Weighted Inverse Weibull Distribution : Statistical Properties and Applications

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Abstract

In this paper, the weighted inverse Weibull (WIW) class of distributions is proposed and studied. This class of distributions contains several models such as: length-biased, hazard and reverse hazard proportional inverse Weibull, proportional inverse Weibull, inverse Weibull, inverse exponential, inverse Rayleigh, and Fréchet distributions as special cases. Properties of these distributions including the behavior of the hazard function, moments, coefficients of variation, skewness, and kurtosis, Rényi entropy and Fisher information are presented. Estimates of the model parameters via method of maximum likelihood (ML) are presented. Extensive simulation study is conducted and numerical examples are given.

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1 Introduction

The inverse Weibull (IW) distribution can be readily applied to modeling processes in reliability, ecology, medicine, branching processes and biological studies. The properties and applications of IW distribution in several areas are given in the literature (Keller [8], Calabria and Pulcini [2], [3], [4], Johnson [7], Khan et al. [9]). A random variable X has an IW distribution if the corresponding probability density function (pdf) is given by

$$f(x;\alpha,\beta)=\beta\alpha^{-\beta}x^{-\beta-1}exp[-(\alpha x)^{-\beta}], \qquad x\geq 0,\, \alpha>0,\, \beta>0. \tag{1}$$

If $\beta=1$, the IW pdf becomes inverse exponential pdf, and when $\beta=2$, the IW pdf is referred to as the inverse Raleigh pdf. The IW cumulative distribution

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function (cdf) is given by

$$F(x;\alpha,\beta) = exp[-(\alpha x)^{-\beta}], \qquad \alpha > 0,\ \beta > 0, \tag{2}$$

where α and β are the scale and shape parameters, respectively. If $\alpha = 1$, we have the Fréchet distribution function.

Motivated by various applications of weighted distributions, (Oluyede [13], Patil and Rao [14]) to biased samples in several areas including reliability, exponential tilting (weighting) in finance and actuarial sciences, we construct and present the statistical properties of this new class of distributions called weighted inverse Weibull (WIW) distribution and apply it to real lifetime data in order to demonstrate its usefulness.

1.1 Weighted Distribution

Weighted distribution can be used to deal with model specification and data interpretation problems. Patil and Rao [14] used weighted distributions as stochastic models in the study of harvesting and predation. Let Y be a non-negative random variable with its natural pdf $f(y;\theta)$, where θ is a parameter in the parameter space Θ, then the pdf of the weighted random variable Y^w is given by:

$$f^w(y;\theta,\beta) = \frac{w(y,\beta)f(y;\theta)}{\omega}, \tag{3}$$

where the weight function $w(y,\beta)$ is a positive function, that may depend on the parameter β, and $0 < \omega = E(w(Y,\beta)) < \infty$ is a normalizing constant. A general class of weight functions $w(y)$ is defined as follows:

$$w(y) = y^k e^{\ell y} F^i(y)\overline{F}^j(y). \tag{4}$$

Setting $k = 0$; $k = j = i = 0$; $\ell = i = j = 0$; $k = \ell = 0$; $i \to i-1$; $j = n-i$; $k = \ell = i = 0$ and $k = \ell = j = 0$ in this weight function, one at a time, implies probability weighted moments, moment-generating functions, moments, order statistics, proportional hazards and proportional reversed hazards, respectively, where $F(y) = P(Y \leq y)$ and $\overline{F}(y) = 1 - F(y)$. If $w(y) = y$, then $Y^* = Y^w$ is called the size-biased version of Y.

The rest of the paper is organized as follows. In section 2, probability weighted moments and the weighted inverse Weibull (WIW) are developed. Some statistical properties, including mode, hazard and reverse hazard functions of the WIW distribution are presented. Glaser's Lemma [6] is applied to the WIW distribution to determine the behavior of the hazard function. Distribution of functions of WIW random variables are presented in section 3. In section 4, moments, entropy measures and Fisher information are given. Section 5 contains estimation of the parameters of the WIW distribution via the methods maximum likelihood. Simulation study is presented in section 6. Numerical examples are given in section 7, followed by concluding remarks.

2 Weighted Proportional Inverse Weibull Distribution

In this section, probability weighted moments (PWMs) of the proportional inverse Weibull distribution and WIW distribution are presented. The mode, hazard and reverse hazard functions are given. The proportional inverse Weibull (IW) distribution has a cdf given by

$$G(x;\alpha,\beta,\gamma)=[F(x)]^{\gamma}=\exp[-\gamma(\alpha x)^{-\beta}], \tag{5}$$

for $\alpha>0$, $\beta>0$, $\gamma>0$, and $x\geq 0$. Let $\alpha^{-\beta}\gamma=\theta$, then PIW distribution reduces to the IW distribution with cdf and pdf

$$G(x;\theta,\beta)=\exp[-\theta x^{-\beta}],$$

and

$$g(x;\theta,\beta)=\theta\beta x^{-\beta-1}\exp[-\theta x^{-\beta}], \tag{6}$$

respectively, for $\theta>0$, $\beta>0$, and $x\geq 0$.

2.1 Probability Weighted Moments

The PWMs of the IW distribution are given by

$$E[X^kG^l(X)\overline{G}^m(X)] = \sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(m+1)}{\Gamma(m+1-j)\Gamma(j+1)}E[X^kG^{l+j}(X)], \tag{7}$$

where

$$E[X^kG^{l+j}(X)]=\int_0^{\infty}x^k e^{-\theta x^{-\beta}(j+l)}\theta\beta x^{-\beta-1}e^{-\theta x^{-\beta}}dx.$$

Now, we make the substitution $u=\theta x^{-\beta}(l+j+1)$, so that

$$\begin{aligned}E[X^kG^{l+j}(X)] &= \theta^{\frac{k}{\beta}}(l+j+1)^{\frac{k}{\beta}-1}\int_0^{\infty}u^{-\frac{k}{\beta}}e^{-u}du\\ &= \theta^{\frac{k}{\beta}}(l+j+1)^{\frac{k}{\beta}-1}\Gamma\left(1-\frac{k}{\beta}\right).\end{aligned}$$

Therefore, PWMs of the IW distribution are

$$\begin{aligned}E[X^kG^l(X)\overline{G}^m(X)] &= \sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(m+1)}{\Gamma(m+1-j)\Gamma(j+1)}\\ &\quad\cdot\theta^{\frac{k}{\beta}}(l+j+1)^{\frac{k}{\beta}-1}\Gamma\left(1-\frac{k}{\beta}\right),\quad \text{for } \beta>k.\end{aligned} \tag{8}$$

Remark: Special cases

1. When $l = m = 0$, the k^{th} noncentral moments of the IW distribution is
$$E[X^k] = \frac{\gamma^{\frac{k}{\beta}}}{\alpha^k}\Gamma\left(1-\frac{k}{\beta}\right), \quad \text{for } \beta > k. \tag{9}$$

2. When $l = k = 0$, we have
$$E[\overline{G}^m(X)] = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(m+1)}{\Gamma(m+1-j)\Gamma(j+1)} \cdot (j+1)^{-1}, \tag{10}$$

3. When $l = 0$, we have
$$E[X^k \overline{G}^m(X)] = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(m+1)}{\Gamma(m+1-j)\Gamma(j+1)} \cdot \theta^{\frac{k}{\beta}} (j+1)^{\frac{k}{\beta}-1} \Gamma\left(1-\frac{k}{\beta}\right), \tag{11}$$
for $\beta > k$.

4. When $m = 0$, we obtain
$$E[X^k G^l(X)] = \theta^{\frac{k}{\beta}} (l+1)^{\frac{k}{\beta}-1} \Gamma\left(1-\frac{k}{\beta}\right), \quad \beta > k. \tag{12}$$

5. When $k = m = 0$, we have $E[G^l(X)] = (l+1)^{-1}$.

6. When $l \to i-1$, $m \to n-i$, we have
$$E[X^k G^{i-1}(X) \overline{G}^{n-i}(X)] = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(n-i+1)}{\Gamma(n-i+1-j)\Gamma(j+1)} \cdot \theta^{\frac{k}{\beta}} (j+i)^{\frac{k}{\beta}-1} \Gamma\left(1-\frac{k}{\beta}\right), \quad \beta > k. \tag{13}$$

7. When $k = 0$, we obtain
$$E[G^l(X)\overline{G}^m(X)] = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(m+1)}{\Gamma(m+1-j)\Gamma(j+1)} \cdot (l+j+1)^{-1}. \tag{14}$$

8. When $k = l = 1$, we get
$$E[XG(X)\overline{G}^m(X)] = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(m+1)}{\Gamma(m+1-j)\Gamma(j+1)} \cdot \theta^{\frac{k}{\beta}} (j+2)^{\frac{1}{\beta}-1} \Gamma\left(1-\frac{1}{\beta}\right), \quad \text{for } \beta > 1. \tag{15}$$

2.2 Weighted Inverse Weibull Distribution

In this section, we present the WIW distribution and some of its properties. We are particularly interested in studying the statistical properties of the WIW distribution with the weight function $w(x) = x^k G^l_{_{IW}}(x)$, compared to those with the weight function when $l = 0$, as well as the parent IW, and its sub-models. We are particularly interested in the distribution obtained via the weight function $w(x) = xG_{_{IW}}(x)$. In general, the WIW pdf is

$$\begin{aligned} g_{_{WIW}}(x) &= \frac{x^k G^l(x)\overline{G}^m(x) g(x)}{E[X^k G^l(X)\overline{G}^m(X)]} \\ &= \frac{x^{k-\beta-1}\theta\beta \left(1-e^{-\theta x^{-\beta}}\right)^m e^{-\theta(l+1)x^{-\beta}}}{\sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(m+1)\,\gamma^{\frac{k}{\beta}}(l+j+1)^{\frac{k}{\beta}-1}}{\Gamma(m+1-j)\Gamma(j+1)} \Gamma\left(1-\frac{k}{\beta}\right)}, \qquad (16) \end{aligned}$$

for $\beta > k$. When $m = 0$, $k = l = 1$ the corresponding WIW pdf and cdf are given by

$$\begin{aligned} g_{_{WIW}}(x) &= \frac{xG(x)g(x)}{E[XG(X)]} \\ &= \frac{\beta\,(2\theta)^{1-\frac{1}{\beta}}\, x^{-\beta} e^{-2\theta x^{-\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)}, \qquad (17) \end{aligned}$$

and

$$\begin{aligned} G_{_{WIW}}(x;\theta,\beta) &= \int_0^x \frac{\beta\,(2\theta)^{1-\frac{1}{\beta}}\, y^{-\beta} e^{-2\theta y^{-\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)} dy \\ &= \frac{\Gamma\left(1-\frac{1}{\beta}, 2\theta x^{-\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)}, \quad \text{for } \beta > 1, \qquad (18) \end{aligned}$$

respectively, where $\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}\,dt$ is the upper incomplete gamma function. Graphs of the WIW pdfs for selected values of the parameters θ and β show different shapes of the curves depending on the values of the parameters [fig. 1]. If the distribution of the a random variable is given by equation (18), we write $X \sim WIW(\theta,\beta) \equiv WIW(\theta,\, \beta,\, 1,\, 1)$

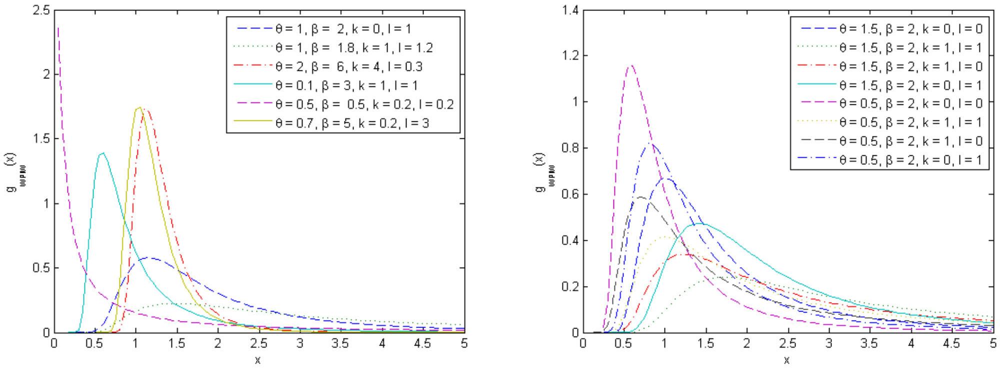

Figure 1: Pdfs of the WIW Distribution

2.2.1 Mode of the WIW Distribution

Consider the WIW pdf given in equation (17). Note that

$$\begin{aligned}\ln(g_{WIW}(x)) &= \left(1-\frac{1}{\beta}\right)\ln(2\theta)+\ln(\beta)-\beta\ln(x)\\ &\quad -2\theta x^{-\beta}-\ln\left(\Gamma\left(1-\frac{1}{\beta}\right)\right). \end{aligned} \tag{19}$$

Differentiating equation (19) with respect to x, we obtain

$$\frac{\partial \ln g_{WIW}(x;\theta,\beta)}{\partial x}=\frac{-\beta}{x}+\frac{2\theta\beta x^{-\beta}}{x}. \tag{20}$$

Now, set equation (20) equal 0 and solve for x, to get

$$x_0=(2\theta)^{\frac{1}{\beta}}. \tag{21}$$

Note, that

$$\frac{\partial^2 \ln g_{WIW}(x;\theta,\beta)}{\partial x^2}=-\frac{-\beta}{x^2}-\frac{2\beta(\beta+1)\theta x^{-\beta}}{x^2}<0. \tag{22}$$

When $0<x<(2\theta)^{\frac{1}{\beta}}$, $\frac{\partial \ln g(x;\theta,\beta)}{\partial x}>0$, so $g_{WIW}(x;\theta,\beta)$ is increasing, and when $x>(2\theta)^{\frac{1}{\beta}}$, $g_{WIW}(x;\theta,\beta,1,1)$ is decreasing, therefore $g_{WIW}(x;\theta,\beta,1,1)$ achieves a maximum when $x_0=(2\theta)^{\frac{1}{\beta}}$, so that x_0 is the mode of WIW distribution.

2.2.2 Hazard and Reverse Hazard Functions

The hazard function of the WIW distribution is given by

$$\lambda_{G_{WIW}}(x;\theta,\beta) = \frac{\beta\,(2\theta)^{1-\frac{1}{\beta}}\,x^{-\beta}e^{-2\theta x^{-\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)-\Gamma\left(1-\frac{1}{\beta},2\theta x^{-\beta}\right)}, \tag{23}$$

for $\theta > 0$, $\beta > 1$, and $x \geq 0$. The behavior of the hazard function of the WIW distribution is established via Glaser's Lemma [6]. Note that

$$\begin{aligned} \eta_{G_{WIW}}(x;\theta,\beta) &= -\frac{g'_{WIW}(x;\theta,\beta)}{g_{WIW}(x;\theta,\beta)} \\ &= (\beta)x^{-1} - 2\theta\beta x^{-\beta-1}, \end{aligned} \tag{24}$$

and

$$\eta'_{G_{WIW}}(x) = (-\beta)x^{-2} + 2\theta\beta(\beta+1)x^{-\beta-2}. \tag{25}$$

Now, $\eta'_{G_{WIW}}(x) = 0$ implies $x_0^* = (2\gamma(\beta+1))^{\frac{1}{\beta}}$, for $\theta > 0$, $\beta > 1$. Note that, when $0 < x < x_0^*$, $\eta'_{G_{WIW}}(x) > 0$, $\eta'_{G_{WIW}}(x_0^*) = 0$ and when $x > x_0^*$, $\eta'_{G_{WIW}}(x) < 0$. Consequently, WIW hazard function has an *upside down bathtub shape*. The graphs of the hazard function show upside down bathtub shape for the selected values of the parameters [fig. 2]. The reverse hazard function

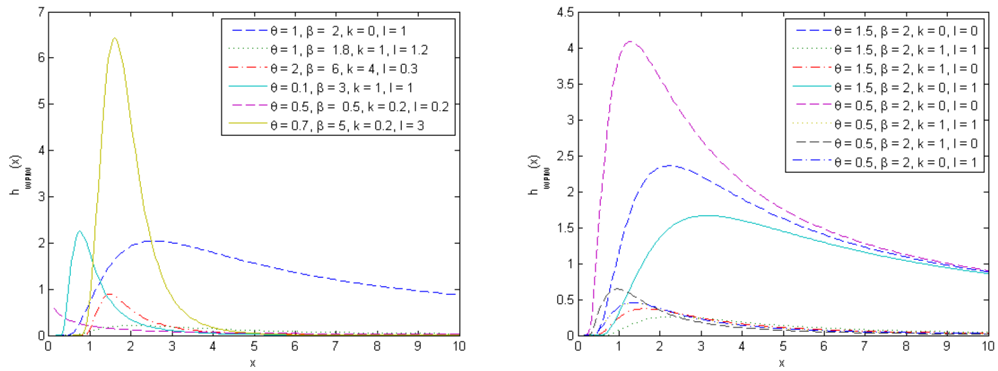

Figure 2: Graphs of Hazard Functions of the WIW Distribution

is given by

$$\tau_{G_{WIW}}(x;\theta,\beta) = \frac{\beta\,(2\theta)^{1-\frac{k}{\beta}}\,x^{-\beta}e^{-2\theta x^{-\beta}}}{\Gamma\left(1-\dfrac{1}{\beta}, 2\theta x^{-\beta}\right)}, \tag{26}$$

for $\theta > 0$, $\beta > 1$, and $x \geq 0$.

3 Distribution of Functions of Random Variables

In this section, distributions of functions of the IW and WIW random variables are presented. Consider the IW pdf:

$$g(x;\theta,\beta) = \theta\beta x^{\beta-1}e^{-\theta x^{\beta}}, \quad x > 0,\ \theta > 0,\ \beta > 0.$$

1. Let $Y = \theta X^{-\beta}$. Then the pdf of Y is

$$\begin{aligned} g_1(y;\theta,\beta) &= \theta\beta x^{-\beta-1} e^{-\theta x^{\beta}} (\theta\beta)^{-1} x^{\beta+1} \\ &= e^{-\theta x^{-\beta}} = e^{-y}, \;\; y > 0, \end{aligned} \tag{27}$$

that is, if $X \sim IW(\theta, \beta)$, then $Y = \theta X^{-\beta} \sim EXP(1)$, unit exponential distribution.

2. Let $X \sim WIW(\theta, \beta)$, that is

$$g_{WIW}(x;\theta,\beta) = \frac{\beta (2\theta)^{1-\frac{1}{\beta}} x^{-\beta} e^{-2\theta x^{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)},$$

and $Y = \theta X^{-\beta}$, then the resulting pdf of Y is given by

$$g_2(y;\beta) = \frac{2^{1-\frac{1}{\beta}} e^{-2y}}{\Gamma\left(1-\frac{1}{\beta}\right)}, \quad \beta > 1, \text{ and } y > 0. \tag{28}$$

Note, that if $l = k = 0$, then $g_1(y) = g_2(y)$, $y > 0$.

3. Now, let $Z = 2\theta X^{-\beta}$, where $X \sim WIW(\theta, \beta)$. The pdf of the random variable Z is given by

$$g(z;\theta,\beta) = \frac{(2\theta z)^{-\frac{1}{\beta}} (2\theta)^{\frac{1}{\beta}} e^{-z}}{\Gamma\left(1-\frac{1}{\beta}\right)} = \frac{z^{1-\frac{1}{\beta}-1} e^{-z}}{\Gamma\left(1-\frac{1}{\beta}\right)}, \tag{29}$$

for $z > 0$, $\beta > 1$. Thus, if $X \sim WIW(\theta, \beta)$, then $Z \sim GAM(1-\frac{1}{\beta}, 1)$.

4 Moments, Entropy and Fisher Information

In this section, we present the moments and related functions as well as entropies and Fisher information for the WIW distribution. The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty.

4.1 Moments and Moment Generating Function

The c^{th} non-central moment of the WIW distribution is given by

$$E(X^c) = \int_0^\infty \frac{\beta (2\theta)^{1-\frac{1}{\beta}} \cdot x^c \cdot x^{-\beta} e^{-2\theta x^{-\beta}}}{\Gamma\left(1-\dfrac{1}{\beta}\right)} dx. \tag{30}$$

Making the substitution $u = 2\theta x^{-\beta}$, $du = -2\theta\beta x^{-\beta-1}dx$, so that $x = \left(\frac{2\theta}{u}\right)^{\frac{1}{\beta}}$, we obtain

$$\begin{aligned} E(X^c) &= \int_0^\infty \frac{(2\theta)^{\frac{c}{\beta}} \cdot u^{-\frac{c+1}{\beta}} exp[-u]}{\Gamma\left(1-\frac{1}{\beta}\right)}\, dx \\ &= \frac{(2\theta)^{\frac{c}{\beta}}\Gamma\left(1-\frac{c+1}{\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)}, \end{aligned} \tag{31}$$

where $\beta > c+1$. Let $\delta_c = \Gamma\left(1-\frac{c+1}{\beta}\right)$, then the mean and variance are $\mu_X = \frac{(2\theta)^{\frac{1}{\beta}}\delta_1}{\delta_0}$, and $\sigma_X^2 = \frac{(2\theta)^{\frac{2}{\beta}}(\delta_0\delta_2 - \delta_1^2)}{\delta_0^2}$, respectively. The coefficient of variation (CV) is $CV = \frac{\sqrt{\delta_0\delta_2 - \delta_1^2}}{\delta_1}$. The coefficient of skewness (CS) is given by $CS = \frac{2\delta_1^3 - 3\delta_0\delta_1\delta_2 + \delta_0^2\delta_3}{[\delta_0\delta_2 - \delta_1^2]^{\frac{3}{2}}}$, and the coefficient of kurtosis (CK) is $CK = \frac{\delta_0^3\delta_4 - 4\delta_0^2\delta_1\delta_3 + 6\delta_0\delta_1^2\delta_2 - 3\delta_1^4}{[\delta_0\delta_2 - \delta_1^2]^2}$.

The graphs of CV, CS and CK versus β are given in figures 3 and 4. We can see decreasing coefficients (CV, CS, CK) for increasing values of β.

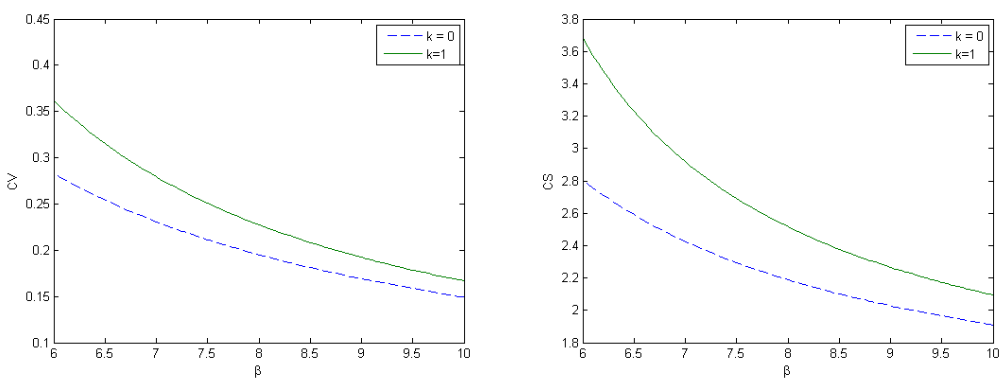

Figure 3: Graphs of CV and CS for WIW Distribution

The moment generating function (MGF) of the WIW distribution is $M_X(t) =$

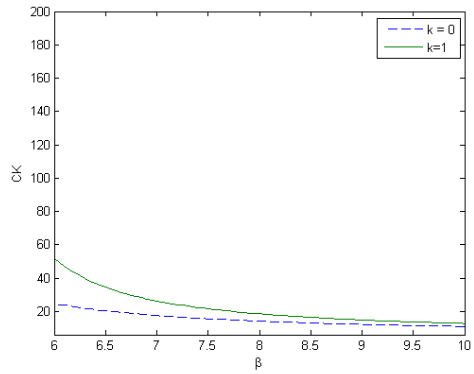

Figure 4: Graphs of CK for WIW Distribution

$\sum_{j=0}^{\infty} \frac{t^j}{j!} E\left[X^j\right]$, where $E\left[X^j\right] = \dfrac{(2\theta)^{\frac{j}{\beta}} \Gamma\left(1 - \dfrac{j+1}{\beta}\right)}{\Gamma\left(1 - \dfrac{1}{\beta}\right)}$, $\beta > j + 1$. Table 1 gives the mode, mean, standard deviation (STD), CV, CS and CK for some values of the parameters θ, β, k and l, $(k, l = 0 \,\text{or}\, 1)$. We can see from Table 1, as

Table 1: Mode, Mean, STD, Coefficients of Variation, Skewness and Kurtosis

θ	β	k	l	Mode	Mean	STD	CV	CS	CK
1.0	5.0	0.0	1.0	6.9882712	1.3373488	0.1764993	0.3657341	3.5350716	48.0915121
0.1	9.0	0.0	0.0	0.6968373	0.8344695	0.0171375	0.1690772	2.0279664	12.2288967
0.5	9.0	1.0	1.0	7.0504293	1.1042834	0.0369783	0.1922976	2.2660506	14.8729807
0.5	6.0	0.0	0.0	2.1690601	1.0056349	0.0634625	0.2827681	2.8055664	24.6781193
1.0	6.0	0.0	0.0	4.3381202	1.1287870	0.0799578	0.2827681	2.8055664	24.6781193
1.5	6.0	0.0	0.0	6.5071802	1.2077041	0.0915288	0.2827681	2.8055664	24.6781193
0.5	6.0	0.0	1.0	4.3381202	1.1287870	0.0799578	0.2827681	2.8055664	24.6781193
1.0	6.0	0.0	1.0	8.6762403	1.2670206	0.1007405	0.2827681	2.8055664	24.6781193
1.5	6.0	0.0	1.0	13.0143605	1.3556021	0.1153191	0.2827681	2.8055664	24.6781193
0.5	6.0	1.0	1.0	4.4510183	1.1996222	0.1311354	0.3621262	3.6848057	51.6558037
1.0	6.0	1.0	1.0	8.9020365	1.3465304	0.1652203	0.3621262	3.6848057	51.6558037
1.5	6.0	1.0	1.0	13.3530548	1.4406706	0.1891300	0.3621262	3.6848057	51.6558037
0.5	6.0	1.0	0.0	2.2255091	1.0687418	0.1040822	0.3621262	3.6848057	51.6558037
1.0	6.0	1.0	0.0	4.4510183	1.1996222	0.1311354	0.3621262	3.6848057	51.6558037
1.5	6.0	1.0	0.0	6.6765274	1.2834916	0.1501126	0.3621262	3.6848057	51.6558037

β increases, the values of CV, CS and CK decrease. We can also see that as θ increases, the values for mode and mean increase, for the selected values of the model parameters.

4.2 Shannon Entropy

Shannon entropy [16] for WIW distribution is given by

$$\begin{aligned} H(g_{_{WIW}}) &= -\int_0^\infty \log\left(\frac{\beta\,(2\theta)^{1-\frac{1}{\beta}}\,x^{-\beta}e^{-2\theta x^{-\beta}}}{\Gamma\left(1-\dfrac{1}{\beta}\right)}\right) \\ &\quad * \left(\frac{\beta\,(2\theta)^{1-\frac{1}{\beta}}\,x^{-\beta}e^{-2\theta x^{-\beta}}}{\Gamma\left(1-\dfrac{1}{\beta}\right)}\right) dx \\ &= -[A+B+C], \end{aligned} \tag{32}$$

where A, B and C are obtained below:

$$\begin{aligned} A &= \log\left(\frac{\beta(2\theta)^{1-\frac{1}{\beta}}}{\delta_0}\right)\int_0^\infty \frac{\beta(2\theta)^{1-\frac{1}{\beta}}x^{-\beta}e^{-2\theta x^{-\beta}}}{\delta_0}\,dx \\ &= \log(\beta) + \left(1-\frac{1}{\beta}\right)\log(2\theta) - \log(\delta_0), \end{aligned} \tag{33}$$

$$\begin{aligned} B &= \int_0^\infty \frac{\beta(2\theta))^{1-\frac{1}{\beta}}x^{-\beta}e^{-2\theta x^{-\beta}}}{\delta_0}\cdot\left(-2\theta x^{-\beta}\right)\,dx \\ &= -\frac{\beta(2\theta)^{2-\frac{1}{\beta}}}{\delta_0}\int_0^\infty e^{-2\theta x^{-\beta}}x^{-2\beta}\,dx. \end{aligned}$$

Let $u = 2\theta x^{-\beta}$, then $du = -2\beta\theta x^{-\beta-1}\,dx$, $x = \left(\dfrac{2\theta}{u}\right)^{\frac{1}{\beta}}$, and we obtain

$$B = -\frac{\Gamma\left(2-\dfrac{1}{\beta}\right)}{\delta_0}. \tag{34}$$

Also,

$$\begin{aligned} C &= \int_0^\infty \frac{\beta(2\theta)^{1-\frac{1}{\beta}}x^{-\beta}e^{-2\theta x^{-\beta}}}{\delta_0}\cdot(-\beta)\log(x)\,dx, \\ &= -\frac{1}{\delta_0}\int_0^\infty [\log(2\theta)-\log(u)]\,e^{-u}u^{-\frac{1}{\beta}}\,du, \end{aligned}$$

where $u = 2\theta x^{-\beta}$. Using the fact that $\Gamma^{(n)}(t) = \int_0^\infty \log^n(x)x^{t-1}\exp(-x)\,dx$, the integral becomes

$$C = \frac{\delta_0'}{\delta_0} - \frac{\delta_0\log(2\theta)}{\delta_0}. \tag{35}$$

Consequently, Shannon entropy for WIW distribution is given by

$$H(g_{_{WIW}}) = \frac{\beta\Gamma\left(2-\frac{1}{\beta}\right) - \beta\delta_0' - \beta\delta_0\log\left(\frac{\beta}{\delta_0}\right) + \delta_0\log(2\theta)}{\beta\delta_0}. \tag{36}$$

4.3 Rényi Entropy

Rényi entropy [15] generalizes Shannon entropy. Renyi entropy of order t, where $t > 0$ and $t \neq 1$ is given by

$$H_R(g) = \frac{1}{1-t}\log\left[\int_0^\infty g^t(x)\,dx\right]. \tag{37}$$

Note that

$$\int_0^\infty g^t_{_{WIW}}(x)\,dx = \int_0^\infty \left[\frac{\beta(2\theta)^{1-\frac{1}{\beta}}x^{-\beta}e^{-2\theta x^{-\beta}}}{\delta_0}\right]^t dx.$$

Let $u = 2\,\theta\,t\,x^{-\beta}$, then the integral becomes

$$\int_0^\infty g^t_{_{WIW}}(x)\,dx = \frac{\beta^{t-1}(2\theta)^{\frac{1-t}{\beta}}}{\delta_0^t}t^{\frac{-\beta t+1}{\beta}}\Gamma\left(\frac{\beta t-1}{\beta}\right). \tag{38}$$

Rényi entropy for WIW distribution reduces to

$$\begin{aligned} H_R(g_{_{WIW}}) &= \log(\beta) + \frac{1}{\beta}\log(2\theta) + \frac{-\beta t+1}{\beta(1-t)}\log(t) \\ &\quad + \frac{1}{1-t}\log\Gamma\left(\frac{\beta t-1}{\beta}\right) - \frac{t}{1-t}\log(\delta_0). \end{aligned} \tag{39}$$

When $k = 1$, $l = 0$, and $k = l = 0$, we obtain Renyi entropy for the length-biased IW and IW distributions, respectively.

4.4 Fisher Information

Let $\Theta = (\theta, \beta)$. Fisher information matrix (FIM) for WIW distribution is given by:

$$I(\Theta) = I(\theta,\beta) = \begin{bmatrix} I_{\theta\theta} & I_{\theta\beta} \\ I_{\theta\beta} & I_{\beta\beta} \end{bmatrix},$$

where the entries of the $I(\theta, \beta)$ are given in Appendix B.

5 Estimation of Parameters

In this section, we obtain estimates of the parameters for the WIW distribution. Method of maximum likelihood (ML) estimation is presented. Asymptotic confidence intervals and likelihood ratio test are also given.

5.1 Maximum Likelihood Estimation

Let $x_1, x_2, \cdots, x_n$ be a random sample from a WIW distribution and $\Theta = (\theta, \beta)$. The log-likelihood function is

$$\begin{aligned}\ln L &= l(\Theta) = n\ln(\beta) + \left(n - \frac{n}{\beta}\right)\ln(2\theta) - n\ln\Gamma\left(1 - \frac{1}{\beta}\right) \\ &\quad -\beta\sum_{i=1}^{n}\ln(x_i) - 2\theta\sum_{i=1}^{n}x_i^{-\beta}.\end{aligned}$$

The normal equations are

$$\begin{aligned}\frac{\partial \ln L}{\partial \theta} &= \left(1 - \frac{1}{\hat{\beta}}\right)\frac{n}{\hat{\theta}} - 2\sum_{i=1}^{n}x_i^{-\hat{\beta}} = 0, \\ \frac{\partial \ln L}{\partial \beta} &= \frac{n}{\hat{\beta}} + \frac{n}{\hat{\beta}^2}\ln(2\hat{\theta}) - \sum_{i=1}^{n}\ln(x_i) - \frac{n}{\hat{\beta}^2}\Psi\left(1 - \frac{1}{\hat{\beta}}\right) \\ &\quad +2\hat{\theta}\sum_{i=1}^{n}\left(x_i^{-\hat{\beta}}\ln(x_i)\right) = 0. \qquad (40)\end{aligned}$$

Therefore, if β is known, we can obtain an estimate for $\hat{\theta}$ from (40)

$$\hat{\theta} = \frac{n(\hat{\beta} - 1)}{2\hat{\beta}\sum_{i=1}^{n}x_i^{-\hat{\beta}}},$$

and if θ is known, we can find an estimate for β using Newton's method. When all parameters are unknown, numerical methods must be used to obtain the MLE $\hat{\theta}$, and $\hat{\beta}$ of the parameters θ, and β, respectively, since the system of equations does not admit any closed form solutions.

5.2 Asymptotic Confidence Intervals and Likelihood Ratio Test

The multivariate normal distribution with mean vector $\underline{\mu}$ and covariance matrix $I(\Theta)$, where $\Theta = (\theta, \beta)$, can be used to obtain confidence intervals and confidence regions for the parameters of the WIW distribution. The approximate $100(1-\delta)\%$ two-sided confidence intervals for the parameters θ, and β are given by

$$\hat{\theta} \pm z_{\delta/2}\left[\widehat{Var(\hat{\theta})}\right]^{1/2} \quad \text{and} \quad \hat{\beta} \pm z_{\delta/2}\left[\widehat{Var(\hat{\beta})}\right]^{1/2}. \tag{41}$$

respectively, where $Var(\cdot)$ are the diagonal elements of $I^{-1}(\hat{\Theta})$ or $J^{-1}(\hat{\Theta})$, corresponding to each parameter, $J(\Theta) = \left[-\frac{\partial^2 l(\Theta)}{\partial\theta_i\partial\theta_j}|_{\Theta=\hat{\Theta}}\right]_{2\text{x}2}$ is the observed information matrix and $z_{\delta/2}$ is the upper $\frac{\delta}{2}^{th}$ percentile of the standard normal distribution.

The likelihood ratio (LR) statistic for testing $\theta = 1$ is given by

$$w = 2[\ln(L(\hat{\theta},\, \hat{\beta},\, 1,\, 1)) - \ln(L(1,\, \tilde{\beta},\, 1,\, 1))], \tag{42}$$

where $\hat{\theta}$, and $\hat{\beta}$ are the unrestricted estimates, and $\tilde{\beta}$ is restricted estimate. The LR test reject the null hypothesis if $w > \chi^2_\epsilon$, where χ^2_ϵ denote the upper $100\epsilon\%$ point of a χ^2 distribution with 1 degrees of freedom.

6 Simulation Study

Various simulation were conducted for different sample sizes (n=50, 100, 200, 300, 500, 1000) to study the performance of IW and WIW distributions. We simulated 1000 samples for Model 1 with the true values of the parameters $\theta = 0.5$, $\beta = 3$, $k = 1$, and $l = 1$, Model 2 with the true values of the parameters $\theta = 1.2$, $\beta = 4.6$, $k = 1$, and $l = 1$, and Model 3 with the true values of the parameters $\theta = 1$, $\beta = 2$, $k = 1$, and $l = 1$. From the results of the simulations presented in Tables 2, 4 and 6, we can see that average bias for the parameters are very small, it is negative for the parameter l in the Model 1, the average bias for the parameters β and k in the Model 2 is negative also. The root mean squared errors (RMSEs) decreases as the sample size n increases. Also, as the sample size gets larger the mean estimates of the parameters gets closer to the true parameter values. When $k = 0$ or $l = 0$, the simulation results are presented in Tables 3, 5 and 7.

7 Applications

In this section, we present examples to illustrate the flexibility of the WIW distribution and its sub-models for data modeling. The first data set from Bjerkedal [1] represents the survival time, in days, of guinea pigs injected with

Table 2: Simulation results for Model 1: Mean estimates, Average Bias and RMSEs

		IW			WIW		
Sample size, n	Parameter	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
50	θ	0.5056	0.00562	0.00946	0.5275	0.02754	0.03327
	β	3.0745	0.07454	0.14025	3.399	0.39902	1.11667
	k	-			1.3346	0.33464	1.54536
	l	-			0.8742	-0.12578	0.11911
100	θ	0.502	0.00199	0.00486	0.5163	0.01626	0.02223
	β	3.0488	0.04880	0.06619	3.2935	0.29346	0.79699
	k	-			1.256	0.25604	1.16051
	l	-			0.8947	-0.10532	0.10025
200	θ	0.5007	0.00074	0.00229	0.5142	0.01418	0.01441
	β	3.0247	0.02471	0.02917	3.153	0.15299	0.46398
	k	-			1.121	0.12101	0.71501
	l	-			0.9492	-0.05076	0.06618
300	θ	0.5003	0.00032	0.00158	0.509	0.00899	0.01089
	β	3.0178	0.01777	0.02126	3.1196	0.11964	0.33067
	k	-			1.1025	0.10249	0.53677
	l	-			0.9609	-0.03905	0.05251
500	θ	0.5006	0.00057	0.00089	0.508	0.008043	0.0063
	β	3.0022	0.00222	0.01074	3.0569	0.056878	0.17724
	k	-			1.0396	0.039614	0.29899
	l	-			0.9797	-0.020335	0.03299
1000	θ	0.5003	0.00033	0.00043	0.5062	0.006174	0.00306
	β	3.0015	0.00151	0.00549	3.0171	0.017092	0.08441
	k	-			1.006	0.006035	0.14517
	l	-			0.9946	-0.005418	0.01981

different doses of tubercle bacilli. It is known that guinea pigs have high susceptibility of human tuberculosis. The data set consists of 72 observations. For the second example, the data is a subset of the breast feeding study from the National Longitudinal Survey of Youth, the complete data set is available in [10]. The data set considered here consists of the times to weaning 927 children of white-race mothers who choose to breast feed their children. The duration of the breast feeding was measured in weeks. Estimates of the parameters of WIW distribution (standard error in parentheses), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC) are given in Table 8 for the first data set and in Table 9 for the second data set. Plots of the fitted densities and histogram of the data are given in figures 5 and 6. Probability plots (Chambers et al. [5]) are also presented in figures 5 and 6. For the probability plot, we plotted for

Table 3: Simulation results for Model 1: Mean estimates, Average Bias and RMSEs of the parameters θ, β with $k=0$ and $l=0$

		WIW, l=0			WIW, k=0		
Sample size, n	Parameter	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
50	θ	0.5331	0.03308	0.10193	0.5129	0.012869	0.00624
	β	3.37	0.37002	1.10579	3.0906	0.090562	0.13476
	k	1.2905	0.29051	1.52578			
	l				0.9751	-0.024869	0.00216
100	θ	0.5015	0.00146	0.0737	0.5068	0.006829	0.003235
	β	3.3378	0.33783	0.83619	3.0373	0.037291	0.058164
	k	1.3033	0.30325	1.18555			
	l				0.9806	-0.019356	0.001479
200	θ	0.5247	0.02473	0.05581	0.5059	0.005931	0.001602
	β	3.1388	0.1388	0.45679	3.0108	0.010799	0.02751
	k	1.1122	0.11225	0.71587			
	l				0.9868	-0.013191	0.000619
300	θ	0.5189	0.018929	0.03914	0.503	0.003011	0.000948
	β	3.0963	0.096287	0.28324	3.0114	0.011392	0.018291
	k	1.0711	0.071086	0.47277			
	l				0.9894	-0.010597	0.000454
500	θ	0.5144	0.014358	0.02661	0.5035	0.003529021	0.000574
	β	3.054	0.054024	0.18532	3.0053	0.005339821	0.011493
	k	1.0414	0.041424	0.30938			
	l				0.9936	-0.006393745	0.000269
1000	θ	0.5022	0.002201	0.01326	0.5003	0.00028	0.000251493
	β	3.0374	0.037408	0.08778	3.0104	0.010412	0.005642621
	k	1.0329	0.032876	0.15123			
	l				0.9978	-0.002225	0.000099492

example,

$$G_{WIW}(x_{(j)}) = \frac{\Gamma(1-\frac{1}{\hat{\beta}}, 2\hat{\theta}x_{(j)}^{-\hat{\beta}})}{\Gamma(1-\frac{1}{\hat{\beta}})}$$

against $\dfrac{j-0.375}{n+0.25}$, $j=1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. We also computed a measure of closeness of each plot to the diagonal line. This measure of closeness is given by the sum of squares

$$SS = \sum_{j=1}^{n} \left[G_{WIW}(x_{(j)}) - \left(\frac{j-0.375}{n+0.25} \right) \right]^2 .$$

For the first dataset, the LR test statistic of the hypothesis H_0: WIW (1, β, 1, 1) against H_a: $WIW(\theta, \beta, 1, 1)$, is $w = 950.7 - 801.7 = 149.0$.

Table 4: Simulation results for Model 2: Mean estimates, Average Bias and RMSEs

		IW			WIW		
Sample size, n	Parameter	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
50	θ	1.2233	0.02328	0.03193	1.2827	0.08274	0.06
	β	4.7348	0.13483	0.28005	4.5406	-0.05935	0.29298
	k	-			0.744	-0.25596	0.67972
	l	-			1.0309	0.03094	0.01757
100	θ	1.2133	0.013254	0.01538	1.2441	0.0441	0.0313
	β	4.6599	0.059884	0.1311	4.5775	-0.02254	0.23071
	k	-			0.871	-0.12903	0.55897
	l	-			1.0147	0.01472	0.01124
200	θ	1.2074	0.007447	0.007283	1.2298	0.02985	0.01594
	β	4.6297	0.029682	0.063478	4.5723	-0.027728	0.20737
	k	-			0.8929	-0.10713	0.4495
	l	-			1.0114	0.011388	0.00733
300	θ	1.2051	0.005126	0.005201	1.2283	0.028268	0.01215
	β	4.6167	0.016657	0.043962	4.5588	-0.04115	0.19049
	k	-			0.8995	-0.1005	0.41477
	l	-			1.0134	0.01339	0.00657
500	θ	1.2027	0.002675	0.002887	1.2224	0.0224	0.00886
	β	4.6136	0.013564	0.026183	4.554	-0.046	0.16117
	k	-			0.9149	-0.085144	0.35118
	l	-			1.0121	0.0121	0.00531
1000	θ	1.2014	0.001405606	0.00142	1.2142	0.014229	0.00501
	β	4.6049	0.004931619	0.01304	4.5746	-0.025379	0.11855
	k	-			0.9455	-0.054508	0.24502
	l	-			1.0054	0.005373	0.00371

The p-value is $2.86\text{x}10^{-34} < 0.001$. Therefore, we reject H_0 in favor of H_a: $WIW(\theta, \beta, 1, 1)$. We can also test the hypothesis H_0: WIW $(1, \beta, 1, 1)$ against H_a: $WIW(\theta, \beta, 1, 1)$, is $w = 836.7 - 801.7 = 35.0$. The p-value is $3.29\text{x}10^{-9} < 0.001$. Therefore, we reject H_0 in favor of H_a: $WIW(\theta, \beta, 1, 1)$. The values of the statistics AIC, AICC and BIC shows that sub-model WIW $(\theta, \beta, 0, 1)$ is a "better" fit for this data. Also, the value of SS given in the probability plot is smallest for this model. For the second dataset, the LR test statistic of the hypothesis H_0: WIW $(1, \beta, 1, 1)$ against H_a: WIW $(\theta, \beta, 1, 1)$, is $w = 7311.3 - 7268.6 = 42.7$. The p-value is < 0.0001. Therefore, we reject H_0 in favor of H_a: WIW $(\theta, \beta, 1, 1)$. According to the values of the statistics AIC, AICC and BIC shows that model WIW $(\theta, \beta, 0, 1)$ is a "better" fit for the second dataset. Also, the value of SS corresponded to this model is the smallest.

Table 5: Simulation results for Model 2: Mean estimates, Average Bias and RMSEs of the parameters θ, β with $k = 0$ and $l = 0$

		WIW, l=0			WIW, k=0		
Sample size, n	Parameter	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
50	θ	1.3143	0.11427	0.11366	1.2421	0.04215	0.02991
	β	4.5703	-0.02975	0.27167	4.7382	0.13819	0.32755
	k	0.8014	-0.19859	0.62554			
	l				1.0124	0.01239	0.0067
100	θ	1.2879	0.08786	0.07876	1.2125	0.01245	0.01009
	β	4.5211	-0.07886	0.25823	4.6621	0.062093	0.14815
	k	0.8228	-0.17723	0.54967			
	l				1.004	0.003983	0.00231
200	θ	1.2649	0.0649	0.06273	1.2091	0.009139	0.005537
	β	4.565	-0.03501	0.21171	4.624	0.023984	0.066618
	k	0.8899	-0.1101	0.47792			
	l				1.0036	0.003556	0.001258
300	θ	1.2409	0.040854	0.04763	1.2075	0.007451	0.00329
	β	4.5706	-0.029358	0.18307	4.6225	0.022453	0.042093
	k	0.9202	-0.079816	0.40167			
	l				1.0026	0.0026	0.000769
500	θ	1.2402	0.040234	0.04053	1.2051	0.005124	0.001949
	β	4.5664	-0.033586	0.15896	4.6102	0.010249	0.025118
	k	0.9222	-0.077767	0.35341			
	l				1.0009	0.000931	0.000502
1000	θ	1.2286	0.028571	0.0278	1.2023	0.002313918	0.000905
	β	4.5766	-0.023371	0.12162	4.6069	0.006890177	0.011887
	k	0.9456	-0.054446	0.256			
	l				1.0009	0.000889	0.00034

8 Concluding Remarks

Statistical properties of the weighted generalized inverse Weibull distribution and its sub-models including the pdf, cdf, moment, hazard function, reverse hazard function, coefficient of variation, skewness and kurtosis, Fisher information, Shannon entropy, Renyi entropy and s-entropy are given. Estimation of the parameters of the models are also presented. Extensive simulation studies is carried out to examine the bias and mean square error of the maximum likelihood estimators of the parameters. Applications of the WIW distribution to real data shows a good fit. The proposed class of distributions contains a large number of distributions with potential applications to several areas of probability and statistics, finance, economics, and medicine.

Table 6: Simulation results for Model 3: Mean estimates, Average Bias and RMSEs

		IW			WIW		
Sample size, n	Parameter	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
50	θ	1.0161	0.01612	0.02519	1.08660	0.08661	0.08435
	β	2.0592	0.05924	0.05569	2.15480	0.15476	0.43824
	k	-			1.08300	0.08303	0.60451
	l	-			1.03870	0.03874	0.03482
100	θ	1.0066	0.00657	0.01123	1.04380	0.04379	0.04818
	β	2.0217	0.02166	0.02607	2.12930	0.12935	0.32394
	k	-			1.08870	0.08873	0.45989
	l	-			1.01550	0.01554	0.02062
200	θ	1.0033	0.00329	0.00620	1.02320	0.02141	0.02522
	β	2.012	0.01205	0.01326	2.09120	0.09122	0.19365
	k	-			1.07590	0.07591	0.26830
	l	-			1.00580	0.00356	0.01040
300	θ	1.0031	0.00312	0.00394	1.02140	0.02323	0.01710
	β	2.0112	0.01122	0.00805	2.04670	0.04673	0.13609
	k	-			1.03140	0.03141	0.19059
	l	-			1.00360	0.00579	0.00650
500	θ	1.0017	0.00231	0.00229	1.00650	0.00646	0.01062
	β	2.0039	0.00387	0.00493	2.05690	0.05695	0.09171
	k	-			1.05270	0.05274	0.12723
	l	-			1.00190	0.00190	0.00452
1000	θ	1.0023	0.00166	0.00107	1.00440	0.00440	0.00530
	β	2.0015	0.00146	0.00230	2.01720	0.01717	0.04023
	k	-			1.01340	0.01344	0.05693
	l	-			1.00090	0.00095	0.00180

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Table 7: Simulation results for Model 3: Mean estimates, Average Bias and RMSEs of the parameters θ, β with $k = 0$ and $l = 0$

		WIW, l=0			WIW, k=0		
Sample size, n	Parameter	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
50	θ	2.138	0.13796	0.44436	1.0263	0.02631	0.02051
	β	1.1749	0.17487	0.35592	2.0569	0.05695	0.05692
	k	1.0979	0.09785	0.62271			
	l				1.0061	0.00613	0.00222
100	θ	1.0692	0.06921	0.19982	1.0114	0.01141	0.00838
	β	2.1335	0.13347	0.31485	2.0298	0.02983	0.02488
	k	1.0648	0.06476	0.43609			
	l				1.0052	0.00522	0.00098
200	θ	1.0361	0.03610	0.12421	1.0061	0.00606	0.00398
	β	2.0993	0.09933	0.21118	2.0105	0.01054	0.01279
	k	1.0836	0.08358	0.29572			
	l				1.0051	0.00509	0.00046
300	θ	1.028	0.02799	0.07702	1.0053	0.00529	0.00267
	β	2.0589	0.05889	0.14201	2.0131	0.01306	0.00845
	k	1.0485	0.04847	0.19445			
	l				1.0035	0.00345	0.00033
500	θ	1.0214	0.02137	0.04729	1.0042	0.00421	0.00163
	β	2.0309	0.03092	0.08440	2.0072	0.00722	0.00519
	k	1.029	0.02904	0.11847			
	l				1.0015	0.00145	0.00027
1000	θ	1.0014	0.00139	0.02373	1.002	0.00205	0.00072
	β	2.0309	0.03093	0.04344	2.0023	0.00229	0.00253
	k	1.0245	0.02448	0.06076			
	l				1.0004	0.00038	0.00023

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Table 8: Estimates of models for Bjerkedal data set

	Estimates					Statistics			
Model	θ	β	k	l	$-2\log L$	AIC	$AICC$	BIC	SS
WIW(θ,β,0,0)	283.84	1.4181	0	0	791.3	795.3	795.5	799.9	0.2572
	125.63	0.1173							
WIW(θ,β,1,1)	679.67	2.057	1	1	801.7	805.7	805.8	810.2	0.42460
	302.43	0.1095							
WIW(θ,1.5,1,1)	64.3826	1.5	1	1	836.7	838.7	838.8	841	2.3415
	13.142								
WIW(1,β,1,1)	1	1.2702	1	1	950.7	952.7	952.8	955	4.1629
		0.03014							
WIW(θ,β,1,0)	1359.33	2.057	1	0	801.7	805.7	805.8	810.2	0.42462
	604.86	0.1095							
WIW(θ,β,0,1)	141.92	1.4148	0	1	791.3	795.3	795.5	799.9	0.2453
	62.8164	0.1173							

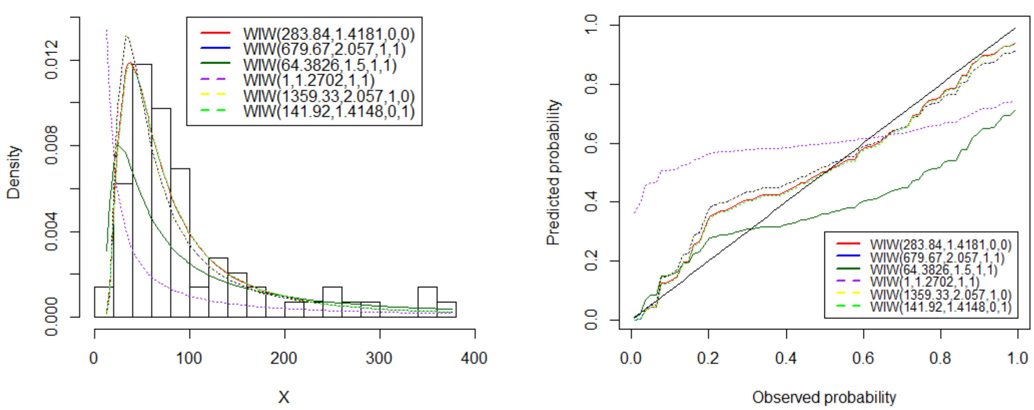

Figure 5: Fitted density and probability plots for guinea pigs survival time

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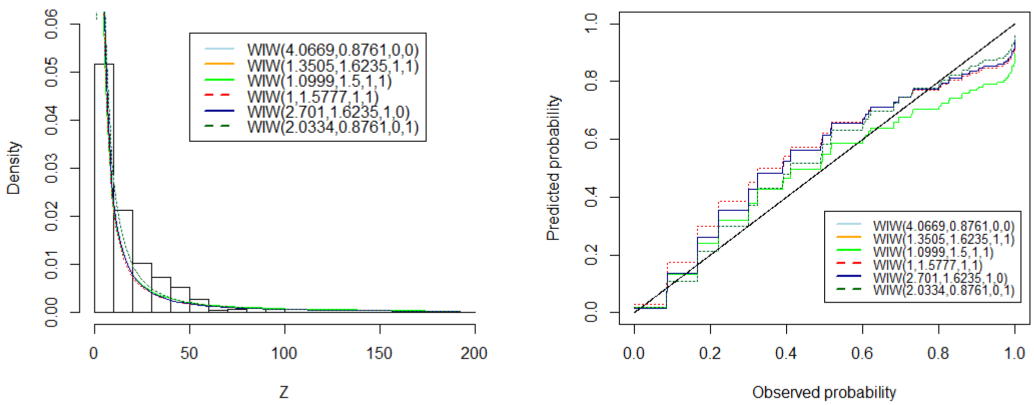

Figure 6: Fitted density and probability plots for breast feeding data

Table 9: Estimates of models for the breast feeding data set

	Estimates				Statistics				
Model	θ	β	k	l	$-2\log L$	*AIC*	*AICC*	*BIC*	*SS*
WIW(θ, β,0,0)	4.0669	0.8761	0	0	7134.3	7138.3	7138.3	7148	2.6436
	(0.1619)	(0.02136)							
WIW(θ, β,1,1)	1.3505	1.6235	1	1	7268.6	7272.6	7272.6	7282.3	5.9571
	(0.08192)	(0.02008)							
WIW(θ,1.5,1,1)	1.0999	1.5	1	1	7311.3	7313.3	7313.3	7318.1	4.8026
	(0.06257)								
WIW(1,β,1,1)	1	1.5777	1	1	7290.7	7292.7	7292.7	7297.6	7.9303
		(0.01685)							
WIW(θ, β,1,0)	2.701	1.6235	1	0	7268.6	7272.6	7272.6	7282.3	5.9571
	(0.1638)	(0.02008)							
WIW(θ, β,0,1)	2.0334	0.8761	0	1	7134.3	7138.3	7138.3	7148	2.6440
	(0.08097)	(0.02136)							

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Appendix A. Some Basic Utility Notions

Here, some basic utility notions and definitions are presented. The n^{th}-order derivative formula of gamma function is given by:

$$\Gamma^{(n)}(s) = \int_0^\infty z^{s-1} (\log z)^n \exp(-z)\, dz. \tag{43}$$

This derivative is used in this paper. The lower incomplete gamma function and the upper incomplete gamma function are

$$\gamma(s,x) = \int_0^x t^{s-1} e^{-t}\, dt \quad \text{and} \quad \Gamma(s,x) = \int_x^\infty t^{s-1} e^{-t}\, dt, \tag{44}$$

respectively. Also, $(1-z)^{b-1} = \sum_{j=0}^\infty \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)\Gamma(j+1)} z^j$, for real non-integer $b > 0$ and $|z| < 1$.

Appendix B. Fisher information

The FI for the WIW distribution, that X contains about the parameters $\Theta = (\theta, \beta)$ is obtained below. We have the following partial derivatives of $\ln[g(x;\theta,\beta)]$ with respect to the parameters:

$$\frac{\partial \ln g_{WIW}(x;\theta,\beta)}{\partial \theta} = \frac{1-\frac{1}{\beta}}{\theta} - 2x^{-\beta}, \tag{45}$$

$$\begin{aligned} \frac{\partial \ln g_{WIW}(x;\theta,\beta)}{\partial \beta} &= \frac{\ln(2\theta)}{\beta^2} + 2\theta x^{-\beta} \ln(x) \\ &\quad + \frac{1}{\beta} - \ln(x) - \frac{\Psi\left(1-\frac{1}{\beta}\right)}{\beta^2}, \end{aligned} \tag{46}$$

We differentiate (45) with respect to θ and β, we obtain:

$$\frac{\partial^2 \ln g_{WIW}(x;\theta,\beta)}{\partial\theta^2} = -\frac{1-\frac{1}{\beta}}{\theta^2}, \tag{47}$$

$$\frac{\partial^2 \ln g_{WIW}(x;\theta,\beta)}{\partial\theta\partial\beta} = \frac{1}{\theta\beta^2} + 2x^{-\beta}\ln(x), \tag{48}$$

Differentiating (46) with respect to β, we get:

$$\begin{aligned} \frac{\partial^2 \ln g_{WIW}(x;\theta,\beta)}{\partial\beta^2} &= -\frac{2\ln(2\theta)}{\beta^3} - 2\theta x^{-\beta}\ln^2(x) \\ &\quad -4\theta x^{-\beta}\ln(x) - \frac{1}{\beta^2} \\ &\quad -\frac{\Psi\left(1,\, 1-\frac{1}{\beta}\right)}{\beta^4} + \frac{2\Psi(1-\frac{1}{\beta})}{\beta^3}. \end{aligned} \tag{49}$$

Now, we compute the following expectations: $E\left[X^{-\beta}\right]$, $E\left[X^{-\beta}\ln(X)\right]$, $E\left[X^{-\beta}\ln^2(X)\right]$ in order to obtain FIM $I(\theta,\beta)$.

$$\begin{aligned} E\left[X^{-\beta}\right] &= \int_0^\infty x^{-\beta} g_{WIW}(x)dx \\ &= \int_0^\infty \frac{x^{-2\beta}[2\theta]^{1-\frac{1}{\beta}}\beta e^{-2\theta x^{-\beta}}\,dx}{\Gamma\left(1-\frac{1}{\beta}\right)}. \end{aligned}$$

Let $u = 2\theta x^{-\beta},\ du = -2\theta\beta x^{-\beta-1}\,dx$ and $x = \left(\frac{2\theta}{u}\right)^{\frac{1}{\beta}}$. The integral becomes:

$$\begin{aligned} E\left[X^{-\beta}\right] &= \int_0^\infty \left(\frac{2\theta}{u}\right)^{\frac{1}{\beta}-1} \frac{(2\theta)^{-\frac{1}{\beta}} e^{-u}\,du}{\Gamma\left(1-\frac{1}{\beta}\right)} \\ &= \frac{(2\theta)^{-1}\,\Gamma\left(2-\frac{1}{\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)}. \end{aligned} \tag{50}$$

$$\begin{aligned} E\left[X^{-\beta}\ln(X)\right] &= \int_0^\infty x^{-\beta}\ln(x) g_{WIW}(x)dx \\ &= \int_0^\infty \frac{x^{-2\beta}(2\theta)^{1-\frac{k}{\beta}}\beta\ln(x)\, e^{-2\theta x^{-\beta}}\,dx}{\Gamma\left(1-\frac{1}{\beta}\right)}. \end{aligned}$$

Making the same substitution $u = 2\theta x^{-\beta}$, we obtain:

$$\begin{aligned} E\left[X^{-\beta}\ln\left(X\right)\right] &= \int_0^\infty \left(\frac{2\theta}{u}\right)^{\frac{1}{\beta}-1} \frac{(2\theta)^{-\frac{1}{\beta}} e^{-u}\,du}{\Gamma\left(1-\frac{1}{\beta}\right)} \\ &\quad * \ln\left(\left(\frac{2\theta}{u}\right)^{\frac{1}{\beta}}\right) du \\ &= \frac{(2\theta)^{-1}\ln\left((2\theta)^{\frac{1}{\beta}}\right)\Gamma\left(2-\frac{1}{\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)} \\ &\quad + \frac{(2\theta)^{-1}\,\Gamma'\left(2-\frac{1}{\beta}\right)}{\beta\Gamma\left(1-\frac{1}{\beta}\right)}. \end{aligned} \tag{51}$$

$$\begin{aligned} E\left[X^{-\beta}\ln^2\left(X\right)\right] &= \int_0^\infty x^{-\beta}\ln^2(x) g_{WIW}(x)dx \\ &= \int_0^\infty \frac{x^{-2\beta}[2\theta]^{1-\frac{1}{\beta}}\beta\ln^2(x)\,e^{-2\theta x^{-\beta}}\,dx}{\Gamma\left(1-\frac{1}{\beta}\right)}. \end{aligned}$$

Making the same substitution one more time we get:

$$\begin{aligned} E\left[X^{-\beta}\ln^2\left(X\right)\right] &= \int_0^\infty \left(\frac{2\theta}{u}\right)^{\frac{1}{\beta}-1} \frac{(2\theta)^{-\frac{1}{\beta}} e^{-u}\,du}{\Gamma\left(1-\frac{1}{\beta}\right)} \\ &\quad * \ln^2\left(\left(\frac{2\theta}{u}\right)^{\frac{1}{\beta}}\right) du \\ &= \frac{(2\theta)^{-1}}{\Gamma\left(1-\frac{1}{\beta}\right)} \int_0^\infty u^{2-\frac{1}{\beta}-1} e^{-u}\left[\ln^2\left((2\theta)^{\frac{1}{\beta}}\right)\right. \\ &\quad \left. * -\frac{2}{\beta}\ln(u)\ln\left((2\theta)^{\frac{1}{\beta}}\right) + \frac{1}{\beta^2}\ln^2(u)\right] du \end{aligned}$$

$$\begin{aligned} &= \frac{(2\theta)^{-1}\ln^2\left((2\theta)^{\frac{1}{\beta}}\right)\Gamma\left(2-\frac{1}{\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)} \\ &\quad -\frac{(2\theta)^{-1}\,2\ln\left((2\theta)^{\frac{1}{\beta}}\right)\Gamma'\left(2-\frac{1}{\beta}\right)}{\beta\Gamma\left(1-\frac{1}{\beta}\right)} \\ &\quad +\frac{(2\theta)^{-1}\,\Gamma^{(2)}\left(2-\frac{1}{\beta}\right)}{\beta^2\Gamma\left(1-\frac{1}{\beta}\right)}. \end{aligned} \tag{52}$$

Now, the entries of FIM are given below:

$$\begin{aligned} I_{\beta\beta} &= \frac{1}{\beta^2}+\frac{2\ln(2\theta)}{\beta^3}+\frac{\Psi\left(1,1-\frac{1}{\beta}\right)}{\beta^4}-\frac{2\Psi\left(1-\frac{1}{\beta}\right)}{\beta^3} \\ &\quad +\frac{\ln^2(2\theta)\Gamma\left(2-\frac{1}{\beta}\right)}{\beta^2\Gamma\left(1-\frac{1}{\beta}\right)}-\frac{2\ln(2\theta)\Gamma'\left(2-\frac{1}{\beta}\right)}{\beta^2\Gamma\left(1-\frac{1}{\beta}\right)} \\ &\quad +\frac{\Gamma^{(2)}\left(2-\frac{1}{\beta}\right)}{\beta^2\Gamma\left(1-\frac{1}{\beta}\right)}+\frac{\ln(2\theta)\Gamma\left(2-\frac{1}{\beta}\right)}{\beta\Gamma\left(1-\frac{1}{\beta}\right)}+\frac{2\Gamma'\left(2-\frac{1}{\beta}\right)}{\beta\Gamma\left(1-\frac{1}{\beta}\right)}, \end{aligned} \tag{53}$$

$$I_{\theta\theta}=\frac{1-\frac{1}{\beta}}{\theta^2}, \tag{54}$$

$$I_{\theta\beta} = -\frac{1}{\beta^2\theta}-\frac{\ln(2\theta)\Gamma\left(2-\frac{1}{\beta}\right)}{\theta\beta\Gamma\left(1-\frac{1}{\beta}\right)}-\frac{\Gamma'\left(2-\frac{1}{\beta}\right)}{\theta\beta\Gamma\left(1-\frac{1}{\beta}\right)}. \tag{55}$$