ON A CERTAIN CLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY FRACTIONAL CALCULUS WITH FIXED POINT

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ABSTRACT .

In this paper, authors study a subclass of harmonic univalent functions defined by fractional calculus operator. We obtain coefficient conditions, extreme points, distortion bounds,and radius of convexity for this class of harmonic univalent functions.We also show that the class studied in this paper is closed under convolution and convex combination.The results obtained for this class reduce to the corresponding results for several known classes in the literature are briefly indicated.

KEYWORDS : Harmonic, Univalent function, Fractional calculus, Fixed Point.

0.1 INTRODUCTION

A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \overline{g}$ where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|, z \in D$. See Clunie and Sheil-Small [3].

Denote by S_H the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit open disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \ g \in S_H$ we may express the analytic functions h and g as

$$h(z) = a_0 z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1}$$

Note that the class S_H reduces to class S of normalized analytic univalent functions if co-analytic part of f i.e. $g \equiv 0$, for this class $f(z)$ may be expressed as

$$f(z) = a_0 z + \sum_{k=2}^{\infty} a_k z^k. \tag{2}$$

For more basic results on hormonic functions one may refer to the following introductory text book by Duren [6] ,(see also [1], [12], [13] and reference there in).

0.2 FRACTIONAL CALCULUS

The following definitions of fractional derivatives and fractional integrals are due to Owa [9] and Srivastava and Owa [16].

Definition 1 The fractional integral of order λ is defined for a function $f(z)$ by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\lambda}} d\xi, \tag{3}$$

where $\lambda > 0, f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z-\xi)^{\lambda-1}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$.

Definition 2 The fractional derivative of order λ is defined for a function $f(z)$ by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi, \tag{4}$$

where $0 \leq \lambda < 1$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z-\xi)^{-\lambda}$ is removed as in Definition 1 above.

Definition 3 Under the hypothesis of Definition 2, the fractional derivative of order $n+\lambda$ is defined for a function $f(z)$ by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z) \tag{5}$$

where $0 \leq \lambda < 1$ and $n \in N_0 = \{0, 1, 2, \ldots\}$. Using Definition 2 and its known extension involving fractional derivative Owa and Srivastava [11] introduced the operator
$\Omega^{\lambda} : A \to A$ as follows.

$$\Omega^{\lambda} f(z) = \Gamma(2-\lambda) a_0 z^{\lambda} D_z^{\lambda} f(z), (\lambda \neq 2, 3, 4, ...) \tag{6}$$

where A denotes the class of functions of form (2) which are analytic in U .

Let $S_H(\alpha,\lambda,\mu,z_0)$ denote the subclass of S_H consisting of functions f of the form (1) satisfying the following condition

$$Re\left(\frac{z(\Omega^\lambda h(z))' - \overline{z(\Omega^\lambda g(z))'}}{\mu\left\{z(\Omega^\lambda h(z))' - \overline{z(\Omega^\lambda g(z))'}\right\} + (1-\mu)\left\{\Omega^\lambda h(z) + \overline{\Omega^\lambda g(z)}\right\}}\right) \geq \alpha \tag{7}$$

where $0 \leq \alpha < 1$, $0 \leq \mu \leq 1$,and $0 \leq \lambda < 1$.

Let $TS_H(\alpha,\lambda,\mu,z_0)$ be the subclass of Let $S_H(\alpha,\lambda,\mu,z_0)$ for which $f(z) = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are of the form

$$h(z) = a_0 z - \sum_{k=2}^{\infty} |a_k|\, z^k, \quad g(z) = \sum_{k=1}^{\infty} |b_k|\, z^k, \quad |b_1| < 1. \tag{8}$$

By specializing the parameters in the subclasses $S_H(\alpha,\lambda,\mu,z_0)$and $TS_H(\alpha,\lambda,\mu,z_0)$ we obtain the following known subclasses of S_H studied earlier by various researchers .

1. $S_H(\alpha,\lambda,0,z_0) \equiv S_H(\alpha,\lambda,z_0)$ and $TS_H(\alpha,\lambda,0,z_0) \equiv TS_H(\alpha,\lambda,z_0)$ studied by Dixit and Porwal [4] , (see also [8]) .
2. $S_H(\alpha,0,\mu,z_0) \equiv S_H(\alpha,\mu,z_0)$ and $TS_H(\alpha,0,\mu,z_0) \equiv TS_H(\alpha,\mu,z_0)$ studied by *Öztürk* et al. [10], (see also [5]) .
3. $S_H(\alpha,0,0,z_0) \equiv S_H^*(\alpha,z_0)$ and $TS_H(\alpha,0,0,z_0) \equiv TS_H^*(\alpha,z_0)$ studied by Jahangiri [7] .
4. $S_H(0,0,0,z_0) \equiv S_H^*(z_0)$ and $TS_H(0,0,0,z_0) \equiv TS_H^*(z_0)$ studied by Silverman [14] , Silverman and Silvia [15] , (see also [2]).

0.3 MAIN RESULTS

We begin with a sufficient coefficient condition for functions in $S_H(\alpha, \lambda, \mu, z_0)$.

Theorem 1 Let $f = h + \overline{g}$ be such that h and g are given by (1). Furthermore, let

$$\sum_{k=1}^{\infty} \left\{ \frac{k(1-\alpha\mu) - \alpha(1-\mu)}{(1-\alpha)} |a_k| + \frac{k(1-\alpha\mu) + \alpha(1-\mu)}{(1-\alpha)} |b_k| \right\} \phi(k,\lambda) \leq 2a_0 \qquad (9)$$

where $a_1 = 1$, $0 \leq \alpha < 1$, $0 \leq \mu \leq 1$, $0 \leq \lambda < 1$ and

$$\phi(k,\lambda) = \left(\frac{\Gamma(k+1)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} \right)$$

Then f is sense-preserving, hormonic univalent in U and $f \in S_H(\alpha, \lambda, \mu, z_0)$.

Proof : First we note that f is locally univalent and sense-preserving in U. This follows from

$$\begin{aligned}
\left| h'(z) \right| &\geq a_0 - \sum_{k=2}^{\infty} k |a_k| r^{(k-1)} \\
&> a_0 - \sum_{k=2}^{\infty} k |a_k| \\
&\geq a_0 - \sum_{k=2}^{\infty} \frac{k(1-\alpha\mu) - \alpha(1-\mu)}{1-\alpha} \phi(k,\lambda) |a_k| \\
&\geq \sum_{k=1}^{\infty} \frac{k(1-\alpha\mu) - \alpha(1-\mu)}{1-\alpha} \phi(k,\lambda) |b_k| \\
&\geq \sum_{k=1}^{\infty} k |b_k| \\
&> \sum_{k=1}^{\infty} k |b_k| r^{(k-1)} \\
&\geq \left| g'(z) \right|
\end{aligned}$$

To show that f is univalent in U. Suppose that $z_1, z_2 \in U$ such that $z_1 \neq z_2$. Then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| = 1-\left|\frac{\sum_{k=1}^{\infty} b_k(z_1^k-z_2^k)}{a_0(z_1-z_2)+\sum\limits_{k=2}^{\infty} a_k(z_1^k-z_2^k)}\right|$$

$$> 1-\frac{\sum_{k=1}^{\infty} k|b_k|}{1-\sum\limits_{k=2}^{\infty} k|a_k|} \geq 1-\frac{\sum_{k=1}^{\infty}\frac{k(1-\alpha\mu)+\alpha(1-\mu)}{1-\alpha}\phi(k,\lambda)|b_k|}{1-\sum\limits_{k=2}^{\infty}\frac{k(1-\alpha\mu)-\alpha(1-\mu)}{1-\alpha}\phi(k,\lambda)|a_k|} \geq 0.$$

Now, we show that $f \in S_H(\alpha,\lambda,\mu,z_0)$. Using the fact that $Re\ \omega \geq \alpha$, if and only if, $|1-\alpha+\omega| \geq |1+\alpha-\omega|$, it suffices to show that

$$|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)| \geq 0, \tag{10}$$

where

$A(z) = z(\Omega^\lambda h(z))' - \overline{z(\Omega^\lambda h(z))'}$

$B(z) = \mu\left\{z(\Omega^\lambda h(z))' - \overline{z(\Omega^\lambda h(z))'}\right\} + (1-\mu)\left\{(\Omega^\lambda h(z)) - \overline{(\Omega^\lambda h(z))}\right\}$

Substituting for $A(z)$ and $B(z)$ in L.H.S. of (10) and performing the simple calculation, we obtain

$$= \left|(2-\alpha)a_0 z + \sum_{k=2}^{\infty}[k+(1-\alpha)\mu k+(1-\alpha)(1-\mu)]\phi(k,\lambda)a_k z^k \right.$$
$$\left. -\overline{\sum_{k=1}^{\infty}[k+(1-\alpha)\mu k-(1-\alpha)(1-\mu)]\phi(k,\lambda)b_k z^k}\right|$$
$$-\left|-\alpha a_0 z + \sum_{k=2}^{\infty}[k-(1-\alpha)\mu k-(1-\alpha)(1-\mu)]\phi(k,\lambda)a_k z^k \right.$$
$$\left. -\overline{\sum_{k=1}^{\infty}[k-(1-\alpha)\mu k+(1-\alpha)(1-\mu)]\phi(k,\lambda)b_k z^k}\right|$$

$$\geq 2(1-\alpha)|a_0||z|[1-\sum_{k=2}^{\infty}\left\{\frac{k(1-\alpha\mu)-\alpha(1-\mu)}{1-\alpha}\right\}\phi(k,\lambda)|a_k|\ |z|^{k-1}$$

$$-\sum_{k=1}^{\infty}\left\{\frac{k(1-\alpha\mu)+\alpha(1-\mu)}{1-\alpha}\right\}\phi(k,\lambda)|b_k|\ |z|^{k-1}]$$

$$> 2(1-\alpha)|a_0||z|[1-\sum_{k=2}^{\infty}\left\{\frac{k(1-\alpha\mu)-\alpha(1-\mu)}{1-\alpha}\right\}\phi(k,\lambda)|a_k|$$

$$-\sum_{k=1}^{\infty}\left\{\frac{k(1-\alpha\mu)+\alpha(1-\mu)}{1-\alpha}\right\}\phi(k,\lambda)|b_k|]$$

≥ 0 using (9) .

The coefficient bounds (9) is sharp because equality holds for the following function

$$f(z)=a_0z+\sum_{k=2}^{\infty}\frac{1-\alpha}{k(1-\alpha\mu)-\alpha(1-\mu)\phi(k,\lambda)}x_kz^k+\sum_{k=1}^{\infty}\frac{1-\alpha}{k(1-\alpha\mu)-\alpha(1-\mu)}\phi(k,\lambda)\overline{y_kz^k} \tag{11}$$

where $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1.$

This completes the proof of Theorem 1.

In the following theorem , it is proved that the condition (9) is also necessary for functions $f=h+\overline{g}$ where h and g are of the form (8).

Theorem 2 : Let the function $f=h+\overline{g}$ be such that h and g are given by (8). Then $f\in TS_H(\alpha,\lambda,\mu,z_0)$, if and only if

$$\sum_{k=1}^{\infty}\left[\{k(1-\alpha\mu)-\alpha(1-\mu)\}\,\phi(k,\lambda)\,|a_k|+\{k(1-\alpha\mu)+\alpha(1-\mu)\}\,\phi(k,\lambda)\,|b_k|\right]\leq 2(1-\alpha)a_0 \tag{12}$$

where $a_1=1$, $0\leq\alpha<1$, $0\leq\mu\leq 1$, $0\leq\lambda<1$ and

$$\phi(k,\lambda)=\left(\frac{\Gamma(k-1)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)}\right)$$

Proof : Since $TS_H(\alpha,\lambda,\mu,z_0) \subset S_H(\alpha,\lambda,\mu,z_0)$ this gives the if part of the theorem. So we only need to prove the "only if" part of the theorem. To this end , for functions f of the form (8) we notice that the condition

$$Re\left(\frac{z(\Omega^\lambda h(z))' - \overline{z(\Omega^\lambda g(z))'}}{\mu\left\{z(\Omega^\lambda h(z))' - \overline{z(\Omega^\lambda g(z))'}\right\} + (1-\mu)\left\{\Omega^\lambda h(z) + \overline{\Omega^\lambda g(z)}\right\}}\right) \geq \alpha$$

is equivalent to

$$\text{Re}\left\{\frac{\begin{array}{c}(1-\alpha)a_0 z - \sum_{k=2}^{\infty}\{k(1-\alpha\mu)-\alpha(1-\mu)\}\,\phi(k,\lambda)|a_k|z^k \\ -\sum_{k=1}^{\infty}\{k(1-\alpha\mu)+\alpha(1-\mu)\}\,\phi(k,\lambda)|b_k|\overline{z}^k\end{array}}{a_0 z - \sum_{k=2}^{\infty}\{k\mu+(1-\mu)\}\,\phi(k,\lambda)|a_k|z^k - \sum_{k=1}^{\infty}\{k\mu-(1-\mu)\}\,\phi(k,\lambda)|b_k|\overline{z}^k}\right\} \geq 0. \quad (13)$$

The above condition must hold for all values of z , $|z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we must have

$$\left\{\frac{\begin{array}{c}(1-\alpha)a_0 - \sum_{k=2}^{\infty}\{k(1-\alpha\mu)-\alpha(1-\mu)\}\,\phi(k,\lambda)|a_k|r^{k-1} \\ -\sum_{k=1}^{\infty}\{k(1-\alpha\mu)+\alpha(1-\mu)\}\,\phi(k,\lambda)|b_k|r^{k-1}\end{array}}{a_0 - \sum_{k=2}^{\infty}\{k\mu+(1-\mu)\}\,\phi(k,\lambda)|a_k|r^{k-1} - \sum_{k=1}^{\infty}\{k\mu-(1-\mu)\}\,\phi(k,\lambda)|b_k|r^{k-1}}\right\} \geq 0. \quad (14)$$

If the condition (12) does not hold then the numerator in (14) is negative for r sufficiently close to 1. Thus there exist a $z_0 = r_0$ in (0,1) for which the quotient in (14) is negative. This contradicts the required condition for $f \in TS_H(\alpha,\lambda,\mu,z_0)$ and so the proof is complete.

Next, we determine the extreme points of closed to convex hull of $TS_H(\alpha,\lambda,\mu,z_0)$ denoted by clco $TS_H(\alpha,\lambda,\mu,z_0)$.

Theorem 3 : If $f \in clco\ \ TS_H(\alpha,\lambda,\mu,z_0)$ if and only if

$$f(z)=\sum_{k=1}^{\infty}(x_kh_k(z)+y_kg_k(z)), \tag{15}$$

where

$$h_1(z)=a_0z,\ h_k(z)=z-\frac{(1-\alpha)a_0}{\{k(1-\alpha\mu)-\alpha(1-\mu)\}\,\phi(k,\lambda)}z^k, \quad (k=2,3,4,\ldots),$$

$$g_k(z)=a_0z+\frac{(1-\alpha)a_0}{\{k(1-\alpha\mu)+\alpha(1-\mu)\}\,\phi(k,\lambda)}\overline{z}^k, \quad (k=1,2,3,4,\ldots),$$

$x_k\ge 0,\ y_k\ge 0,\ \sum_{k=1}^{\infty}(x_k+y_k)=1$. In particular, the extreme points of $TS_H(\alpha,\lambda,\mu,z_0)$ are $\{h_k\}$ and $\{g_k\}$.

Proof : For functiom f of the form (15) we have

$$f(z)=\sum_{k=1}^{\infty}(x_kh_k(z)+y_kg_k(z))$$

$$=\sum_{k=1}^{\infty}(x_k+y_k)a_0z-\sum_{k=2}^{\infty}\frac{(1-\alpha)a_0}{\{k(1-\alpha\mu)-\alpha(1-\mu)\}\,\phi(k,\lambda)}x_kz^k$$
$$+\sum_{k=1}^{\infty}\frac{(1-\alpha)a_0}{\{k(1-\alpha\mu)+\alpha(1-\mu)\}\,\phi(k,\lambda)}y_k\overline{z}^k$$

Then

$$\sum_{k=2}^{\infty}\frac{k(1-\alpha\mu)-\alpha(1-\mu)}{(1-\alpha)}\phi(k,\lambda)\left\{\frac{(1-\alpha)a_0}{[k(1-\alpha\mu)-\alpha(1-\mu)]\,\phi(k,\lambda)}x_k\right\}$$
$$+\sum_{k=1}^{\infty}\frac{k(1-\alpha\mu)+\alpha(1-\mu)}{(1-\alpha)}\phi(k,\lambda)\left\{\frac{(1-\alpha)a_0}{[k(1-\alpha\mu)+\alpha(1-\mu)]\,\phi(k,\lambda)}y_k\right\}$$
$$=\sum_{k=2}^{\infty}x_k+\sum_{k=1}^{\infty}y_k=1-x_1\le 1$$

and so $f\in clcoTS_H(\alpha,\lambda,\mu,z_0)$.

Conversely, suppose that $f \in clco\ \ TS_H(\alpha,\lambda,\mu,z_0)$. Set

$$x_k=\frac{[k(1-\alpha\mu)-\alpha(1-\mu)]\phi(k,\lambda)}{(1-\alpha)}\left|a_k\right|, \qquad (k=2,3,4.......)$$

and

$$y_k = \frac{[k(1-\alpha\mu)+\alpha(1-\mu)]\phi(k,\lambda)}{(1-\alpha)} |b_k|, \qquad (k = 1, 2, 3, ...)$$

Then note that by the Theorem (2),

$0 \le x_k \le 1, \quad (k = 2, 3, 4....)$ and $0 \le y_k \le 1, \quad (k = 1, 2, 3, ...)$

We define

$$x_1 = 1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k$$

and note that by Theorem (2), $x_1 \ge 0$. Consequently, we obtain

$$f(z) = \sum_{k=1}^{\infty} (x_k h_k(z) + y_k g_k(z))$$

as required.

The following theorem gives the bounds for functions in $TS_H(\alpha, \lambda, \mu, z_0)$ which yields a covering result for this class.

Theorem 4 : Let $f \in TS_H(\alpha, \lambda, \mu, z_0)$. Then

$$|f(z)| \le (a_0 + |b_1|)r + \left\{\frac{2-\lambda}{2}\right\} \left\{\frac{(1-\alpha)a_0}{2-\alpha(1+\mu)} - \frac{1+\alpha(1-2\mu)}{2-\alpha(1+\mu)}|b_1|\right\} r^2,$$

$$|z| = r < 1$$

and

$$|f(z)| \ge (a_0 + |b_1|)r - \left\{\frac{2-\lambda}{2}\right\} \left\{\frac{(1-\alpha)a_0}{2-\alpha(1+\mu)} - \frac{1+\alpha(1-2\mu)}{2-\alpha(1+\mu)}|b_1|\right\} r^2,$$

$$|z| = r < 1$$

Proof : We only prove that the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f \in TS_H(\alpha,\lambda,\mu,z_0)$. Taking the absolute value of f we obtain

$$|f(z)| \leq (a_0+|b_1|)r + \sum_{k=2}^{\infty} (|a_k|+|b_k|)\, r^2$$

$$= (a_0+|b_1|)r + \frac{a_0(1-\alpha)}{\{2-\alpha(1+\mu)\}\,\phi(2,\lambda)} \sum_{k=2}^{\infty} \frac{\{2-\alpha(1+\mu)\}\,\phi(2,\lambda)}{1-\alpha} (|a_k|+|b_k|)\, r^2$$

$$\leq (a_0+|b_1|)r + \frac{a_0(1-\alpha)(2-\lambda)}{2\,\{2-\alpha(1+\mu)\}} \sum_{k=2}^{\infty} \left(\frac{k(1-\alpha\mu)-\alpha(1-\mu)}{1-\alpha} |a_k| + \frac{k(1-\alpha\mu)+\alpha(1-\mu)}{1-\alpha} |b_k| \right) \phi(k,\lambda) r^2$$

$$\leq (a_0+|b_1|)r + \frac{a_0(1-\alpha)(2-\lambda)}{2\,\{2-\alpha(1+\mu)\}} \left(1 - \frac{1+\alpha(1-2\mu)}{1-\alpha} |b_k| \right) r^2$$

$$\leq (a_0+|b_1|)r + \left(\frac{a_0(1-\alpha)}{2-\alpha(1+\mu)} - \frac{1+\alpha(1-2\mu)}{2-\alpha(1+\mu)} |b_k| \right) \left\{ \frac{2-\lambda}{2} \right\} r^2.$$

0.4 CONVOLUTION AND CONVEX COMBINATIONS

In this section, we show that the class $TS_H(\alpha,\lambda,\mu,z_0)$ is closed under convolution and convex combination. Now we need the following definition of convolution of two hormonic functions. Let the functions $f(z)$ and $F(z)$ be defined by

$$f(z) = a_0 z - \sum_{k=2}^{\infty} |a_k|\, z^k + \sum_{k=1}^{\infty} |b_k|\, \overline{z}^k$$

and

$$F(z) = a_0 z - \sum_{k=2}^{\infty} |A_k|\, z^k + \sum_{k=1}^{\infty} |B_k|\, \overline{z}^k$$

we define their convolution

$$(f*F)(z) = f(z)*F(z) = a_0 z - \sum_{k=2}^{\infty} |a_k A_k|\, z^k + \sum_{k=1}^{\infty} |b_k B_k|\, \overline{z}^k \tag{16}$$

Using this definition, we show that the class $TS_H(\alpha,\lambda,\mu,z_0)$ is closed under convolution.

Theorem 5 : For $0 \leq \beta \leq \alpha < 1$, let $f \in TS_H(\alpha, \lambda, \mu, z_0)$ and $F \in TS_H(\beta, \lambda, \mu, z_0)$.Then

$$(f * F)(z) \in TS_H(\alpha, \lambda, \mu, z_0) \subset TS_H(\beta, \lambda, \mu, z_0)$$

Proof : Let

$$f(z) = a_0 z - \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| \overline{z}^k$$

be in $TS_H(\alpha, \lambda, \mu, z_0)$ and

$$F(z) = a_0 z - \sum_{k=2}^{\infty} |A_k| z^k + \sum_{k=1}^{\infty} |B_k| \overline{z}^k$$

be in $TS_H(\beta, \lambda, \mu, z_0)$.Then the convolution $(f * F)(z)$ is given by (16).We wish to show that the coefficients of $(f * F)(z)$ satisfy the required condition given in Theorem (2). For $F \in TS_H(\beta, \lambda, \mu, z_0)$ we note that $|A_k| \leq 1$ and $|B_k| \leq 1$. Now, for the convolution function $(f * F)(z)$, we obtain

$$\sum_{k=2}^{\infty} \left\{ \frac{k(1-\alpha\mu) - \alpha(1-\mu)}{1-\alpha} \right\} \phi(k,\lambda) |a_k A_k| + \sum_{k=1}^{\infty} \left\{ \frac{k(1-\alpha\mu) + \alpha(1-\mu)}{1-\alpha} \right\} \phi(k,\lambda) |b_k B_k|$$

$$\leq \sum_{k=2}^{\infty} \left\{ \frac{k(1-\alpha\mu) - \alpha(1-\mu)}{1-\alpha} \right\} \phi(k,\lambda) |a_k| + \sum_{k=1}^{\infty} \left\{ \frac{k(1-\alpha\mu) + \alpha(1-\mu)}{1-\alpha} \right\} \phi(k,\lambda) |b_k|$$

$\leq a_0 \quad (Since \;\; f \in TS_H(\alpha, \lambda, \mu, z_0))$.

Therefore $(f * F)(z) \in TS_H(\alpha, \lambda, \mu, z_0)$.

0.5 A FAMILY OF CLASS PRESERVING INTEGRAL OPERATOR

Let $f(z) = h(z) + \overline{g(z)}$ be defined by (1). Then $F(z)$ defined by the relation

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} h(t)dt + \overline{\frac{c+1}{z^c} \int_0^z t^{c-1} g(t)dt} \qquad (c > -1). \tag{17}$$

Theorem 6 : Let $f(z) = h(z) + \overline{g(z)} \in S_H$ be given by (8) and $f \in TS_H(\alpha, \lambda, \mu, z_0)$, $0 \leq \alpha < 1, \quad 0 \leq \lambda < 1, \quad and \quad 0 \leq \mu \leq 1$. Then $F(z)$ defined by (17) is also in the class $TS_H(\alpha, \lambda, \mu, z_0)$.

Proof : From the representation (17) of $F(z)$, it follows that

$$F(z) = a_0 z - \sum_{k=2}^{\infty} \frac{c+1}{c+k} |a_k| z^k + \sum_{k=1}^{\infty} \frac{c+1}{c+k} |b_k| \overline{z}^k$$

Since $f \in TS_H(\alpha, \lambda, \mu, z_0)$, we have

$$\sum_{k=2}^{\infty} \left\{ \frac{k(1-\alpha\mu) - \alpha(1-\mu)}{1-\alpha} \right\} \phi(k,\lambda) |a_k| + \sum_{k=1}^{\infty} \left\{ \frac{k(1-\alpha\mu) + \alpha(1-\mu)}{1-\alpha} \right\} \phi(k,\lambda) |b_k| \leq a_0 \tag{18}$$

Now

$$\sum_{k=2}^{\infty} \left\{ \frac{k(1-\alpha\mu) - \alpha(1-\mu)}{1-\alpha} \right\} \phi(k,\lambda) \left(\frac{c+1}{c+k} \right) |a_k| + \sum_{k=1}^{\infty} \left\{ \frac{k(1-\alpha\mu) + \alpha(1-\mu)}{1-\alpha} \right\} \phi(k,\lambda) \left(\frac{c+1}{c+k} \right) |b_k|$$

$$\leq \sum_{k=2}^{\infty} \left\{ \frac{k(1-\alpha\mu) - \alpha(1-\mu)}{1-\alpha} \right\} \phi(k,\lambda) |a_k| + \sum_{k=1}^{\infty} \left\{ \frac{k(1-\alpha\mu) + \alpha(1-\mu)}{1-\alpha} \right\} \phi(k,\lambda) |b_k|$$

$\leq a_0$

Thus $F(z) \in TS_H(\alpha, \lambda, \mu, z_0)$.

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