infinity:

$$lim_{m\to\infty}(\sqrt{p_{m+1}} - \sqrt{p_m}) = 0 \tag{7}$$

Finally, as the quantity $\sqrt{p_{m+1}} - \sqrt{p_m}$ reaches a maximum of $\sqrt{11} - \sqrt{7} = 0,67... < 1$ for $m = 4$ before tending to zero as proved above in (7), this proves Andrica's conjecture. □

4.3 Brocard's conjecture (1904)

This conjecture states that for $m \geq 2$:

$$\pi(p_{m+1}^2) - \pi(p_m^2) \geq 4$$

Proof. As the minimum distance between two odd primes is $d_{min} = p_{m+1} - p_m = 2$ for the case of twin primes and that, for any N, we have:

$$(N+1) - (N-1) = 2$$

and:

$$(N+1)^2 - (N-1)^2 = 4N \tag{8}$$

it is therefore possible to consider the minimum case of twin (odd) primes with:

$$p_m = (N-1)$$
$$p_{m+1} = (N+1)$$

where N is inevitably even, so that from (8):

$$p_{m+1}^2 - p_m^2 = 4N$$

Sierpiński's matrix S(N) extended up to line N+7 for an even N between twin primes, can then be written as in Table 12.

Here, we have to consider four points.

First point, Table 12 is obtained for the case of twin primes.

Second point, no couple of two consecutive primes can be nearer than twin primes.

Third point, in Table 12 the square numbers $(N-3)^2$, $(N-2)^2$, $(N-1)^2 = p_m^2$, $(N+1)^2 = p_{m+1}^2$, $(N+2)^2$ and $(N+3)^2$ of general equation:

$$z = (N \pm \sqrt{C})^2$$

where z is integer only when the column number C is a square number, are located on a parabola of horizontal axis that is described by the equation:

$$L = (N+1) \pm 2\sqrt{C}$$

where C is the column number and L is the line number.

Fourth point, the already proved Sierpiński's and Legendre's conjectures provide at least one prime number per line in Table 12, up to p_{m+1}^2.

Table 12. Sierpiński's matrix S(N) extended up to line N+7 for twin primes

...	...	...	...	$(N+3)^2$	...	N^2+7N
...	...	...	...	...	...	N^2+6N
...	$(N+2)^2$	...	...	...	...	N^2+5N
...	...	...	...	...	...	N^2+4N
p_{m+1}^2	...	...	...	...	...	N^2+3N
...	proved Leg.'s	conj.	...	...	...	N^2+2N
...	proved Leg.'s	conj	...	...	...	N^2+N
...	proved Sierp.'s	conj.	...	...	...	N^2
p_m^2	proved Sierp.'s	conj.	...	...	...	N^2-N
...	...	...	...	...	...	N^2-2N
...	$(N-2)^2$	...	...	...	...	N^2-3N
...	...	...	...	...	...	N^2-4N
...	...	...	...	$(N-3)^2$	...	N^2-5N
...	...	...	...	...	...	...
...	proved Sierp.'s	conj.	...	...	...	...
...	...	...	...	...	...	...
2N+1	...	...	...	...	...	3N
p_{m+1}	...	...	...	...	...	2N
1	...	...	...	...	p_m	N(even)
C1...	C4...	...	...	C9...	C(N-1)	C(N=102)

Taking the four points into account, as p_{m+1}^2 and p_m^2 are located four lines away in Table 12 when p_m and p_{m+1} are twin primes, and, as each of these four lines contains at least one prime number, as Sierpiński's and Legendre's conjectures have been proved, we can conclude that there is always at least 4 prime numbers between p_m^2 and p_{m+1}^2. This proves Brocard's conjecture. □

5 Conclusion

This article provides the proofs of six conjectures: Oppermann's, Sierpiński's, Legendre's, the one on the gaps $d_m = p_{m+1} - p_m$ between consecutive primes as well as Andrica's and Brocard's conjectures. It shows a method to solve the overlapping three first conjectures by linking them into a larger one, the D conjecture, that includes the property of recurrence.

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References

[1] Oppermann L., Om vor Kundskab om Primtallenes Maengde mellem givne Graendser, *Oversigt over det Kongelige Danske Videnskabernes Selskabs Forhandlinger og dets Medlemmers Arbejder*, (1882), 169-179.

[2] Schinzel A., Sierpiński W., Sur certaines hypothèses concernant les nombres premiers. Remarque., *Acta Arithm.*, **4 and 5**, (1958 and 1959), 185 - 208 in **4** and 259 in **5**.

[3] Hoheisel G., Primezahlenprobleme in der Analysis, *Sitzungberichte Berliner Akad. d. W.*, (1930), 580 - 588.

[4] Andrica D., Note on a conjecture in prime number theory, *Studia Univ. Babes-Bolyai Math.*, **31**(4), (1986), 44 - 48.